C Magnetostatics

©2000/01 Franz Wegner Universität Heidelberg

In this chapter we consider magnetostatics starting from the equations, which were derived at the beginning of section (3.a) for time independent currents.

9 Magnetic Induction and Vector Potential

9.a Ampere's Law

From

$$\operatorname{curl} \mathbf{B}(\mathbf{r}) = \frac{4\pi}{\mathrm{c}} \mathbf{j}(\mathbf{r}) \tag{9.1}$$

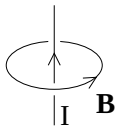
one obtains

$$\int d\mathbf{f} \cdot \operatorname{curl} \mathbf{B}(\mathbf{r}) = \frac{4\pi}{\mathrm{c}} \int d\mathbf{f} \cdot \mathbf{j}(\mathbf{r}), \tag{9.2}$$

which can be written by means of Stokes' theorem (B.56)

$$\oint d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = \frac{4\pi}{\mathrm{c}} I. \tag{9.3}$$

The line integral of the magnetic induction $\mathbf{B}$ along a closed line yields $4\pi/\mathrm{c}$ times the current through the line.Here the corkscrew rule applies: If the current moves in the direction of the corkscrew, then the magnetic induction has the direction in which the corkscrew rotates.



9.b Magnetic Flux

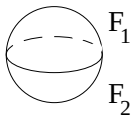
The magnetic flux Ψ^{m} through an oriented area F is defined as the integral

$$\Psi^{\mathrm{m}} = \int_F d\mathbf{f} \cdot \mathbf{B}(\mathbf{r}). \tag{9.4}$$

The magnetic flux depends only on the boundary ∂F of the area. To show this we consider the difference of the flux through two areas F_1 and F_2 with the same boundary and obtain

$$\Psi_1^{\mathrm{m}} - \Psi_2^{\mathrm{m}} = \int_{F_1} d\mathbf{f} \cdot \mathbf{B}(\mathbf{r}) - \int_{F_2} d\mathbf{f} \cdot \mathbf{B}(\mathbf{r}) = \int_F d\mathbf{f} \cdot \mathbf{B}(\mathbf{r}) = \int_V d^3r \operatorname{div} \mathbf{B}(\mathbf{r}) = 0 \tag{9.5}$$

by means of the divergence theorem (B.59) and $\operatorname{div} \mathbf{B}(\mathbf{r}) = 0$.Suppose F_1 and F_2 are oriented in the same direction (for example upwards). Then the closed surface F is composed of F_1 and F_2, where F_2 is now oriented in the opposite direction.Then F has a definite orientation (for example outwards) and includes the volume V.



9.c Field of a Current Distribution

From $\operatorname{curl}\operatorname{curl}\mathbf{B}(\mathbf{r}) = (4\pi/\mathrm{c})\operatorname{curl}\mathbf{j}(\mathbf{r})$ due to (B.26)

$$\operatorname{curl}\operatorname{curl}\mathbf{B}(\mathbf{r}) = \operatorname{grad}\operatorname{div}\mathbf{B}(\mathbf{r}) - \triangle\mathbf{B}(\mathbf{r}) \tag{9.6}$$

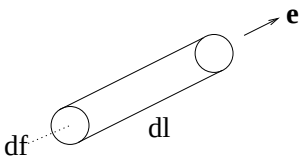
and $\operatorname{div}\mathbf{B}(\mathbf{r}) = 0$ one obtains

$$\triangle\mathbf{B}(\mathbf{r}) = -\frac{4\pi}{\mathrm{c}}\operatorname{curl}\mathbf{j}(\mathbf{r}) \tag{9.7}$$

with the solution

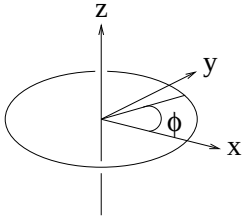
$$\mathbf{B}(\mathbf{r}) = \frac{1}{\mathrm{c}}\int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{curl}'\mathbf{j}(\mathbf{r}') = -\frac{1}{\mathrm{c}}\int d^3r' \left(\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{j}(\mathbf{r}') = \frac{1}{\mathrm{c}}\int d^3r' \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3}\times\mathbf{j}(\mathbf{r}'), \tag{9.8}$$

where we have used (B.63) at the second equals sign. The last expression is called the law of Biot and Savart.If the extension of a wire perpendicular to the direction of the current is negligible (filamentary wire) then one can approximate $d^3r'\mathbf{j}(\mathbf{r}') = df'dl'j(\mathbf{r}')\mathbf{e} = Id\mathbf{r}'$ and obtains



$$\mathbf{B}(\mathbf{r}) = \frac{I}{\mathrm{c}}\int \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3}\times d\mathbf{r}' \tag{9.9}$$

As an example we consider the induction in the middle axis of a current along a circle



$$\mathbf{r} = z\mathbf{e}_\mathrm{z}, \quad \mathbf{r}' = (R\cos\phi, R\sin\phi, z') \quad d\mathbf{r}' = (-R\sin\phi, R\cos\phi, 0)d\phi \tag{9.10}$$

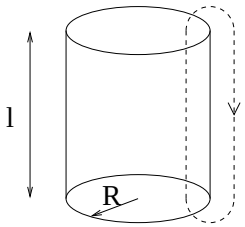
$$(\mathbf{r}'-\mathbf{r})\times d\mathbf{r}' = (R(z-z')\cos\phi, R(z-z')\sin\phi, R^2)d\phi \tag{9.11}$$

$$\mathbf{B}(0,0,z) = \frac{2\pi IR^2\mathbf{e}_\mathrm{z}}{\mathrm{c}(R^2+(z-z')^2)^{3/2}}. \tag{9.12}$$

Starting from this result we calculate the field in the axis of a coil. The number of windings be N and it extends from $z' = -l/2$ to $z' = +l/2$. Then we obtain

$$\mathbf{B}(0,0,z) = \int_{-l/2}^{+l/2}\frac{Ndz'}{l}\frac{2\pi IR^2\mathbf{e}_\mathrm{z}}{\mathrm{c}(R^2+(z-z')^2)^{3/2}} = \frac{2\pi IN}{cl}\mathbf{e}_\mathrm{z}\left(\frac{\frac{l}{2}-z}{\sqrt{R^2+(\frac{l}{2}-z)^2}} + \frac{\frac{l}{2}+z}{\sqrt{R^2+(\frac{l}{2}+z)^2}}\right). \tag{9.13}$$

If the coil is long, $R \ll l$, then one may neglect R^2 and obtains inside the coil



$$\mathbf{B} = \frac{4\pi IN}{cl}\mathbf{e}_\mathrm{z}. \tag{9.14}$$

At the ends of the coil the field has decayed to one half of its intensity inside the coil. From Ampere's law one obtains by integration along the path described in the figure

$$\oint d\mathbf{r}\cdot\mathbf{B} = \frac{4\pi}{\mathrm{c}}IN. \tag{9.15}$$

Thus inside the coil one obtains the induction (9.14), whereas the magnetic induction outside is comparatively small.

9.d Vector Potential

We now rewrite the expression for the magnetic induction

$$\mathbf{B}(\mathbf{r}) = -\frac{1}{\mathrm{c}}\int d^3r'\left(\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{j}(\mathbf{r}') = \frac{1}{\mathrm{c}}\int d^3r'\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\times\mathbf{j}(\mathbf{r}') = \operatorname{curl}\mathbf{A}(\mathbf{r}) \tag{9.16}$$

with

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\mathrm{c}} \int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{9.17}$$

One calls $\mathbf{A}$ the vector potential. We show, that $\mathbf{A}$ is divergence-free

$$\begin{aligned} \mathrm{div}\,\mathbf{A}(\mathbf{r}) &= \frac{1}{\mathrm{c}} \int d^3r' \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \cdot \mathbf{j}(\mathbf{r}') = -\frac{1}{\mathrm{c}} \int d^3r' \left(\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \cdot \mathbf{j}(\mathbf{r}') \\ &= \frac{1}{\mathrm{c}} \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \mathbf{j}(\mathbf{r}') = 0. \end{aligned} \tag{9.18}$$

At the third equals sign we have performed a partial integration (B.62). Finally we have used $\mathrm{div}\,\mathbf{j}(\mathbf{r}) = 0$.

9.e Force Between Two Circuits

Finally, we consider the force between two circuits. The force exerted by circuit (1) on circuit (2) is

$$\begin{aligned} \mathbf{K}_2 &= \frac{1}{\mathrm{c}} \int d^3r \mathbf{j}_2(\mathbf{r}) \times \mathbf{B}_1(\mathbf{r}) = \frac{1}{\mathrm{c}^2} \int d^3r d^3r' \mathbf{j}_2(\mathbf{r}) \times \left(\left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \times \mathbf{j}_1(\mathbf{r}') \right) \\ &= \frac{1}{\mathrm{c}^2} \int d^3r d^3r' \big(\mathbf{j}_1(\mathbf{r}') \cdot \mathbf{j}_2(\mathbf{r})\big) \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\mathrm{c}^2} \int d^3r d^3r' \Big(\left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \cdot \mathbf{j}_2(\mathbf{r}) \Big) \mathbf{j}_1(\mathbf{r}') \end{aligned} \tag{9.19}$$

where (B.14) has been applied. Since due to (B.62)

$$\int d^3r \big(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\big) \cdot \mathbf{j}_2(\mathbf{r}) = -\int d^3r \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla \cdot \mathbf{j}_2(\mathbf{r}) \tag{9.20}$$

and $\mathrm{div}\,\mathbf{j}_2(\mathbf{r}) = 0$, one obtains finally for the force

$$\mathbf{K}_2 = \frac{1}{\mathrm{c}^2} \int d^3r d^3r' \big(\mathbf{j}_1(\mathbf{r}') \cdot \mathbf{j}_2(\mathbf{r})\big) \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{9.21}$$

The force acting on circuit (1) is obtained by exchanging 1 and 2. Simultaneously, one can exchange $\mathbf{r}$ and $\mathbf{r}'$. One sees then, that

$$\mathbf{K}_1 = -\mathbf{K}_2 \tag{9.22}$$

holds.

10 Loops of Current as Magnetic Dipoles

10.a Localized Current Distribution and Magnetic Dipole

We consider a distribution of currents which vanishes outside a sphere of radius R ($\mathbf{j}(\mathbf{r}') = \mathbf{0}$ for $r' > R$) and determine the magnetic induction $\mathbf{B}(\mathbf{r})$ for $r > R$. We may expand the vector potential $\mathbf{A}(\mathbf{r})$ (9.17) similar to the electric potential $\Phi(\mathbf{r})$ in section (4)

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\mathrm{c}} \int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{cr} \int d^3r' \mathbf{j}(\mathbf{r}') + \frac{x_\alpha}{cr^3} \int d^3r' x'_\alpha \mathbf{j}(\mathbf{r}') + ... \tag{10.1}$$

Since no current flows through the surface of the sphere one obtains

$$0 = \int d\mathbf{f} \cdot g(\mathbf{r})\mathbf{j}(\mathbf{r}) = \int d^3r \,\mathrm{div}\,(g(\mathbf{r})\mathbf{j}(\mathbf{r})) = \int d^3r \,\mathrm{grad}\, g(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}) + \int d^3r g(\mathbf{r}) \,\mathrm{div}\mathbf{j}(\mathbf{r}), \tag{10.2}$$

where the integrals are extended over the surface and the volume of the sphere, respectively. From the equation of continuity (1.12,3.1) it follows that

$$\int d^3r \,\mathrm{grad}\, g(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}) = 0. \tag{10.3}$$

This is used to simplify the integral in the expansion (10.1). With $g(\mathbf{r}) = x_\alpha$ one obtains

$$\int d^3r j_\alpha(\mathbf{r}) = 0. \tag{10.4}$$

Thus the first term in the expansion vanishes. There is no contribution to the vector potential decaying like $1/r$ in magnetostatics, that is no magnetic monopole. With $g(\mathbf{r}) = x_\alpha x_\beta$ one obtains

$$\int d^3r \big(x_\alpha j_\beta(\mathbf{r}) + x_\beta j_\alpha(\mathbf{r})\big) = 0. \tag{10.5}$$

Thus we can rewrite

$$\int d^3r x_\alpha j_\beta = \frac{1}{2} \int d^3r (x_\alpha j_\beta - x_\beta j_\alpha) + \frac{1}{2} \int d^3r (x_\alpha j_\beta + x_\beta j_\alpha). \tag{10.6}$$

The second integral vanishes, as we have seen. The first one changes its sign upon exchanging the indices α and β. One introduces

$$\int d^3r x_\alpha j_\beta = \frac{1}{2} \int d^3r (x_\alpha j_\beta - x_\beta j_\alpha) = \mathrm{c}\epsilon_{\alpha,\beta,\gamma} m_\gamma \tag{10.7}$$

and calls the resulting vector

$$\mathbf{m} = \frac{1}{2\mathrm{c}} \int d^3r' (\mathbf{r}' \times \mathbf{j}(\mathbf{r}')) \tag{10.8}$$

magnetic dipole moment. Then one obtains

$$A_\beta(\mathbf{r}) = \frac{x_\alpha}{cr^3} \mathrm{c}\epsilon_{\alpha,\beta,\gamma} m_\gamma + ... \tag{10.9}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m} \times \mathbf{r}}{r^3} + ... \tag{10.10}$$

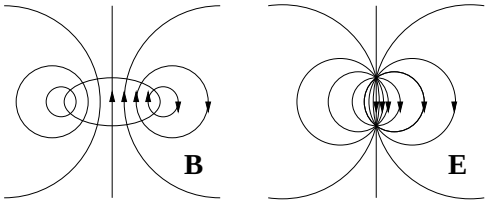
with $\mathbf{B}(\mathbf{r}) = \mathrm{curl}\,\mathbf{A}(\mathbf{r})$ one obtains

$$\mathbf{B}(\mathbf{r}) = \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r}) - \mathbf{m}r^2}{r^5} + ... \tag{10.11}$$

This is the field of a magnetic dipole. It has the same form as the electric field of an electric dipole (4.12)

$$\mathbf{E}(\mathbf{r}) = -\operatorname{grad}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right) = \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r}) - \mathbf{p}r^2}{r^5}, \tag{10.12}$$

but there is a difference at the location of the dipole. This can be seen in the accompanying figure. Calculate the $\delta^3(\mathbf{r})$-contribution to both dipolar moments. Compare (B.71).



10.b Magnetic Dipolar Moment of a Current Loop

The magnetic dipolar moment of a current on a closed curve yields

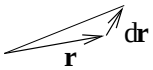
$$\mathbf{m} = \frac{I}{2\mathrm{c}}\int \mathbf{r}\times d\mathbf{r} = \frac{I}{\mathrm{c}}\mathbf{f}, \tag{10.13}$$

e.g.

$$m_\mathrm{z} = \frac{I}{2\mathrm{c}}\int (x dy - y dx) = \frac{I}{\mathrm{c}} f_\mathrm{z}. \tag{10.14}$$

Here f_α is the projection of the area inside the loop onto the plane spanned by the two other axes

$$d\mathbf{f} = \frac{1}{2}\mathbf{r}\times d\mathbf{r}. \tag{10.15}$$



If $\mathbf{j} = \sum_i q_i \mathbf{v}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$, then the magnetic moment reads

$$\mathbf{m} = \frac{1}{2\mathrm{c}}\sum_i q_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i \frac{q_i}{2m_i\mathrm{c}}\mathbf{l}_i, \tag{10.16}$$

where m_i is the mass and $\mathbf{l}_i$ the angular momentum. If only one kind of charges is dealt with, then one has

$$\mathbf{m} = \frac{q}{2m\mathrm{c}}\mathbf{l}. \tag{10.17}$$

This applies for orbital currents. For spins, however, one has

$$\mathbf{m} = \frac{q}{2mc} g\mathbf{s}, \tag{10.18}$$

where $\mathbf{s}$ is the angular momentum of the spin. The gyromagnetic factor for electrons is $g = 2.0023$ and the components of the spin $\mathbf{s}$ are $\pm\hbar/2$. Since in quantum mechanics the orbital angular momentum assumes integer multiples of $\hbar$, one introduces as unit for the magnetic moment of the electron Bohr's magneton, $\mu_B = \frac{\mathrm{e}_0\hbar}{2\mathrm{m}_0\mathrm{c}} = 0.927\cdot 10^{-20}$ dyn$^{1/2}$ cm^2.

10.c Force and Torque on a Dipole in an External Magnetic Field

10.c.α Force

An external magnetic induction $\mathbf{B}_\mathrm{a}$ exerts on a loop of a current the Lorentz force

$$\mathbf{K} = \frac{1}{\mathrm{c}}\int d^3r \mathbf{j}(\mathbf{r})\times\mathbf{B}_\mathrm{a}(\mathbf{r}) = -\frac{1}{\mathrm{c}}\mathbf{B}_\mathrm{a}(0)\times\int d^3r\mathbf{j}(\mathbf{r}) - \frac{1}{\mathrm{c}}\frac{\partial\mathbf{B}_\mathrm{a}}{\partial x_\alpha}\times\int d^3r x_\alpha j_\beta(\mathbf{r})\mathbf{e}_\beta - \ldots = -\frac{\partial\mathbf{B}_\mathrm{a}}{\partial x_\alpha}\times\mathbf{e}_\beta\epsilon_{\alpha,\beta,\gamma}m_\gamma. \tag{10.19}$$

We rewrite $m_\gamma \epsilon_{\alpha,\beta,\gamma} \mathbf{e}_\beta = m_\gamma \mathbf{e}_\gamma \times \mathbf{e}_\alpha = \mathbf{m} \times \mathbf{e}_\alpha$ and find

$$\mathbf{K} = -\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha} \times (\mathbf{m} \times \mathbf{e}_\alpha) = (\mathbf{m} \cdot \frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}) \mathbf{e}_\alpha - (\mathbf{e}_\alpha \cdot \frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}) \mathbf{m}. \tag{10.20}$$

The last term vanishes because of $\operatorname{div} \mathbf{B} = 0$. The first term on the right hand side can be written $(\mathbf{m} \cdot \frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}) \mathbf{e}_\alpha = m_\gamma \frac{\partial B_{\mathrm{a},\gamma}}{\partial x_\alpha} \mathbf{e}_\alpha = m_\gamma \frac{\partial B_{\mathrm{a},\alpha}}{\partial x_\gamma} \mathbf{e}_\alpha = (\mathbf{m}\nabla) \mathbf{B}_\mathrm{a}$, where we have used $\operatorname{curl} \mathbf{B}_\mathrm{a} = \mathbf{0}$ in the region of the dipole. Thus we obtain the force

$$\mathbf{K} = (\mathbf{m} \operatorname{grad}) \mathbf{B}_\mathrm{a} \tag{10.21}$$

acting on the magnetic dipole in analogy to (4.35), where we obtained the force $(\mathbf{p} \operatorname{grad}) \mathbf{E}_\mathrm{a}$ acting on an electric dipole.

10.c.β Torque

The torque on a magnetic dipole is given by

$$\mathbf{M}_\mathrm{mech} = \frac{1}{\mathrm{c}} \int d^3 r \mathbf{r} \times (\mathbf{j} \times \mathbf{B}_\mathrm{a}) = -\frac{1}{\mathrm{c}} \mathbf{B}_\mathrm{a} \int d^3 r (\mathbf{r} \cdot \mathbf{j}) + \frac{1}{\mathrm{c}} \int d^3 r (\mathbf{B}_\mathrm{a} \cdot \mathbf{r}) \mathbf{j}. \tag{10.22}$$

The first integral vanishes, which is easily seen from (10.3) and $g = r^2/2$. The second integral yields

$$\mathbf{M}_\mathrm{mech} = \frac{1}{\mathrm{c}} \mathbf{e}_\beta B_{\mathrm{a},\alpha} \int d^3 r x_\alpha j_\beta = B_{\mathrm{a},\alpha} \mathbf{e}_\beta \epsilon_{\alpha,\beta,\gamma} m_\gamma = \mathbf{m} \times \mathbf{B}_\mathrm{a}. \tag{10.23}$$

Analogously the torque on an electric dipole was $\mathbf{p} \times \mathbf{E}_\mathrm{a}$, (4.36).
From the law of force one concludes the energy of a magnetic dipole in an external field as

$$U = -\mathbf{m} \cdot \mathbf{B}_\mathrm{a}. \tag{10.24}$$

This is correct for permanent magnetic moments. However, the precise derivation of this expression becomes clear only when we treat the law of induction (section 13).

11 Magnetism in Matter

In a similar way as we separated the polarization charges from freely accessible charges, we divide the current density into a freely moving current density $\mathbf{j}_\mathrm{f}$ and the density of the magnetization current $\mathbf{j}_\mathrm{M}$, which may come from orbital currents of electrons

$$\mathbf{j}(\mathbf{r}) = \mathbf{j}_\mathrm{f}(\mathbf{r}) + \mathbf{j}_\mathrm{M}(\mathbf{r}). \tag{11.1}$$

We introduce the magnetization as the density of magnetic dipoles

$$\mathbf{M} = \frac{\sum \mathbf{m}_i}{\Delta V} \tag{11.2}$$

and conduct the continuum limit

$$\sum_i \mathbf{m}_i f(\mathbf{r}_i) \to \int d^3r' \mathbf{M}(\mathbf{r}') f(\mathbf{r}'). \tag{11.3}$$

Then using (10.10) we obtain for the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\mathrm{c}} \int d^3r' \frac{\mathbf{j}_\mathrm{f}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int d^3r' \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{11.4}$$

The second integral can be rewritten

$$\int d^3r' \mathbf{M}(\mathbf{r}') \times \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \times \mathbf{M}(\mathbf{r}'), \tag{11.5}$$

so that one obtains

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\mathrm{c}} \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} (\mathbf{j}_\mathrm{f}(\mathbf{r}') + \mathrm{c}\,\mathrm{curl}\,'\mathbf{M}(\mathbf{r}')). \tag{11.6}$$

Thus one interprets $\mathbf{j}_\mathrm{M}(\mathbf{r}') = \mathrm{c}\,\mathrm{curl}\,'\mathbf{M}(\mathbf{r}')$ as the density of the magnetization current. Then one obtains for the magnetic induction

$$\mathrm{curl}\,\mathbf{B}(\mathbf{r}) = \frac{4\pi}{\mathrm{c}} \mathbf{j}_\mathrm{f}(\mathbf{r}) + 4\pi\,\mathrm{curl}\,\mathbf{M}(\mathbf{r}). \tag{11.7}$$

Now one introduces the magnetic field strength

$$\mathbf{H}(\mathbf{r}) := \mathbf{B}(\mathbf{r}) - 4\pi \mathbf{M}(\mathbf{r}) \tag{11.8}$$

for which Maxwell's equation

$$\mathrm{curl}\,\mathbf{H}(\mathbf{r}) = \frac{4\pi}{\mathrm{c}} \mathbf{j}_\mathrm{f}(\mathbf{r}) \tag{11.9}$$

holds. Maxwell's equation $\mathrm{div}\,\mathbf{B}(\mathbf{r}) = 0$ remains unchanged.
One obtains for paramagnetic and diamagnetic materials in not too strong fields

$$\mathbf{M} = \chi_\mathrm{m} \mathbf{H}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mu = 1 + 4\pi\chi_\mathrm{m}, \tag{11.10}$$

where χ_m is the magnetic susceptibility and μ the permeability. In superconductors (of first kind) one finds complete diamagnetism $\mathbf{B} = \mathbf{0}$. There the magnetic induction is completely expelled from the interior by surface currents.
In analogy to the arguments for the dielectric displacement and the electric field one obtains that the normal component B_n is continuous, and in the absence of conductive currents also the tangential components $\mathbf{H}_\mathrm{t}$ are continuous across the boundary.
In the Gaussian system of units the fields $\mathbf{M}$ and $\mathbf{H}$ are measured just as $\mathbf{B}$ in $\mathrm{dyn}^{1/2}\ \mathrm{cm}^{-1}$, whereas in SI-units $\mathbf{B}$ is measured in $\mathrm{Vs/m}^2$, $\mathbf{H}$ and $\mathbf{M}$ in A/m. The conversion factors for $\mathbf{H}$ and $\mathbf{M}$ differ by a factor 4π. For more information see appendix A.