Classical Electrodynamics

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English Edition

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I appreciate being informed of misprints.

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Books:

Becker, Sauter: Theorie der Elektrizität I
Jackson, Classical Electrodynamics
Landau, Lifschitz: Lehrbuch der Theoretischen Physik II: Klassische Feldtheorie
Panofsky, Phillips, Classical Electricity and Magnetism
Sommerfeld: Vorlesungen über Theoretische Physik III: Elektrodynamik
Stratton, Electromagnetic Theory
Stumpf, Schuler: Elektrodynamik

A Basic Equations

Introductory Remarks
I assume that the student is already somewhat familiar with classical electrodynamics from an introductory course. Therefore I start with the complete set of equations and from this set I spezialize to various cases of interest.
In this manuscript I will use GAUSSian units instead of the SI-units. The connection between both systems and the motivation for using GAUSSIAN units will be given in the next section and in appendix A.
Formulae for vector algebra and vector analysis are given in appendix B. A warning to the reader: Sometimes (B.11, B.15, B.34-B.50 and exercise after B.71) he/she should insert the result by him/herself. He/She is requested to perform the calculations by him/herself or should at least insert the results given in this script.

1 Basic Equations of Electrodynamics

Electrodynamics describes electric and magnetic fields, their generation by charges and electric currents, their propagation (electromagnetic waves), and their reaction on matter (forces).

1.a Charges and Currents

1.a.α Charge Density

The charge density ρ is defined as the charge Δq per volume element ΔV

$$\rho(\mathbf{r}) = \lim_{\Delta V\to 0} \frac{\Delta q}{\Delta V} = \frac{\mathrm{d}q}{\mathrm{d}V}. \tag{1.1}$$

Therefore the charge q in the volume V is given by

$$q = \int_V \mathrm{d}^3r \rho(\mathbf{r}). \tag{1.2}$$

If the charge distribution consists of point charges q_i at points $\mathbf{r}_i$, then the charge density is given by the sum

$$\rho(\mathbf{r}) = \sum_i q_i \delta^3(\mathbf{r}_i - \mathbf{r}), \tag{1.3}$$

where DIRAC's delta-function (correctly delta-distribution) has the property

$$\int_V \mathrm{d}^3r f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{r}_0) = \begin{cases} f(\mathbf{r}_0) & \text{if } \mathbf{r}_0 \in V \\ 0 & \text{if } \mathbf{r}_0 \notin V \end{cases}. \tag{1.4}$$

Similarly one defines the charge density per area $\sigma(\mathbf{r})$ at boundaries and surfaces as charge per area

$$\sigma(\mathbf{r}) = \frac{\mathrm{d}q}{\mathrm{d}f}, \tag{1.5}$$

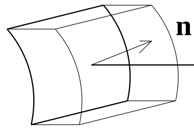
similarly the charge density on a line.

1.a.β Current and Current Density

The current I is the charge dq that flows through a certain area F per time dt,

$$I = \frac{\mathrm{d}q}{\mathrm{d}t}. \tag{1.6}$$

Be $\mathbf{v}(\mathbf{r}, t)$ the average velocity of the charge carriers and $\mathbf{n}$ the unit vector normal to the area element. Then $\mathbf{v}\mathrm{d}t$ is the distance vector traversed during time dt. Multiplied by $\mathbf{n}$ one obtains the thickness of the layer $\mathbf{v}\cdot\mathbf{n}\mathrm{d}t$ of the carriers which passed the surface during time dt.Multiplied by the surface element df one obtains the volume of the charge, which flows through the area. Additional multiplication by ρ yields the charge dq which passes during time dt the surface df



$$\mathrm{d}q = \int_F \mathbf{v}\mathrm{d}t\cdot\mathbf{n}\mathrm{d}f\rho \tag{1.7}$$

$$I = \mathrm{d}q/\mathrm{d}t = \int_F \mathbf{v}(\mathbf{r}, t)\rho(\mathbf{r}, t)\cdot\mathbf{n}(\mathbf{r})\mathrm{d}f = \int_F \mathbf{j}(\mathbf{r}, t)\cdot\mathrm{d}\mathbf{f} \tag{1.8}$$

with the current density $\mathbf{j} = \rho\mathbf{v}$ and the oriented area element $\mathrm{d}\mathbf{f} = \mathbf{n}\mathrm{d}f$.

1.a.γ Conservation of Charge and Equation of Continuity

The charge q in a fixed volume V

$$q(t) = \int_V \mathrm{d}^3r\rho(\mathbf{r}, t) \tag{1.9}$$

changes as a function of time by

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \int_V \mathrm{d}^3r\frac{\partial\rho(\mathbf{r}, t)}{\partial t}. \tag{1.10}$$

This charge can only change, if some charge flows through the surface ∂V of the volume, since charge is conserved. We denote the current which flows outward by I. Then

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = -I(t) = -\int_{\partial V} \mathbf{j}(\mathbf{r}, t)\cdot\mathrm{d}\mathbf{f} = -\int_V \mathrm{d}^3r\,\mathrm{div}\,\mathbf{j}(\mathbf{r}, t), \tag{1.11}$$

where we make use of the divergence theorem (B.59). Since (1.10) and (1.11) hold for any volume and volume element, the integrands in the volume integrals have to be equal

$$\frac{\partial\rho(\mathbf{r}, t)}{\partial t} + \mathrm{div}\,\mathbf{j}(\mathbf{r}, t) = 0. \tag{1.12}$$

This equation is called the equation of continuity. It expresses in differential form the conservation of charge.

1.b MAXWELL's Equations

The electric charges and currents generate the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic induction $\mathbf{B}(\mathbf{r}, t)$. This relation is described by the four MAXWELL Equations

$$\mathrm{curl}\,\mathbf{B}(\mathbf{r}, t) - \frac{\partial\mathbf{E}(\mathbf{r}, t)}{c\partial t} = \frac{4\pi}{c}\mathbf{j}(\mathbf{r}, t) \tag{1.13}$$

$$\mathrm{div}\,\mathbf{E}(\mathbf{r}, t) = 4\pi\rho(\mathbf{r}, t) \tag{1.14}$$

$$\mathrm{curl}\,\mathbf{E}(\mathbf{r}, t) + \frac{\partial\mathbf{B}(\mathbf{r}, t)}{c\partial t} = \mathbf{0} \tag{1.15}$$

$$\mathrm{div}\,\mathbf{B}(\mathbf{r}, t) = 0. \tag{1.16}$$

These equations named after MAXWELL are often called MAXWELL's Equations in the vacuum. However, they are also valid in matter. The charge density and the current density contain all contributions, the densities of free charges and polarization charges, and of free currents and polarization- and magnetization currents.
Often one requires as a boundary condition that the electric and the magnetic fields vanish at infinity.

1.c Coulomb and Lorentz Force

The electric field **E** and the magnetic induction **B** exert a force **K** on a charge q located at **r**, moving with a velocity **v**

$$\mathbf{K} = q\mathbf{E}(\mathbf{r}) + q\frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{r}). \tag{1.17}$$

Here **E** and **B** are the contributions which do not come from q itself. The fields generated by q itself exert the reaction force which we will not consider further.
The first contribution in (1.17) is the Coulomb force, the second one the Lorentz force. One has c= 299 792 458 m/s. Later we will see that this is the velocity of light in vacuum. (It has been defined with the value given above in order to introduce a factor between time and length.) The force acting on a small volume ΔV can be written as

$$\Delta\mathbf{K} = \mathbf{k}(\mathbf{r})\Delta V \tag{1.18}$$

$$\mathbf{k}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) + \frac{1}{c}\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}). \tag{1.19}$$

k is called the density of force. The electromagnetic force acting on the volume V is given by

$$\mathbf{K} = \int_V \mathrm{d}^3 r\, \mathbf{k}(\mathbf{r}). \tag{1.20}$$

2 Dimensions and Units

2.a Gaussian Units

In this course we use Gaussian units. We consider the dimensions of the various quantities. From the equation of continuity (1.12) and Maxwell's equations (1.13 to 1.16) one obtains

$$
\begin{aligned}
[\rho]/[t] &= [j]/[x] && (2.1)\\
[B]/[x] &= [E]/([c][t]) = [j]/[c] && (2.2)\\
[E]/[x] &= [B]/([c][t]) = [\rho]. && (2.3)
\end{aligned}
$$

From this one obtains

$$
\begin{aligned}
[j] &= [\rho][x]/[t] && (2.4)\\
[E] &= [\rho][x] && (2.5)\\
[B] &= [\rho][c][t] = [\rho][x]^2/([c][t]), && (2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
[c] &= [x]/[t] && (2.7)\\
[B] &= [\rho][x]. && (2.8)
\end{aligned}
$$

From (2.7) one sees that c really has the dimension of a velocity. In order to determine the dimensions of the other quantities we still have to use expression (1.19) for the force density k

$$[k] = [\rho][E] = [\rho]^2[x]. \qquad (2.9)$$

From this one obtains

$$
\begin{aligned}
[\rho]^2 &= [k]/[x] = \text{dyn cm}^{-4} && (2.10)\\
[\rho] &= \text{dyn}^{1/2}\ \text{cm}^{-2} && (2.11)\\
[E] = [B] &= \text{dyn}^{1/2}\ \text{cm}^{-1} && (2.12)\\
[j] &= \text{dyn}^{1/2}\ \text{cm}^{-1}\ \text{s}^{-1} && (2.13)\\
[q] &= [\rho][x]^3 = \text{dyn}^{1/2}\ \text{cm} && (2.14)\\
[I] &= [j][x]^2 = \text{dyn}^{1/2}\ \text{cm s}^{-1}. && (2.15)
\end{aligned}
$$

2.b Other Systems of Units

The unit for each quantity can be defined independently. Fortunately, this is not used extensively.
Besides the Gaussian system of units a number of other cgs-systems is used as well as the SI-system (international system of units, Giorgi-system). The last one is the legal system in many countries (e.g. in the US since 1894, in Germany since 1898) and is used for technical purposes.
Whereas all electromagnetic quantities in the Gaussian system are expressed in cm, g und s, the Giorgi-system uses besides the mechanical units m, kg and s two other units, A (ampere) und V (volt). They are not independent, but related by the unit of energy

$$1\ \text{kg m}^2\ \text{s}^{-2} = 1\ \text{J} = 1\ \text{W s} = 1\ \text{A V s}. \qquad (2.16)$$

The conversion of the conventional systems of units can be described by three conversion factors ϵ_0, μ_0 and ψ. The factors ϵ_0 and μ_0 (known as the dielectric constant and permeability constant of the vacuum in the SI-system) and the interlinking factor

$$\gamma = c\sqrt{\epsilon_0\mu_0} \qquad (2.17)$$

can carry dimensions whereas ψ is a dimensionless number. One distinguishes between rational systems $\psi = 4\pi$) and non-rational systems ($\psi = 1$) of units. The conversion factors of some conventional systems of units are

System of Units	ϵ_0	μ_0	γ	ψ
Gaussian	1	1	c	1
electrostatic (esu)	1	c^{-2}	1	1
electromagnetic (emu)	c^{-2}	1	1	1
Heaviside-Lorentz	1	1	c	4π
Giorgi (SI)	$(c^2\mu_0)^{-1}$	$\frac{4\pi}{10^7}\frac{\mathrm{Vs}}{\mathrm{Am}}$	1	4π

The quantities introduced until now are expressed in Gaussian units by those of other systems of units (indicated by an asterisk) in the following way

$$\mathbf{E} = \sqrt{\psi\epsilon_0}\mathbf{E}^* \qquad 1\ \mathrm{dyn}^{1/2}\ \mathrm{cm}^{-1} \hat{=} 3\cdot 10^4 \mathrm{V/m} \tag{2.18}$$

$$\mathbf{B} = \sqrt{\psi/\mu_0}\mathbf{B}^* \qquad 1\ \mathrm{dyn}^{1/2}\ \mathrm{cm}^{-1} \hat{=} 10^{-4} \mathrm{Vs/m}^2 \tag{2.19}$$

$$q = \frac{1}{\sqrt{\psi\epsilon_0}} q^* \qquad 1\ \mathrm{dyn}^{1/2}\ \mathrm{cm} \hat{=} 10^{-9}/3\mathrm{As}, \text{ similarly } \rho, \sigma, I, j. \tag{2.20}$$

An example of conversion: The Coulomb-Lorentz-force can be written

$$\mathbf{K} = q(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}) = \frac{q^*}{\sqrt{\psi\epsilon_0}}(\sqrt{\psi\epsilon_0}\mathbf{E}^* + \frac{\sqrt{\psi}}{c\sqrt{\mu_0}}\mathbf{v}\times\mathbf{B}^*) = q^*(\mathbf{E}^* + \frac{1}{c\sqrt{\epsilon_0\mu_0}}\mathbf{v}\times\mathbf{B}^*) = q^*(\mathbf{E}^* + \frac{1}{\gamma}\mathbf{v}\times\mathbf{B}^*). \tag{2.21}$$

The elementary charge e_0 is $4.803\cdot 10^{-10}$ $\mathrm{dyn}^{1/2}$ cm in Gaussian units and $1.602\cdot 10^{-19}$ As in SI-units. The electron carries charge $-e_0$, the proton e_0, a nucleus with Z protons the charge Ze_0, quarks the charges $\pm e_0/3$ and $\pm 2e_0/3$.
The conversion of other quantities is given where they are introduced. A summary is given in Appendix A.

2.c Motivation for Gaussian Units

In the SI-system the electrical field **E** and the dielectric displacement **D** as well as the magnetic induction **B** and the magnetic field **H** carry different dimensions. This leads easily to the misleading impression that these are independent fields. On a microscopic level one deals only with two fields, **E** and **B**, (1.13-1.16) (Lorentz 1892). However, the second set of fields is introduced only in order to extract the polarization and magnetization contributions of charges and currents in matter from the total charges and currents, and to add them to the fields. (Section 6 and 11).
This close relation is better expressed in cgs-units, where **E** and **D** have the same dimension, as well as **B** and **H**.
Unfortunately, the Gaussian system belongs to the irrational ones, whereas the SI-system is a rational one, so that in conversions factors 4π appear. I would have preferred to use a rational system like that of Heaviside and Lorentz. However, in the usual textbooks only the SI-system and the Gaussian one are used. I do not wish to offer the electrodynamics in a system which in practice is not used in other textbooks.