4. Exercise sheet for the lecture Statistical Physics

Hand in your solutions on Thursday, 11.11.2010 or Friday, 12.11.2010 in the exercise group.

Exercise 4.1: *Model for a rubber band* **(10 points)**

A simple picture of a rubber band is a single long chain of linking groups of atoms, with the links oriented in any direction. When the rubber band is pulled so that the chain of atoms is completely linear there is only one possible arrangement and the entropy is zero; when the rubber band is all tangled up there are a huge number of arrangements of the links leading to a large entropy. We simplify matters by assuming that the links can lie in only two directions,

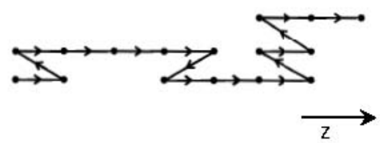

Fig. 1: A collection of links representing a simplified model of a rubber band. The links lie in the $+z$ and $-z$ direction. The links in $-z$ direction are drawn with an angle so that the arrangement is seen more clearly. The quantum state shown here would be $|+,-,+,+,+,+,-,+,+,+,-,+,-,+,+\rangle$

either in direction of increasing z or in opposite direction (see Fig. 1). Start at one end of the chain (the end with the smaller value of z) and count how many lie along the $+z$ direction ($= n_+$) and how many lie along the $-z$ direction ($= n_-$). There can be different ways of ending up with n_+ links in the $+z$ direction. Each such arrangement can be represented as a different quantum state (see Fig. 1). The total extension of the rubber band, the distance from one end of the chain to the other, is l. Let d be the length of the links. The quantum state shown in Fig. 1 has an extension of $7d$. There are many other quantum states with this length (how many?). The complete set of quantum states of length $7d$ make up the macrostate. We are interested in much longer molecules with many more links than 15. Let there be n_+ links going to the right and n_- going to the left.

If F is the force of tension of the rubber band, the change of internal energy of the rubber band is $dU = TdS + Fdl$ from which we get for the force of tension F

$$\frac{F}{T} = -\left(\frac{\partial S}{\partial l}\right)_U = -k_B \left(\frac{\partial \ln(\Omega)}{\partial l}\right)_U$$

with $\Omega =$ the number of microstates corresponding to a total length of the rubber band of l.

a) Find a relation between n_-, n_+ and N and expresse l in terms of n_+, N and d. Determine $\Omega(l) = \Omega(N, n_+)$. (2 point)

b) Determine the entropy $S = k_B \ln(\Omega)$ of the system as a function of l, d and N for large $N \gg 1$. (4 points)
Hint: It is useful to define a parameter $x := \frac{l}{Nd}$. Nd is the length that occurs when all the links point the same way, that is the maximum length of the chain molecule that makes up the rubber band. Thus x is a measure of how much the band has been stretched.

c) Determine the force of tension F as a function of the temperature T, the total length l and total number of links N and the link length d. Show, that for small values of $x = \frac{l}{Nd}$ one gets
$$F \approx \frac{k_B T l}{N d^2}$$
What happens with a rubber band under tension if it is heated? Give a physical interpretation. (4 points)

Exercise 4.2: *Non interacting spins on a lattice* **(10 points)**

Suppose there are N particles placed on lattice sites with each particle having a spin $\frac{1}{2}$ with an associated magnetic moment μ_B. The system is placed in a homogenous, constant external magnetic field $\vec{B} = B\,\vec{e}_z$. The Hamilton-operator is then given by

$$\hat{H} = -\sum_{i=1}^{N} \hat{\vec{\mu}}_i \cdot \vec{B} = -2\mu_B\,B \sum_{i=1}^{N} \hat{S}_i^z$$

where μ_B = Bohr magneton.

The eigenstates of $\hat{H}$

$$\hat{H}|\sigma_1\sigma_2...\sigma_N\rangle = -2\mu_B\,B \sum_{i=1}^{N} \hat{S}_i^z\,|\sigma_1\sigma_2...\sigma_N\rangle$$

fulfill

$$\hat{S}_i^z\,|\sigma_1\sigma_2...\sigma_N\rangle = \sigma_i|\sigma_1\sigma_2\ldots\sigma_N\rangle$$

with $\sigma_i \in \{-\frac{1}{2}, +\frac{1}{2}\}$.

a) Let n_1 be the number of particles with spin up and $n_2 = N - n_1$ the number of particles with spin down. Give the possible energy eigenvalues $U = E(n_1, N)$ (=total energy of the system) in terms of N, n_1 and $\epsilon := \mu_B B$. What is their degree of degeneracy?
. (2 points)

b) Determine the entropy S as a function of $x = \frac{U}{N\epsilon}$ for large $N \gg 1$ (Hint: Use *Stirlings* approximation). Give a sketch of $\frac{S}{Nk_B}$ as a function of x. (2 points)

c) The temperature is given by the equation $\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_V$. Give the temperature of the system as a function of ϵ and x. Sketch $T(x)$. What do you get for the sign of the temperature T in dependence on x? (3 points)

d) Use the result of c) to express the total energy U of the system in terms of N, ϵ and T. (2 points)

e) Calculate the everage total magnetic moment of the system $M = \mu_B(n_1 - n_2)$ as a function of μ_B, N, B and the temperature T. This is the equation of state for the magnetization of the system made up of non-interacting spin 1/2 particles in a magnetic induction field, B. (1 point)