9. Exercise sheet for the lecture Statistical Physics

Hand in your solutions on Thursday, 16.12.2010 or Friday, 17.12.2010 in the exercise group.

Exercise 9.1 *Ultra-relativistic ideal Fermi gas* **(3 points)**

An ultra-relativistic ideal Fermi gas is in a volume V. The chemical potential is μ. Consider in the following only the special case $T = 0$. Calculate the mean particle number $\bar{N}$ and the mean total energy $\bar{E}$. Express $\bar{E}$ in terms of $\bar{N}$ and μ.

Hint: Ultra-relativistic means that you could neglect the mass term in the relativistic formula for the energy of the particles: $\epsilon(\vec{p}) = \sqrt{m^2c^4 + \vec{p}^2c^2} \approx pc$

Exercise 9.2 *Fermi gas: Electrons in a metal* **(7 points)**

A metal contains conduction electrons in a volume V. The temperature is $T = 0$, the chemical potential is $\mu = \epsilon_F$ and the magnetic moment of the electrons is μ_m.
A magnetic field $\vec{B}$ is switched on, which is so weak that it only has to be considered to first order in the following.

a) Calculate the mean electron number $N_\pm$ for electrons with spin parallel and antiparallel to the magnetic field, respectively. (4 points)

b) Calculate the mean magnetisation M and the susceptibility $\chi = \left(\frac{\partial M}{\partial B}\right)_{N,V,T=0}$. (1 point)

c) Express the chemical potential in terms of the mean total number of electrons $N = N_+ + N_-$ and by this means, eliminate the chemical potential in the formula for the susceptibility. (2 point)

Exercise 9.3 *Liquid ^{3}He* **(2 point)**

Liquid ^{3}He is an ideal Fermi gas: Determine the value of the chemical potential $\mu = \epsilon_F$ for ^{3}He at $T = 0$. The density of the liquid is $0.081 g/cm^3$.

Exercise 9.4 *Single particle density of states* **(3 points)**

The single particle density of states $\Omega_1(\epsilon)$ for non-relativistic fermions is given by (see lecture)

$$\Omega_1(\epsilon) = \frac{g_I V}{2\pi^2 N} \int_0^\infty d|k||k|^2 \delta\left(\epsilon - \frac{\hbar^2|k|^2}{2M}\right) \qquad (*)$$

It was shown in the lecture, that for $\epsilon > 0$

$$\Omega_1(\epsilon) = \frac{3}{2}\epsilon_F^{-3/2}\epsilon^{1/2} \qquad (**)$$

gives the right result for the mean energy $\bar{E}$ of the system:

$$\bar{E} = \int_0^{\epsilon_F} d\epsilon\, \epsilon\, \Omega_1(\epsilon) = \frac{3}{5}\epsilon_F \quad .$$

Proof (**) by evaluating (*).

Exercise 9.5 *Pressure of a Fermi gas* **(5 points)**

a) By evaluating $J = -k_B T \ln Z_{gc} = -k_B T \sum_\nu \ln\left(1 + e^{-\beta(E_\nu - \mu)}\right)$ and using $P = -J/V$ show that the pressure of a Fermi gas is given by

$$P = \frac{2k_B T}{3\pi^2}\left(\frac{2mk_B T}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{x^4}{e^{x^2-\eta}+1}\,dx$$

with $\eta = \frac{\mu}{k_B T}$. (3 points)

b) Now consider the special case of $T = 0$. Show that the pressure of the Fermi gas at absolute zero of temperature is given by

$$P = \frac{2}{5} n \epsilon_F \,.$$

(2 points)

Before I came here I was confused about this subject. Having listened to your lecture I am still confused. But on a higher level. *Enrico Fermi*