Appendices

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A Connection between different Systems of Units

Besides the Gaussian system of units a number of other cgs-systems is used as well as the SI-system (international system of units, Giorgi-system).
Whereas all electromagnetic quantities in the Gaussian system are expressed in cm, g und s, the Giorgi-system uses besides the mechanical units m, kg und s two other units, A (ampere) und V (volt). They are not independent, but related by the unit of energy

$$1 \text{ kg m}^2 \text{ s}^{-2} = 1 \text{ J} = 1 \text{ W s} = 1 \text{ A V s}. \tag{A.1}$$

The conversion of some conventional systems of units can be described by three conversion factors ϵ_0, μ_0 and ψ. The factors ϵ_0 and μ_0 (known as the dielectric constant and permeability constant of the vacuum in the SI-system) and the interlinking factor

$$\gamma = c\sqrt{\epsilon_0\mu_0} \tag{A.2}$$

can carry dimensions whereas ψ is a dimensionless number. One distinguishes between rational systems $\psi = 4\pi$) and non-rational systems ($\psi = 1$). The conversion factors of some conventional systems of units are

System of Units	ϵ_0	μ_0	γ	ψ
Gaussian	1	1	c	1
Electrostatic (esu)	1	c^{-2}	1	1
Electromagnetic (emu)	c^{-2}	1	1	1
Heaviside-Lorentz	1	1	c	4π
Giorgi (SI)	$(c^2\mu_0)^{-1}$	$\frac{4\pi}{10^7}\frac{\text{Vs}}{\text{Am}}$	1	4π

The field intensities are expressed in Gaussian units by those of other systems (indicated by an asterisk) in the following way

$$\begin{aligned}
\mathbf{E} &= \sqrt{\psi\epsilon_0}\mathbf{E}^* && \text{analogously electric potential} \\
\mathbf{D} &= \sqrt{\psi/\epsilon_0}\mathbf{D}^* && \\
\mathbf{P} &= 1/\sqrt{\psi\epsilon_0}\mathbf{P}^* && \text{analogously charge, current and their densities, electric moments} \\
\mathbf{B} &= \sqrt{\psi/\mu_0}\mathbf{B}^* && \text{analogously vector potential, magnetic flux} \\
\mathbf{H} &= \sqrt{\psi\mu_0}\mathbf{H}^* && \\
\mathbf{M} &= \sqrt{\mu_0/\psi}\mathbf{M}^* && \text{analogously magnetic moments}
\end{aligned} \tag{A.3}$$

One has for the quantities connected with conductivity and resistance

$$\begin{aligned}
\sigma &= 1/(\psi\epsilon_0)\sigma^* && \text{analogously capacity} \\
R &= \psi\epsilon_0 R^* && \text{analogously inductance}
\end{aligned} \tag{A.4}$$

For the electric and magnetic susceptibilities one obtains

$$\chi = \chi^*/\psi. \tag{A.5}$$

We obtain the following equations for arbitrary systems of units (the * has now been removed): MAXWELL's equations in matter read now

$$\operatorname{curl}\mathbf{H} = \frac{1}{\gamma}(\dot{\mathbf{D}} + \frac{4\pi}{\psi}\mathbf{j}_f), \quad (A.6)$$

$$\operatorname{div}\mathbf{D} = \frac{4\pi}{\psi}\rho_f, \quad (A.7)$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{\gamma}\dot{\mathbf{B}}, \quad (A.8)$$

$$\operatorname{div}\mathbf{B} = 0. \quad (A.9)$$

The material equations read

$$\mathbf{D} = \epsilon_0\mathbf{E} + \frac{4\pi}{\psi}\mathbf{P}, \quad (A.10)$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \frac{4\pi}{\psi}\mathbf{M}. \quad (A.11)$$

For the LORENTZ force one obtains

$$\mathbf{K} = q(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{\gamma}) \quad (A.12)$$

For the energy density u and the POYNTING vector $\mathbf{S}$ on obtains

$$u = \frac{\psi}{4\pi}\int(\mathbf{E}\cdot d\mathbf{D} + \mathbf{H}\cdot d\mathbf{B}), \quad (A.13)$$

$$\mathbf{S} = \frac{\psi\gamma}{4\pi}\mathbf{E}\times\mathbf{H}. \quad (A.14)$$

Whereas in GAUSSian units all the fields **E**, **D**, **P**, **B**, **H** und **M** are measured in units

$$\sqrt{\text{dyn}}/\text{cm} = \sqrt{\text{erg/cm}^3} \quad (A.15)$$

the GIORGI system measures **E** in V/m, **D** and **P** in As/m^2, **B** in Vs/m^2, **H** and **M** in A/m. Depending on the quantity 1 $\text{dyn}^{1/2}\ \text{cm}^{-1}$ in units of the GAUSSIAN system corresponds to (analogously for the quantities listed in (A.3) and (A.4))

$$\mathbf{E} = 3\cdot 10^4\ \text{V/m} \quad (A.16)$$

$$\mathbf{D} = 10^{-5}/(12\pi)\ \text{As/m}^2 \quad (A.17)$$

$$\mathbf{P} = 10^{-5}/3\ \text{As/m}^2 \quad (A.18)$$

$$\mathbf{B} = 10^{-4}\ \text{Vs/m}^2 \quad (A.19)$$

$$\mathbf{H} = 10^3/(4\pi)\ \text{A/m} \quad (A.20)$$

$$\mathbf{M} = 10^3\ \text{A/m}. \quad (A.21)$$

For resistors one has $c^{-1}\hat{=}30\Omega$. For precise calculations the factors 3 (including the 3 in $12 = 4\cdot 3$) are to be replaced by the factor 2.99792458. This number multiplied by 10^8 m/s is the speed of light.
There are special names for the following often used units in the GAUSSian and electromagnetic system

magnetic induction	1 $\text{dyn}^{1/2}\ \text{cm}^{-1}$=1 G (Gauß)
magnetic field intensity	1 $\text{dyn}^{1/2}\ \text{cm}^{-1}$=1 Oe (Oerstedt)
magnetic flux	1 $\text{dyn}^{1/2}$ cm =1 Mx (Maxwell)

The following quantities besides Ampere and Volt have their own names in the SI-system:

charge	1 As =1 C (Coulomb)
resistance	1 V/A =1 Ω (Ohm)
conductance	1 A/V =1 S (Siemens)
capacitance	1 As/V =1 F (Farad)
inductivity	1 Vs/A =1 H (Henry)
magnetic flux	1 Vs =1 Wb (Weber)
magnetic induction	1 Vs/m^2=1 T (Tesla).

Historically the international or SI system was derived from the electromagnetic system. Since the units of this system where inconveniently large or small one introduced as unit for the current $1\ \mathrm{A} = 10^{-1}\ \mathrm{dyn}^{1/2}$ and for the voltage $1\ \mathrm{V} = 10^{8}\ \mathrm{dyn}^{1/2}\ \mathrm{cm\ s}^{-1}$. GIORGI realized that changing to mks-units one obtains the relation (A.1). However, one changed also from non-rational to rational units.

B Formulae for Vector Calculus

The reader is asked to solve the exercises B.11, B.15, B.34-B.50 and the exercise after B.71 by her- or himself or to take the results from the script where they are used.

B.a Vector Algebra

B.a.α Summation Convention and Orthonormal Basis

We use the summation convention which says that summation is performed over all indices, which appear twice in a product. Therefore

$$\mathbf{a} = a_\alpha \mathbf{e}_\alpha \tag{B.1}$$

stands for

$$\mathbf{a} = \sum_{\alpha=1}^{3} a_\alpha \mathbf{e}_\alpha = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

In the following we assume that the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in (B.1) represent an orthonormal and space independent right-handed basis. Then a_1, a_2, a_3 are the components of the vector $\mathbf{a}$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

B.a.β Scalar Product

The scalar product is defined by

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = a_\alpha b_\alpha, \tag{B.2}$$

in particular we have

$$\mathbf{e}_\alpha\cdot\mathbf{e}_\beta = \delta_{\alpha,\beta} = \begin{cases} 1 \text{ for } \alpha=\beta \\ 0 \text{ for } \alpha\neq\beta \end{cases} \tag{B.3}$$

with the Kronecker symbol $\delta_{\alpha,\beta}$ which is symmetric in its indices, and

$$\mathbf{a}\cdot\mathbf{e}_\alpha = a_\alpha. \tag{B.4}$$

B.a.γ Vector Product

The vector product is given by

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = \epsilon_{\alpha,\beta,\gamma} a_\alpha b_\beta \mathbf{e}_\gamma = (a_2b_3-a_3b_2)\mathbf{e}_1 + (a_3b_1-a_1b_3)\mathbf{e}_2 + (a_1b_2-a_2b_1)\mathbf{e}_3 \tag{B.5}$$

with the total antisymmetric Levi-Civita symbol

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} +1 & \text{for } (\alpha,\beta,\gamma) = (1,2,3),(2,3,1),(3,1,2) \\ -1 & \text{for } (\alpha,\beta,\gamma) = (1,3,2),(2,1,3),(3,2,1) \\ 0 & \text{otherwise} \end{cases} \tag{B.6}$$

Using determinants it can be written

$$\epsilon_{\alpha,\beta,\gamma} = \begin{vmatrix} \delta_{\alpha,1} & \delta_{\beta,1} & \delta_{\gamma,1} \\ \delta_{\alpha,2} & \delta_{\beta,2} & \delta_{\gamma,2} \\ \delta_{\alpha,3} & \delta_{\beta,3} & \delta_{\gamma,3} \end{vmatrix}. \tag{B.7}$$

From (B.5) one obtains by multiplication with a_α, b_β and $\mathbf{e}_\gamma$ and summation

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & b_1 & \mathbf{e}_1 \\ a_2 & b_2 & \mathbf{e}_2 \\ a_3 & b_3 & \mathbf{e}_3 \end{vmatrix}. \tag{B.8}$$

In particular one obtains

$$\mathbf{a}\times\mathbf{a} = \mathbf{0} \tag{B.9}$$

and

$$\mathbf{e}_\alpha \times \mathbf{e}_\beta = \epsilon_{\alpha,\beta,\gamma}\mathbf{e}_\gamma. \tag{B.10}$$

Express the sum

$$\epsilon_{\alpha,\beta,\gamma}\epsilon_{\zeta,\eta,\gamma} = \tag{B.11}$$

by means of Kronecker deltas.

B.a.δ Multiple Products

For the scalar triple product one has

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \epsilon_{\alpha,\beta,\gamma}a_\alpha b_\beta c_\gamma = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{B.12}$$

One has

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{b},\mathbf{a},\mathbf{c}] = -[\mathbf{c},\mathbf{b},\mathbf{a}]. \tag{B.13}$$

For the vector triple product one has

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\mathbf{c})\mathbf{b} - (\mathbf{a}\mathbf{b})\mathbf{c}. \tag{B.14}$$

Express the quadruple product

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \tag{B.15}$$

by means of (B.11) or (B.14) in terms of scalar products.

B.b Vector Analysis

B.b.α Differentiation in Space, Del Operator

Differentiation in space is performed by means of the del-Operator ∇. It is a differential operator with vector properties. In cartesian coordinates it is written

$$\nabla = \mathbf{e}_\alpha\partial_\alpha, \tag{B.16}$$

where ∂_α stands for $\partial/\partial x_\alpha$. One calls

$$\nabla\Phi(\mathbf{r}) = \mathbf{e}_\alpha\partial_\alpha\Phi(\mathbf{r}) = \operatorname{grad}\Phi(\mathbf{r}) \tag{B.17}$$

the gradient,

$$(\mathbf{b}(\mathbf{r})\nabla)\mathbf{a}(\mathbf{r}) = b_\alpha(\mathbf{r})\partial_\alpha\mathbf{a}(\mathbf{r}) = (\mathbf{b}(\mathbf{r})\operatorname{grad})\mathbf{a}(\mathbf{r}) \tag{B.18}$$

the vector gradient

$$\nabla\mathbf{a}(\mathbf{r}) = \partial_\alpha a_\alpha(\mathbf{r}) = \operatorname{div}\mathbf{a}(\mathbf{r}) \tag{B.19}$$

the divergence and

$$\nabla\times\mathbf{a}(\mathbf{r}) = (\mathbf{e}_\alpha\times\mathbf{e}_\beta)\partial_\alpha a_\beta(\mathbf{r}) = \epsilon_{\alpha,\beta,\gamma}\partial_\alpha a_\beta(\mathbf{r})\mathbf{e}_\gamma = \operatorname{curl}\mathbf{a}(\mathbf{r}) \tag{B.20}$$

the curl.

B.b.β Second Derivatives, Laplacian

As far as differentiations do commute one has

$$\nabla \times \nabla = \mathbf{0}, \tag{B.21}$$

from which

$$\begin{aligned} \text{curl grad}\,\Phi(\mathbf{r}) &= \mathbf{0}, && \text{(B.22)} \\ \text{div curl}\,\mathbf{a}(\mathbf{r}) &= 0 && \text{(B.23)} \end{aligned}$$

follows. The scalar product

$$\nabla \cdot \nabla = \partial_\alpha \partial_\alpha = \triangle \tag{B.24}$$

is called the Laplacian. Therefore one has

$$\text{div grad}\,\Phi(\mathbf{r}) = \triangle\Phi(\mathbf{r}). \tag{B.25}$$

One obtains

$$\triangle\mathbf{a}(\mathbf{r}) = \text{grad div}\,\mathbf{a}(\mathbf{r}) - \text{curl curl}\,\mathbf{a}(\mathbf{r}), \tag{B.26}$$

by replacing **a** and **b** by ∇ in (B.14) and bringing the vector **c** always to the right.

B.b.γ Derivatives of Products

Application of the del operator onto a product of two factors yields according to the product rule two contributions. In one contribution one differentiates the first factor and keeps the second one constant, in the other contribution one differentiates the second factor and keeps the first constant. Then the expressions have to be rearranged, so that the constant factors are to the left, those to be differentiated to the right of the del operator. In doing this one has to keep the vector character of the del in mind. Then one obtains

$$\begin{aligned} \text{grad}\,(\Phi\Psi) &= \Phi\,\text{grad}\,\Psi + \Psi\,\text{grad}\,\Phi && \text{(B.27)} \\ \text{div}\,(\Phi\mathbf{a}) &= \Phi\,\text{div}\,\mathbf{a} + \mathbf{a}\cdot\text{grad}\,\Phi && \text{(B.28)} \\ \text{curl}\,(\Phi\mathbf{a}) &= \Phi\,\text{curl}\,\mathbf{a} + (\text{grad}\,\Phi)\times\mathbf{a} && \text{(B.29)} \\ \text{div}\,(\mathbf{a}\times\mathbf{b}) &= \mathbf{b}\cdot\text{curl}\,\mathbf{a} - \mathbf{a}\cdot\text{curl}\,\mathbf{b} && \text{(B.30)} \\ \text{curl}\,(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}\,\text{div}\,\mathbf{b} - \mathbf{b}\,\text{div}\,\mathbf{a} + (\mathbf{b}\,\text{grad}\,)\mathbf{a} - (\mathbf{a}\,\text{grad}\,)\mathbf{b} && \text{(B.31)} \\ \text{grad}\,(\mathbf{a}\cdot\mathbf{b}) &= \mathbf{a}\times\text{curl}\,\mathbf{b} + \mathbf{b}\times\text{curl}\,\mathbf{a} + (\mathbf{b}\,\text{grad}\,)\mathbf{a} + (\mathbf{a}\,\text{grad}\,)\mathbf{b} && \text{(B.32)} \\ \triangle(\Phi\Psi) &= \Phi\triangle\Psi + \Psi\triangle\Phi + 2(\text{grad}\,\Phi)\cdot(\text{grad}\,\Psi). && \text{(B.33)} \end{aligned}$$

B.c Special Expressions

Calculate for $r = |\mathbf{r}|$ and constant vector **c**

$$\begin{aligned} \text{grad}\,r^2 &= && \text{(B.34)} \\ \text{div}\,\mathbf{r} &= && \text{(B.35)} \\ \text{curl}\,\mathbf{r} &= && \text{(B.36)} \\ \text{grad}\,(\mathbf{c}\cdot\mathbf{r}) &= && \text{(B.37)} \\ (\mathbf{c}\,\text{grad}\,)\mathbf{r} &= && \text{(B.38)} \\ \text{grad}\,f(r) &= && \text{(B.39)} \\ \text{div}\,(\mathbf{c}\times\mathbf{r}) &= && \text{(B.40)} \\ \text{curl}\,(\mathbf{c}\times\mathbf{r}) &= && \text{(B.41)} \\ \text{grad}\,\frac{1}{r} &= && \text{(B.42)} \end{aligned}$$

$$\operatorname{div} \frac{\mathbf{c}}{r} = \tag{B.43}$$

$$\operatorname{curl} \frac{\mathbf{c}}{r} = \tag{B.44}$$

$$\operatorname{div} \frac{\mathbf{r}}{r^3} = \tag{B.45}$$

$$\operatorname{curl} \frac{\mathbf{r}}{r^3} = \tag{B.46}$$

$$\operatorname{grad} \frac{\mathbf{c}\cdot\mathbf{r}}{r^3} = \tag{B.47}$$

$$\operatorname{div} \frac{\mathbf{c}\times\mathbf{r}}{r^3} = \tag{B.48}$$

$$\operatorname{curl} \frac{\mathbf{c}\times\mathbf{r}}{r^3} = \tag{B.49}$$

$$\operatorname{grad} \frac{1}{|\mathbf{r}-\mathbf{c}|} = \quad , \tag{B.50}$$

with the exception of singular points.

B.d Integral Theorems

B.d.α Line Integrals

For a scalar or a vector field $A(\mathbf{r})$ one has

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} (\mathrm{d}\mathbf{r}\nabla) A(\mathbf{r}) = A(\mathbf{r}_2) - A(\mathbf{r}_1), \tag{B.51}$$

that is

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{d}\mathbf{r} \operatorname{grad} \Phi(\mathbf{r}) = \Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1), \tag{B.52}$$

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} (\mathrm{d}\mathbf{r} \operatorname{grad}) \mathbf{a}(\mathbf{r}) = \mathbf{a}(\mathbf{r}_2) - \mathbf{a}(\mathbf{r}_1). \tag{B.53}$$

B.d.β Surface Integrals

According to Stokes a surface integral over F of the form

$$\int_F (\mathrm{d}\mathbf{f} \times \nabla) A(\mathbf{r}) = \oint_{\partial F} \mathrm{d}\mathbf{r} A(\mathbf{r}) \tag{B.54}$$

can be rewritten as a line integral over the curve ∂F bounding the surface. The direction is given by the right-hand rule; that is, if the thumb of your right hand points in the direction of d**f**, your fingers curve in the direction d**r** of the line integral,

$$\int_F \mathrm{d}\mathbf{f} \times \operatorname{grad} \Phi(\mathbf{r}) = \oint_{\partial F} \mathrm{d}\mathbf{r} \Phi(\mathbf{r}), \tag{B.55}$$

$$\int_F \mathrm{d}\mathbf{f} \cdot \operatorname{curl} \mathbf{a}(\mathbf{r}) = \oint_{\partial F} \mathrm{d}\mathbf{r} \cdot \mathbf{a}(\mathbf{r}). \tag{B.56}$$

B.d.γ Volume Integrals

According to Gauss a volume integral of the form

$$\int_V d^3r \nabla A(\mathbf{r}) = \int_{\partial V} d\mathbf{f} A(\mathbf{r}) \tag{B.57}$$

can be converted into an integral over the surface ∂V of the volume. The vector d**f** points out of the volume. In particular one has

$$\int_V d^3r \operatorname{grad} \Phi(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \Phi(\mathbf{r}), \tag{B.58}$$

$$\int_V d^3r \operatorname{div} \mathbf{a}(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \cdot \mathbf{a}(\mathbf{r}), \tag{B.59}$$

$$\int_V d^3r \operatorname{curl} \mathbf{a}(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \times \mathbf{a}(\mathbf{r}). \tag{B.60}$$

B.d.δ Volume Integrals of Products

If one substitutes products for $\Phi(\mathbf{r})$ or $\mathbf{a}(\mathbf{r})$ in equations (B.58-B.60) and applies equations (B.27-B.30), then one obtains

$$\int_V d^3r \Phi(\mathbf{r}) \operatorname{grad} \Psi(\mathbf{r}) + \int_V d^3r \Psi(\mathbf{r}) \operatorname{grad} \Phi(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \Phi(\mathbf{r}) \Psi(\mathbf{r}), \tag{B.61}$$

$$\int_V d^3r \Phi(\mathbf{r}) \operatorname{div} \mathbf{a}(\mathbf{r}) + \int_V d^3r \mathbf{a}(\mathbf{r}) \cdot \operatorname{grad} \Phi(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \cdot \mathbf{a}(\mathbf{r}) \Phi(\mathbf{r}), \tag{B.62}$$

$$\int_V d^3r \Phi(\mathbf{r}) \operatorname{curl} \mathbf{a}(\mathbf{r}) + \int_V d^3r (\operatorname{grad} \Phi(\mathbf{r})) \times \mathbf{a}(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \times \mathbf{a}(\mathbf{r}) \Phi(\mathbf{r}), \tag{B.63}$$

$$\int_V d^3r \mathbf{b}(\mathbf{r}) \cdot \operatorname{curl} \mathbf{a}(\mathbf{r}) - \int_V d^3r \mathbf{a}(\mathbf{r}) \cdot \operatorname{curl} \mathbf{b}(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \cdot (\mathbf{a}(\mathbf{r}) \times \mathbf{b}(\mathbf{r})). \tag{B.64}$$

These equations allow the transformation of a volume integral into another one and a surface integral. This is the generalization of integration by parts from one dimension to three. In many cases the surface integral vanishes in the limit of infinite volume, so that the equations (B.61-B.64) allow the conversion from one volume integral into another one.
If one replaces $\mathbf{a}(\mathbf{r})$ in (B.62) by $\operatorname{curl} \mathbf{a}(\mathbf{r})$ or $\mathbf{b}(\mathbf{r})$ in (B.64) by $\operatorname{grad} \Phi(\mathbf{r})$, then one obtains with (B.22) and (B.23)

$$\int_V d^3r \operatorname{grad} \Phi(\mathbf{r}) \cdot \operatorname{curl} \mathbf{a}(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \cdot (\mathbf{a}(\mathbf{r}) \times \operatorname{grad} \Phi(\mathbf{r})) = \int_{\partial V} d\mathbf{f} \cdot (\Phi(\mathbf{r}) \operatorname{curl} \mathbf{a}(\mathbf{r})). \tag{B.65}$$

Similarly one obtains from (B.63)

$$\int_V d^3r \operatorname{grad} \Phi(\mathbf{r}) \times \operatorname{grad} \Psi(\mathbf{r}) = \int_{\partial V} d\mathbf{f} \times (\operatorname{grad} \Psi(\mathbf{r})) \Phi(\mathbf{r}) = -\int_{\partial V} d\mathbf{f} \times (\operatorname{grad} \Phi(\mathbf{r})) \Psi(\mathbf{r}). \tag{B.66}$$

If one replaces $\mathbf{a}(\mathbf{r})$ in (B.59) by $\Phi \operatorname{grad} \Psi - \Psi \operatorname{grad} \Phi$, then one obtains Green's theorem

$$\int_V d^3r (\Phi(\mathbf{r}) \triangle \Psi(\mathbf{r}) - \Psi(\mathbf{r}) \triangle \Phi(\mathbf{r})) = \int_{\partial V} d\mathbf{f} \cdot (\Phi(\mathbf{r}) \operatorname{grad} \Psi(\mathbf{r}) - \Psi(\mathbf{r}) \operatorname{grad} \Phi(\mathbf{r})). \tag{B.67}$$

B.e The Laplacian of $1/r$ and Related Expressions

B.e.α The Laplacian of $1/r$

For $r \neq 0$ one finds $\triangle(1/r) = 0$. If one evaluates the integral over a sphere of radius R by use of (B.59),

$$\int \triangle(\frac{1}{r})\mathrm{d}^3r = \int \mathrm{d}\mathbf{f}\cdot \operatorname{grad}(\frac{1}{r}) = -\int \mathbf{r}r\mathrm{d}\Omega\cdot\frac{\mathbf{r}}{r^3} = -4\pi \tag{B.68}$$

with the solid-angle element $\mathrm{d}\Omega$, then one obtains -4π. Therefore one writes

$$\triangle(\frac{1}{r}) = -4\pi\delta^3(\mathbf{r}), \tag{B.69}$$

where DIRAC's delta "function" $\delta^3(\mathbf{r})$ (actually a distribution) has the property

$$\int_V \mathrm{d}^3r f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{r}_0) = \begin{cases} f(\mathbf{r}_0) & \text{if } \mathbf{r}_0 \in V \\ 0 & \text{otherwise.} \end{cases} \tag{B.70}$$

From

$$\triangle\frac{\mathbf{c}}{|\mathbf{r}-\mathbf{r}'|} = \mathbf{c}\triangle\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\mathbf{c}\delta^3(\mathbf{r}-\mathbf{r}')$$

one obtains with (B.26,B.43,B.44)

$$4\pi\mathbf{c}\delta^3(\mathbf{r}-\mathbf{r}') = -\operatorname{grad}\operatorname{div}\frac{\mathbf{c}}{|\mathbf{r}-\mathbf{r}'|} + \operatorname{curl}\operatorname{curl}\frac{\mathbf{c}}{|\mathbf{r}-\mathbf{r}'|} = \operatorname{grad}\frac{\mathbf{c}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \operatorname{curl}\frac{\mathbf{c}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{B.71}$$

Determine the δ-function contributions in (B.45) to (B.49). What is the dimension of $\delta^3(\mathbf{r})$?

B.e.β Representation of a Vector Field as a Sum of an Irrotational and a Divergence-free Field

We rewrite the vector field $\mathbf{a}(\mathbf{r})$ in the form

$$\mathbf{a}(\mathbf{r}) = \int \mathrm{d}^3r'\mathbf{a}(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}') \tag{B.72}$$

and obtain from (B.71), since $\mathbf{a}(\mathbf{r}')$ does not depend on $\mathbf{r}$

$$\mathbf{a}(\mathbf{r}) = \frac{1}{4\pi}\int \mathrm{d}^3r' \operatorname{grad}\frac{\mathbf{a}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{1}{4\pi}\int \mathrm{d}^3r' \operatorname{curl}\frac{\mathbf{a}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{B.73}$$

which may be written as

$$\mathbf{a}(\mathbf{r}) = -\operatorname{grad}\Phi(\mathbf{r}) + \operatorname{curl}\mathbf{A}(\mathbf{r}) \tag{B.74}$$

with

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\int \mathrm{d}^3r'\frac{\mathbf{a}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{B.75}$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int \mathrm{d}^3r'\frac{\mathbf{a}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{B.76}$$

If the integrals (B.75) and (B.76) exist, then one obtains in this way a representation of $\mathbf{a}(\mathbf{r})$ as sum of the irrotational field $-\operatorname{grad}\Phi(\mathbf{r})$ and the divergence free field $\operatorname{curl}\mathbf{A}(\mathbf{r})$. With (B.48) one finds

$$\operatorname{div}\mathbf{A}(\mathbf{r}) = 0. \tag{B.77}$$

C Spherical Harmonics

C.a Eigenvalue Problem and Separation of Variables

We are looking for the eigen functions Y of

$$\Delta_\Omega Y(\theta,\phi) = \lambda Y(\theta,\phi) \tag{C.1}$$

with

$$\Delta_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{C.2}$$

where the operators (multiplication by functions and differentiation) apply from right to left (compare 5.16). One introduces the ansatz

$$Y = g(\cos\theta)h(\phi). \tag{C.3}$$

With

$$\xi = \cos\theta, \quad \frac{dg}{d\theta} = -\sin\theta\frac{dg}{d\cos\theta} = -\sqrt{1-\xi^2}\frac{dg}{d\xi} \tag{C.4}$$

one obtains by insertion into the eigenvalue equation and division by $h(\phi)$

$$\frac{d}{d\xi}((1-\xi^2)\frac{dg}{d\xi}) + \frac{g(\xi)}{1-\xi^2}(\frac{d^2h(\phi)}{d\phi^2}/h(\phi)) = \lambda g(\xi). \tag{C.5}$$

This equation can only be fulfilled, if $d^2h(\phi)/d\phi^2/h(\phi)$ is constant. Since moreover one requires $h(\phi+2\pi) = h(\phi)$, it follows that

$$h(\phi) = e^{im\phi} \text{ with integer } m. \tag{C.6}$$

This reduces the differential equation for g to

$$\frac{d}{d\xi}((1-\xi^2)\frac{dg}{d\xi}) - \frac{m^2 g(\xi)}{1-\xi^2} = \lambda g(\xi). \tag{C.7}$$

C.b Associated Legendre Functions

Considering that (at least for positive m) the factor $e^{im\phi}$ comes from the analytic function $(x+iy)^m = r^m(\sin\theta)^m e^{im\phi}$, it seems appropriate to extract a factor $(\sin\theta)^m$ out of g

$$g(\xi) = (\sin\theta)^m G(\xi) = (1-\xi^2)^{m/2}G(\xi), \tag{C.8}$$

so that one obtains the equation

$$-m(m+1)G(\xi) - 2(m+1)\xi G'(\xi) + (1-\xi^2)G'' = \lambda G(\xi) \tag{C.9}$$

for G.
For G we may assume a Taylor expansion

$$G(\xi) = \sum_k a_k\xi^k, \quad G'(\xi) = \sum_k k a_k\xi^{k-1}, \quad G''(\xi) = \sum_k k(k-1)a_k\xi^{k-2} \tag{C.10}$$

and find by comparison of the coefficients

$$[m(m+1) + 2(m+1)k + k(k-1) + \lambda]a_k = (k+2)(k+1)a_{k+2}. \tag{C.11}$$

If we put

$$\lambda = -l(l+1), \tag{C.12}$$

then the recurrence formula reads

$$\frac{a_{k+2}}{a_k} = \frac{(m+k+l+1)(m+k-l)}{(k+1)(k+2)}. \tag{C.13}$$

The series expansion comes to an end at finite k, if the numerator vanishes, in particular for integer not negative $k = l - m$. We continue to investigate this case. Without closer consideration we mention that in the other cases the function Y develops a nonanalyticity at $\cos\theta = \pm 1$.
The leading term has then the coefficient a_{l-m}. Application of the recurrence formula yields

$$\begin{aligned} a_{l-m-2} &= -\frac{(l-m)(l-m-1)}{(2l-1)2} a_{l-m} \\ &= -\frac{(l-m)(l-m-1)l}{(2l-1)2l} a_{l-m}, \end{aligned} \tag{C.14}$$

$$\begin{aligned} a_{l-m-4} &= \frac{(l-m)(l-m-1)(l-m-2)(l-m-3)}{(2l-1)(2l-3)2\cdot 4} a_{l-m} \\ &= \frac{(l-m)(l-m-1)(l-m-2)(l-m-3)l(l-1)}{(2l-1)(2l-3)2l(2l-2)2} a_{l-m}, \end{aligned} \tag{C.15}$$

$$a_{l-m-2k} = (-)^k \frac{(l-m)!l!(2l-2k)!}{(l-m-2k)!(l-k)!(2l)!k!} a_{l-m}. \tag{C.16}$$

Conventionally one chooses

$$a_{l-m} = \frac{(-)^m(2l)!}{(l-m)!2^l l!}. \tag{C.17}$$

Then it follows that

$$G(\xi) = \frac{(-)^m}{2^l l!} \sum_k \frac{(2l-2k)!}{(l-m-2k)!} \frac{l!}{k!(l-k)!} (-)^k \xi^{l-m-2k} \tag{C.18}$$

$$= \frac{(-)^m}{2^l l!} \sum_k \binom{l}{k} (-)^k \frac{d^{l+m}\xi^{2l-2k}}{d\xi^{l+m}} = \frac{(-)^m}{2^l l!} \frac{d^{l+m}(\xi^2-1)^l}{d\xi^{l+m}}. \tag{C.19}$$

The solutions $g(\xi)$ in the form

$$P_l^m(\xi) = (1-\xi^2)^{m/2} \frac{(-)^m}{2^l l!} \frac{d^{l+m}}{d\xi^{l+m}} (\xi^2-1)^l \tag{C.20}$$

are called associated LEGENDRE functions. $Y_{lm}(\theta,\phi)$ is given by $P_l^m(\cos\theta)e^{im\phi}$ apart from the normalization.
The differential equation for g depends only on m^2, but not on the sign of m. Therefore we compare P_l^m and P_l^{-m}. Be $m \geq 0$, then it follows that

$$\begin{aligned} \frac{d^{l-m}}{d\xi^{l-m}}(\xi^2-1)^l &= \sum_{k=0}^{l-m} \binom{l-m}{k} \frac{d^k(\xi-1)^l}{d\xi^k} \frac{d^{l-m-k}(\xi+1)^l}{d\xi^{l-m-k}} \\ &= \sum_{k=0}^{l-m} \frac{(l-m)!l!l!}{k!(l-m-k)!(l-k)!(m+k)!} (\xi-1)^{l-k}(\xi+1)^{m+k}, \end{aligned} \tag{C.21}$$

$$\begin{aligned} \frac{d^{l+m}}{d\xi^{l+m}}(\xi^2-1)^l &= \sum_{k=0}^{l-m} \binom{l+m}{k+m} \frac{d^{m+k}(\xi-1)^l}{d\xi^{m+k}} \frac{d^{l-k}(\xi+1)^l}{d\xi^{l-k}} \\ &= \sum_{k=0}^{l-m} \frac{(l+m)!l!l!}{(m+k)!(l-k)!(l-m-k)!k!} (\xi-1)^{l-k-m}(\xi+1)^k. \end{aligned} \tag{C.22}$$

Comparison shows

$$P_l^{-m}(\xi) = \frac{(l-m)!}{(l+m)!}(-)^m P_l^m(\xi), \tag{C.23}$$

that is, apart from the normalization both solutions agree.

C.c Orthogonality and Normalization

We consider the normalization integral

$$N_{lml'm'} = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta P_l^m(\cos\theta) e^{-im\phi} P_{l'}^{m'}(\cos\theta) e^{im'\phi}. \tag{C.24}$$

The integration over ϕ yields

$$\begin{aligned} N_{lml'm'} &= 2\pi\delta_{mm'} \int_{-1}^{+1} P_l^m(\xi) P_{l'}^m(\xi) d\xi \\ &= 2\pi\delta_{mm'}(-)^m \frac{(l'+m)!}{(l'-m)!} \int_{-1}^{+1} P_l^m(\xi) P_{l'}^{-m}(\xi) d\xi \\ &= \frac{2\pi(l'+m)!}{(l'-m)!} \frac{\delta_{mm'}}{2^{2l} l!^2} I_m^{ll'} \end{aligned} \tag{C.25}$$

with

$$I_m^{ll'} = (-)^m \int_{-1}^{+1} \frac{d^{l+m}(\xi^2-1)^l}{d\xi^{l+m}} \frac{d^{l'-m}(\xi^2-1)^{l'}}{d\xi^{l'-m}} d\xi. \tag{C.26}$$

Partial integration yields

$$I_m^{ll'} = (-)^m \left[\frac{d^{l+m}(\xi^2-1)^l}{d\xi^{l+m}} \frac{d^{l'-m-1}(\xi^2-1)^{l'}}{d\xi^{l'-m-1}} \right]_{-1}^{+1} + I_{m+1}^{ll'}. \tag{C.27}$$

The first factor in square brackets contains at least $-m$, the second $m+1$ zeroes at $\xi = \pm 1$. The contents in square brackets vanishes therefore. Thus $I_m^{ll'}$ is independent of m for $-l \le m \le l'$. For $l' > l$ it follows that $I_m^{ll'} = I_{l'}^{ll'} = 0$, since the first factor of the integrand of $I_{l'}^{ll'}$ vanishes. For $l' < l$ one obtains $I_m^{ll'} = I_{-l}^{ll'} = 0$, since the second factor of the integrand of $I_{-l}^{ll'}$ vanishes. For $l = l'$ we evaluate

$$I_m^{ll} = I_l^{ll} = (-)^l \int_{-1}^{+1} \frac{d^{2l}(\xi^2-1)^l}{d\xi^{2l}} (\xi^2-1)^l d\xi. \tag{C.28}$$

The first factor in the integrand is the constant $(2l)!$

$$I_m^{ll} = (2l)! \int_{-1}^{+1} (1-\xi^2)^l d\xi. \tag{C.29}$$

The last integral yields $2^{2l+1} l!^2/(2l+1)!$ (one obtains this by writing the integrand $(1+\xi)^l(1-\xi)^l$ and performing partial integration l times, by always differentiating the power of $1-\xi$ and integrating that of $1+\xi$. This yields the norm

$$N_{lml'm'} = 2\pi \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{ll'} \delta_{m,m'}. \tag{C.30}$$

Thus the normalized spherical harmonics read

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}. \tag{C.31}$$

C.d Remark on Completeness

If we expand a function f which is analytic in the Cartesian coordinates x, y, z in the vicinity of the origin in a Taylor expansion

$$f(\mathbf{r}) = \sum_{ijk} a_{ijk} x^i y^j z^k = \sum_n r^n f_n(\theta,\phi), \tag{C.32}$$

then the contributions proportional to r^n are contained in those with $i + j + k = n$. These are in total $(n + 1) + n + (n - 1) + ... = (n + 2)(n + 1)/2$ terms

$$f_n(\theta, \phi) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} a_{n-j-k,j,k} (\frac{x}{r})^{n-j-k} (\frac{y}{r})^{j} (\frac{z}{r})^{k}. \tag{C.33}$$

On the other hand we may represent the function f_n equally well by the functions $Y_{lm}(\theta, \phi) = \sqrt{...} P_l^{|m|}(\cos\theta) e^{im\phi}$, since they can be written $(\sin\theta)^{|m|} e^{im\phi} = ((x \pm iy)/r)^{|m|}$ multiplied by a polynomial in $\cos\theta$ of order $l - |m|$. The appearing powers of the $\cos\theta$ can be written $(\cos\theta)^{l-|m|-2k} = (z/r)^{l-|m|-2k}((x^2 + y^2 + z^2)/r^2)^k$. In addition we introduce a factor $((x^2 + y^2 + z^2)/r^2)^{(n-l)/2}$. Then we obtain contributions for $l = n, n - 2, n - 4, ...$ Since m runs from $-l$ to l, one obtains in total $(2n + 1) + (2n - 3) + (2n - 7) + ... = (n + 2)(n + 1)/2$ linearly independent (since orthogonal) contributions. Therefore the space of these functions has the same dimension as that of the f_n's. Thus we may express each f_n as a linear combination of the spherical harmonics.