B Electrostatics

3 Electric Field, Potential, Energy of the Field

3.a Statics

First we consider the time-independent problem: Statics. This means, the quantities depend only on their location, $\rho = \rho(\mathbf{r})$, $\mathbf{j} = \mathbf{j}(\mathbf{r})$, $\mathbf{E} = \mathbf{E}(\mathbf{r})$, $\mathbf{B} = \mathbf{B}(\mathbf{r})$. Then the equation of continuity (1.12) and MAXWELL's equations (1.13-1.16) separate into two groups

$$\begin{array}{cc} \operatorname{div}\mathbf{j}(\mathbf{r}) = 0 & \\ \operatorname{curl}\mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r}) & \operatorname{div}\mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r}) \\ \operatorname{div}\mathbf{B}(\mathbf{r}) = 0 & \operatorname{curl}\mathbf{E}(\mathbf{r}) = \mathbf{0} \\ \text{magnetostatics} & \text{electrostatics} \\ \mathbf{k}_{\mathrm{ma}} = \frac{1}{c}\mathbf{j}(\mathbf{r})\times\mathbf{B}(\mathbf{r}) & \mathbf{k}_{\mathrm{el}} = \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) \end{array} \tag{3.1}$$

The first group of equations contains only the magnetic induction $\mathbf{B}$ and the current density $\mathbf{j}$. It describes magnetostatics. The second group of equations contains only the electric field $\mathbf{E}$ and the charge density ρ. It is the basis of electrostatics. The expressions for the corresponding parts of the force density $\mathbf{k}$ is given in the last line.

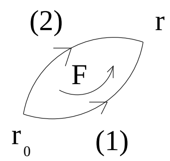
3.b Electric Field and Potential

3.b.α Electric Potential

Now we introduce the electric Potential $\Phi(\mathbf{r})$. For this purpose we consider the path integral over $\mathbf{E}$ along to different paths (1) and (2) from $\mathbf{r}_0$ to $\mathbf{r}$

$$\int_{\substack{\mathbf{r}_0\\(1)}}^{\mathbf{r}} d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) = \int_{\substack{\mathbf{r}_0\\(2)}}^{\mathbf{r}} d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) + \oint d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}), \tag{3.2}$$

where the last integral has to be performed along the closed path from $\mathbf{r}_0$ along (1) to $\mathbf{r}$ and from there in opposite direction along (2) to $\mathbf{r}_0$.This later integral can be transformed by means of STOKES' theorem (B.56) into the integral $\int d\mathbf{f}\cdot\operatorname{curl}\mathbf{E}(\mathbf{r})$ over the open surface bounded by (1) and (2), which vanishes due to MAXWELL's equation $\operatorname{curl}\mathbf{E}(\mathbf{r}) = \mathbf{0}$ (3.1).



Therefore the integral (3.2) is independent of the path and one defines the electric potential

$$\Phi(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) + \Phi(\mathbf{r}_0). \tag{3.3}$$

The choice of $\mathbf{r}_0$ and of $\Phi(\mathbf{r}_0)$ is arbitrary, but fixed. Therefore $\Phi(\mathbf{r})$ is defined apart from an arbitrary additive constant. From the definition (3.3) we have

$$d\Phi(\mathbf{r}) = -d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}), \quad \mathbf{E}(\mathbf{r}) = -\operatorname{grad}\Phi(\mathbf{r}). \tag{3.4}$$

3.b.β Electric Flux and Charge

From $\operatorname{div}\mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r})$, (3.1) one obtains

$$\int_V \mathrm{d}^3r \operatorname{div}\mathbf{E}(\mathbf{r}) = 4\pi \int_V \mathrm{d}^3r\rho(\mathbf{r}) \tag{3.5}$$

and therefore with the divergence theorem (B.59)

$$\int_{\partial V} \mathrm{d}\mathbf{f}\cdot\mathbf{E}(\mathbf{r}) = 4\pi q(V), \tag{3.6}$$

id est the electric flux of the field $\mathbf{E}$ through the surface equals 4π times the charge q in the volume V.
This has a simple application for the electric field of a rotational invariant charge distribution $\rho(\mathbf{r}) = \rho(r)$ with $r = |\mathbf{r}|$. For reasons of symmetry the electric field points in radial direction, $\mathbf{E} = E(r)\mathbf{r}/r$

$$4\pi r^2 E(r) = 4\pi \int_0^r \rho(r')r'^2\mathrm{d}r'\mathrm{d}\Omega = (4\pi)^2 \int_0^r \rho(r')r'^2\mathrm{d}r', \tag{3.7}$$

so that one obtains

$$E(r) = \frac{4\pi}{r^2}\int_0^r \rho(r')r'^2 dr' \tag{3.8}$$

for the field.
As a special case we consider a point charge in the origin. Then one has

$$4\pi r^2 E(r) = 4\pi q, \quad E(r) = \frac{q}{r^2}, \quad \mathbf{E}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}q. \tag{3.9}$$

The potential depends only on r for reasons of symmetry. Then one obtains

$$\operatorname{grad}\Phi(r) = \frac{\mathbf{r}}{r}\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = -\mathbf{E}(\mathbf{r}), \tag{3.10}$$

which after integration yields

$$\Phi(r) = \frac{q}{r} + \text{const.} \tag{3.11}$$

3.b.γ Potential of a Charge Distribution

We start out from point charges q_i at locations $\mathbf{r}_i$. The corresponding potential and the field is obtained from (3.11) und (3.10) by shifting $\mathbf{r}$ by $\mathbf{r}_i$

$$\Phi(\mathbf{r}) = \sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|} \tag{3.12}$$

$$\mathbf{E}(\mathbf{r}) = -\operatorname{grad}\Phi(\mathbf{r}) = \sum_i \frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3}. \tag{3.13}$$

We change now from point charges to the charge density $\rho(\mathbf{r})$. To do this we perform the transition from $\sum_i q_i f(\mathbf{r}_i) = \sum_i \Delta V\rho(\mathbf{r}_i)f(\mathbf{r}_i)$ to $\int \mathrm{d}^3r'\rho(\mathbf{r}')f(\mathbf{r}')$, which yields

$$\Phi(\mathbf{r}) = \int \mathrm{d}^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{3.14}$$

From $\mathbf{E} = -\operatorname{grad}\Phi$ and $\operatorname{div}\mathbf{E} = 4\pi\rho$ one obtains Poisson's equation

$$\triangle\Phi(\mathbf{r}) = -4\pi\rho(\mathbf{r}). \tag{3.15}$$

Please distinguish $\triangle = \nabla\cdot\nabla$ and Δ =Delta. We check eq. (3.15). First we determine

$$\nabla\Phi(\mathbf{r}) = \int \mathrm{d}^3r'\rho(\mathbf{r}')\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} = \int \mathrm{d}^3a\,\rho(\mathbf{r}+\mathbf{a})\frac{\mathbf{a}}{a^3} \tag{3.16}$$

and

$$\triangle\Phi(\mathbf{r}) = \int d^3a(\nabla\rho(\mathbf{r}+\mathbf{a}))\cdot\frac{\mathbf{a}}{a^3} = \int_0^\infty da\int d\Omega_a\frac{\partial\rho(\mathbf{r}+\mathbf{a})}{\partial a} = \int d\Omega_a(\rho(\mathbf{r}+\infty\mathbf{e}_a)-\rho(\mathbf{r})) = -4\pi\rho(\mathbf{r}), \quad (3.17)$$

assuming that ρ vanishes at infinity. The three-dimensional integral over a has been separated by the integral over the radius a and the solid angle Ω_a, $d^3a = a^2 da d\Omega$ (compare section 5).
From Poisson's equation one obtains

$$\triangle\Phi(\mathbf{r}) = \int d^3r'\rho(\mathbf{r}')\triangle\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\rho(\mathbf{r}) = -4\pi\int d^3r'\rho(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}') \quad (3.18)$$

and from the equality of the integrands

$$\triangle\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \quad (3.19)$$

3.c Coulomb Force and Field Energy

The force acting on the charge q_i at $\mathbf{r}_i$ is

$$\mathbf{K}_i = q_i\mathbf{E}_i(\mathbf{r}_i). \quad (3.20)$$

Here $\mathbf{E}_i$ is the electric field without that generated by the charge q_i itself. Then one obtains the Coulomb force

$$\mathbf{K}_i = q_i\sum_{j\neq i}\frac{q_j(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^3}. \quad (3.21)$$

From this equation one realizes the definition of the unit of charge in Gauss's units, 1 dyn$^{1/2}$ cm is the charge, which exerts on the same amount of charge in the distance of 1 cm the force 1 dyn.
The potential energy is

$$U = \frac{1}{2}\sum_i\sum_{j\neq i}\frac{q_iq_j}{|\mathbf{r}_i-\mathbf{r}_j|} = \frac{1}{2}\sum_i q_i\Phi_i(\mathbf{r}_i). \quad (3.22)$$

The factor 1/2 is introduced since each pair of charges appears twice in the sum. E.g., the interaction energy between charge 1 and charge 2 is contained both in $i = 1, j = 2$ and $i = 2, j = 1$. Thus we have to divide by 2. The contribution from q_i is excluded from the potential Φ_i. The force is then as usually

$$\mathbf{K}_i = -\,\mathrm{grad}_{\mathbf{r}_i}U. \quad (3.23)$$

In the continuum one obtains by use of (B.62)

$$U = \frac{1}{2}\int d^3r\rho(\mathbf{r})\Phi(\mathbf{r}) = \frac{1}{8\pi}\int d^3r\,\mathrm{div}\,\mathbf{E}(\mathbf{r})\Phi(\mathbf{r}) = \frac{1}{8\pi}\int_F d\mathbf{f}\cdot\mathbf{E}(\mathbf{r})\Phi(\mathbf{r}) - \frac{1}{8\pi}\int d^3r\mathbf{E}(\mathbf{r})\cdot\,\mathrm{grad}\,\Phi(\mathbf{r}), \quad (3.24)$$

where no longer the contribution from the charge density at the same location has to be excluded from Φ, since it is negligible for a continuous distribution. F should include all charges and may be a sphere of radius R. In the limit $R\to\infty$ one obtains $\Phi\propto 1/R$, $E\propto 1/R^2$, $\int_F\propto 1/R\to 0$. Then one obtains the electrostatic energy

$$U = \frac{1}{8\pi}\int d^3rE^2(\mathbf{r}) = \int d^3r\,u(\mathbf{r}) \quad (3.25)$$

with the energy density

$$u(\mathbf{r}) = \frac{1}{8\pi}E^2(\mathbf{r}). \quad (3.26)$$

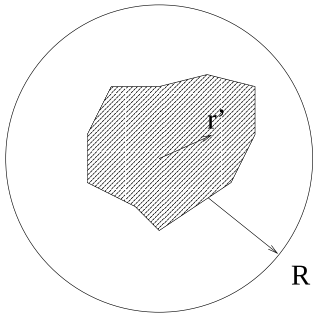
Classical Radius of the Electron As an example we consider the "classical radius of an electron" R_0: One assumes that the charge is homogeneously distributed on the surface of the sphere of radius R. The electric field energy should equal the energy m_0c^2, where m_0 is the mass of the electron.

$$\frac{1}{8\pi}\int_{R_0}^\infty\left(\frac{e_0}{r^2}\right)^2 r^2 dr d\Omega = \frac{e_0^2}{2R_0} = m_0c^2 \quad (3.27)$$

yields $R_0 = 1.4\cdot 10^{-13}$ cm. The assumption of a homogeneous distribution of the charge inside the sphere yields a slightly different result.
From scattering experiments at high energies one knows that the extension of the electron is at least smaller by a factor of 100, thus the assumption made above does not apply.

4 Electric Dipole and Quadrupole

A charge distribution $\rho(\mathbf{r}')$ inside a sphere of radius R around the origin is given. We assume $\rho(\mathbf{r}') = 0$ outside the sphere.



4.a The Field for $r > R$

The potential of the charge distribution is

$$\Phi(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.1}$$

We perform a TAYLOR-expansion in $\mathbf{r}'$, i.e. in the three variables x'_1, x'_2 und x'_3

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{(-\mathbf{r}'\nabla)^l}{l!}\frac{1}{r} = \frac{1}{r} - (\mathbf{r}'\nabla)\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\nabla)(\mathbf{r}'\nabla)\frac{1}{r} - ... \tag{4.2}$$

At first we have to calculate the gradient of $1/r$

$$\nabla\frac{1}{r} = -\frac{\mathbf{r}}{r^3}, \text{ since } \nabla f(r) = \frac{\mathbf{r}}{r}f'(r), \tag{4.3}$$

solve (B.39, B.42). Then one obtains

$$(\mathbf{r}'\nabla)\frac{1}{r} = -\frac{\mathbf{r}'\cdot\mathbf{r}}{r^3}. \tag{4.4}$$

Next we calculate (B.47)

$$\nabla\frac{\mathbf{c}\cdot\mathbf{r}}{r^3} = \frac{1}{r^3}\,\mathrm{grad}\,(\mathbf{c}\cdot\mathbf{r}) + (\mathbf{c}\cdot\mathbf{r})\,\mathrm{grad}\left(\frac{1}{r^3}\right) = \frac{\mathbf{c}}{r^3} - \frac{3(\mathbf{c}\cdot\mathbf{r})\mathbf{r}}{r^5} \tag{4.5}$$

using (B.27) and the solutions of (B.37, B.39). Then we obtain the TAYLOR-expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2r'^2}{2r^5} + ... \tag{4.6}$$

At first we transform $3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2r'^2$

$$3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2r'^2 = x'_\alpha x'_\beta(3x_\alpha x_\beta - r^2\delta_{\alpha,\beta}) = (x'_\alpha x'_\beta - \frac{1}{3}r'^2\delta_{\alpha,\beta})(3x_\alpha x_\beta - r^2\delta_{\alpha,\beta}) \tag{4.7}$$

because of $\delta_{\alpha,\beta}(3x_\alpha x_\beta - \delta_{\alpha,\beta}r^2) = 3x_\alpha x_\alpha - r^2\delta_{\alpha,\alpha} = 0$. Here and in the following we use the summation convention, i.e. we sum over all indices (of components), which appear twice in a product in (4.7), that is over α and β.
We now introduce the quantities

$$q = \int d^3r'\rho(\mathbf{r}') \qquad \text{charge} \tag{4.8}$$

$$\mathbf{p} = \int d^3r'\mathbf{r}'\rho(\mathbf{r}') \qquad \text{dipolar moment} \tag{4.9}$$

$$Q_{\alpha,\beta} = \int d^3r'(x'_\alpha x'_\beta - \frac{1}{3}\delta_{\alpha,\beta}r'^2)\rho(\mathbf{r}') \qquad \text{components of the quadrupolar moment} \tag{4.10}$$

and obtain the expansion for the potential and the electric field

$$\Phi(\mathbf{r}) = \frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + Q_{\alpha,\beta}\frac{3x_\alpha x_\beta - r^2\delta_{\alpha,\beta}}{2r^5} + O(\frac{1}{r^4}) \tag{4.11}$$

$$\mathbf{E}(\mathbf{r}) = -\,\mathrm{grad}\,\Phi(\mathbf{r}) = \frac{q\mathbf{r}}{r^3} + \frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - \mathbf{p}r^2}{r^5} + O(\frac{1}{r^4}) \tag{4.12}$$

4.b Transformation Properties

The multipole moments are defined with respect to a given point, for example with respect to the origin. If one shifts the point of reference by $\mathbf{a}$, i.e. $\mathbf{r}'_1 = \mathbf{r}' - \mathbf{a}$, then one finds with $\rho_1(\mathbf{r}'_1) = \rho(\mathbf{r}')$

$$q_1 = \int \mathrm{d}^3r'_1 \rho_1(\mathbf{r}'_1) = \int \mathrm{d}^3r' \rho(\mathbf{r}') = q \tag{4.13}$$

$$\mathbf{p}_1 = \int \mathrm{d}^3r'_1 \mathbf{r}'_1 \rho_1(\mathbf{r}'_1) = \int \mathrm{d}^3r' (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}') = \mathbf{p} - \mathbf{a}q. \tag{4.14}$$

The total charge is independent of the point of reference. The dipolar moment is independent of the point of reference if $q = 0$ (pure dipol), otherwise it depends on the point of reference. Similarly one finds that the quadrupolar moment is independent of the point of reference, if $q = 0$ and $\mathbf{p} = 0$ (pure quadrupole).
The charge q is invariant under rotation (scalar) $x'_{1,\alpha} = D_{\alpha,\beta} x'_\beta$, where D is a rotation matrix, which describes an orthogonal transformation. The dipole $\mathbf{p}$ transforms like a vector

$$p_{1,\alpha} = \int \mathrm{d}^3r' D_{\alpha,\beta} x'_\beta \rho(\mathbf{r}') = D_{\alpha,\beta} p_\beta \tag{4.15}$$

and the quadrupole Q like a tensor of rank 2

$$Q_{1,\alpha,\beta} = \int \mathrm{d}^3r' (D_{\alpha,\gamma} x'_\gamma D_{\beta,\delta} x'_\delta - \frac{1}{3}\delta_{\alpha,\beta} r'^2)\rho(\mathbf{r}'). \tag{4.16}$$

Taking into account that due to the orthogonality of D

$$\delta_{\alpha,\beta} = D_{\alpha,\gamma} D_{\beta,\gamma} = D_{\alpha,\gamma}\delta_{\gamma,\delta} D_{\beta,\delta}, \tag{4.17}$$

it follows that

$$Q_{1,\alpha,\beta} = D_{\alpha,\gamma} D_{\beta,\delta} Q_{\gamma,\delta}, \tag{4.18}$$

that is the transformation law for tensors of second rank.

4.c Dipole

The prototype of a dipole consists of two charges of opposite sign, q at $\mathbf{r}_0 + \mathbf{a}$ and $-q$ at $\mathbf{r}_0$.

$$\mathbf{p} = q\mathbf{a}. \tag{4.19}$$

Therefore the corresponding charge distribution is

$$\rho(\mathbf{r}) = q(\delta^3(\mathbf{r} - \mathbf{r}_0 - \mathbf{a}) - \delta^3(\mathbf{r} - \mathbf{r}_0)). \tag{4.20}$$

We perform now the TAYLOR expansion in $\mathbf{a}$

$$\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}_0) - q\mathbf{a}\cdot\nabla\delta^3(\mathbf{r} - \mathbf{r}_0) + \frac{q}{2}(\mathbf{a}\cdot\nabla)^2\delta^3(\mathbf{r} - \mathbf{r}_0) + ... - q\delta^3(\mathbf{r} - \mathbf{r}_0), \tag{4.21}$$

where the first and the last term cancel. We consider now the limit $\mathbf{a} \to 0$, where the product $q\mathbf{a} = \mathbf{p}$ is kept fixed. Then we obtain the charge distribution of a dipole $\mathbf{p}$ at location $\mathbf{r}_0$

$$\rho(\mathbf{r}) = -\mathbf{p}\cdot\nabla\delta^3(\mathbf{r} - \mathbf{r}_0) \tag{4.22}$$

and its potential

$$\begin{aligned}\Phi(\mathbf{r}) &= \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = -\mathbf{p}\cdot\int d^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|}\,\mathrm{grad}'\delta^3(\mathbf{r}' - \mathbf{r}_0) = \mathbf{p}\cdot\int d^3r'\,\mathrm{grad}'\frac{1}{|\mathbf{r} - \mathbf{r}'|}\delta^3(\mathbf{r}' - \mathbf{r}_0) \\ &= \mathbf{p}\cdot\int d^3r' \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\delta^3(\mathbf{r}' - \mathbf{r}_0) = \frac{\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3},\end{aligned} \tag{4.23}$$

where equation (B.61) is used and (B.50) has to be solved.

4.d Quadrupole

The quadrupole is described by the second moment of the charge distribution.

4.d.α Symmetries

Q is a symmetric tensor

$$Q_{\alpha,\beta} = Q_{\beta,\alpha}. \tag{4.24}$$

It can be diagonalized by an orthogonal transformation similarly as the tensor of inertia. Further from definition (4.10) it follows that

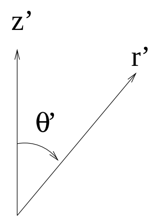
$$Q_{\alpha,\alpha} = 0, \tag{4.25}$$

that is the trace of the quadrupole tensor vanishes. Thus the tensor does not have six, but only five independent components.

4.d.β Symmetric Quadrupole

A special case is the symmetric quadrupole. Its charge distribution depends only on z and on the distance from the z-axis, $\rho = \rho(z, \sqrt{x^2+y^2})$. It obeys

$$Q_{x,y} = Q_{x,z} = Q_{y,z} = 0, \tag{4.26}$$



because $\rho(x,y,z) = \rho(-x,y,z) = \rho(x,-y,z)$. Furthermore one has

$$Q_{x,x} = Q_{y,y} = -\frac{1}{2}Q_{z,z} =: -\frac{1}{3}\hat{Q}. \tag{4.27}$$

The first equality follows from $\rho(x,y,z) = \rho(y,x,z)$, the second one from the vanishing of the trace of Q. The last equality-sign gives the definition of $\hat{Q}$.

One finds

$$\hat{Q} = \frac{3}{2}Q_{z,z} = \int d^3r'(\frac{3}{2}z'^2 - \frac{1}{2}r'^2)\rho(\mathbf{r}') = \int d^3r'r'^2P_2(\cos\theta')\rho(\mathbf{r}') \tag{4.28}$$

with the Legendre polynomial $P_2(\xi) = \frac{3}{2}\xi^2 - \frac{1}{2}$. We will return to the Legendre polynomials in the next section and in appendix C.

As an example we consider the stretched quadrupole with two charges q at $\pm a\mathbf{e}_z$ and a charge $-2q$ in the origin. Then we obtain $\hat{Q} = 2qa^2$. The different charges contribute to the potential of the quadrupole

$$\Phi(\mathbf{r}) = -\frac{1}{3}\hat{Q}\frac{3x^2-r^2}{2r^5} - \frac{1}{3}\hat{Q}\frac{3y^2-r^2}{2r^5} + \frac{2}{3}\hat{Q}\frac{3z^2-r^2}{2r^5} = \frac{\hat{Q}P_2(\cos\theta)}{r^3}. \tag{4.29}$$

4.e Energy, Force and Torque on a Multipole in an external Field

A charge distribution $\rho(\mathbf{r})$ localized around the origin is considered in an external electric potential $\Phi_{\mathrm{a}}(\mathbf{r})$, which may be generated by an external charge distribution ρ_{a}. The interaction energy is then given by

$$U = \int d^3r\rho(\mathbf{r})\Phi_{\mathrm{a}}(\mathbf{r}). \tag{4.30}$$

No factor 1/2 appears in front of the integral, which might be expected in view of this factor in (3.24), since besides the integral over $\rho(\mathbf{r})\Phi_{\mathrm{a}}(\mathbf{r})$ there is a second one over $\rho_{\mathrm{a}}(\mathbf{r})\Phi(\mathbf{r})$, which yields the same contribution. We now expand the external potential and obtain for the interaction energy

$$\begin{aligned} U &= \int d^3r\rho(\mathbf{r})\left\{\Phi_{\mathrm{a}}(0) + \mathbf{r}\nabla\Phi_{\mathrm{a}}|_{r=0} + \frac{1}{2}x_\alpha x_\beta\ \nabla_\alpha\nabla_\beta\Phi_{\mathrm{a}}\Big|_{r=0} + ...\right\} \\ &= q\Phi_{\mathrm{a}}(0) + \mathbf{p}\cdot\nabla\Phi_{\mathrm{a}}|_{r=0} + \frac{1}{2}\left(Q_{\alpha,\beta} + \frac{1}{3}\delta_{\alpha,\beta}\int d^3r\rho(\mathbf{r})r^2\right)\nabla_\alpha\nabla_\beta\Phi_{\mathrm{a}}\Big|_{r=0} + ... \end{aligned} \tag{4.31}$$

The contribution proportional to the integral over $\rho(\mathbf{r})r^2$ vanishes, since $\nabla_\alpha\nabla_\alpha\Phi_\text{a} = \triangle\Phi_\text{a} = -4\pi\rho_\text{a}(\mathbf{r}) = 0$, since there are no charges at the origin, which generate Φ_a. Therefore we are left with the potential of interaction

$$U = q\Phi_\text{a}(0) - \mathbf{p}\cdot\mathbf{E}_\text{a}(0) + \frac{1}{2}Q_{\alpha,\beta}\nabla_\alpha\nabla_\beta\Phi_\text{a} + ... \tag{4.32}$$

For example we can now determine the potential energy between two dipoles, $\mathbf{p}_\text{b}$ in the origin and $\mathbf{p}_\text{a}$ at $\mathbf{r}_0$. The dipole $\mathbf{p}_\text{a}$ generates the potential

$$\Phi_\text{a}(\mathbf{r}) = \frac{\mathbf{p}_\text{a}\cdot(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}. \tag{4.33}$$

Then the interaction energy yields (compare B.47)

$$U_\text{a,b} = \mathbf{p}_\text{b}\cdot\nabla\Phi_\text{a}|_{r=0} = \frac{\mathbf{p}_\text{a}\cdot\mathbf{p}_\text{b}}{r_0^3} - \frac{3(\mathbf{p}_\text{a}\cdot\mathbf{r}_0)(\mathbf{p}_\text{b}\cdot\mathbf{r}_0)}{r_0^5}. \tag{4.34}$$

The force on the dipole in the origin is then given by

$$\mathbf{K} = \int d^3r\rho(\mathbf{r})\mathbf{E}_\text{a}(\mathbf{r}) = \int d^3r\rho(\mathbf{r})(\mathbf{E}_\text{a}(0) + x_\alpha\nabla_\alpha\mathbf{E}_\text{a}|_{r=0} + ...) = q\mathbf{E}_\text{a}(0) + (\mathbf{p}\cdot\text{ grad })\mathbf{E}_\text{a}(0) + ... \tag{4.35}$$

The torque on a dipole in the origin is given by

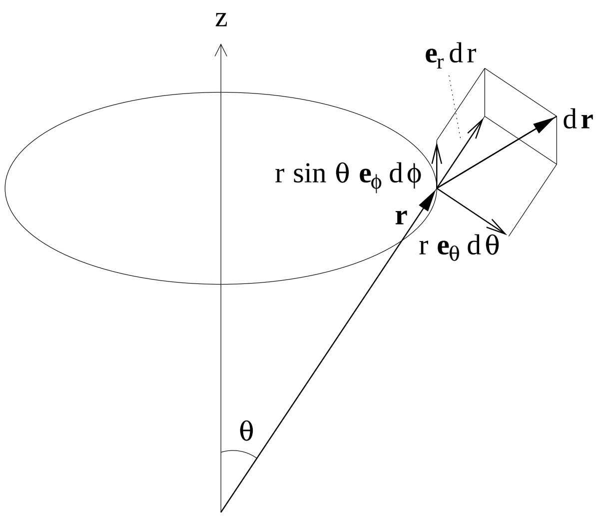
$$\mathbf{M}_\text{mech} = \int d^3r'\rho(\mathbf{r}')\mathbf{r}'\times\mathbf{E}_\text{a}(\mathbf{r}') = \mathbf{p}\times\mathbf{E}_\text{a}(0) + ... \tag{4.36}$$

5 Multipole Expansion in Spherical Coordinates

5.a POISSON Equation in Spherical Coordinates

We first derive the expression for the Laplacian operator in spherical coordinates

$$x = r\sin\theta\cos\phi \quad (5.1)$$
$$y = r\sin\theta\sin\phi \quad (5.2)$$
$$z = r\cos\theta. \quad (5.3)$$



Initially we use only that we deal with curvilinear coordinates which intersect at right angles, so that we may write

$$\mathrm{d}\mathbf{r} = g_r\mathbf{e}_r\mathrm{d}r + g_\theta\mathbf{e}_\theta\mathrm{d}\theta + g_\phi\mathbf{e}_\phi\mathrm{d}\phi \quad (5.4)$$

where the $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ constitute an orthonormal space dependent basis. Easily one finds

$$g_r = 1, \quad g_\theta = r, \quad g_\phi = r\sin\theta. \quad (5.5)$$

The volume element is given by

$$\mathrm{d}^3r = g_r\mathrm{d}rg_\theta\mathrm{d}\theta g_\phi\mathrm{d}\phi = r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\phi = r^2\mathrm{d}r\mathrm{d}\Omega \quad (5.6)$$

with the element of the solid angle

$$\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi. \quad (5.7)$$

5.a.α The Gradient

In order to determine the gradient we consider the differential of the function $\Phi(\mathbf{r})$

$$\mathrm{d}\Phi(\mathbf{r}) = \frac{\partial\Phi}{\partial r}\mathrm{d}r + \frac{\partial\Phi}{\partial\theta}\mathrm{d}\theta + \frac{\partial\Phi}{\partial\phi}\mathrm{d}\phi, \quad (5.8)$$

which coincides with $(\operatorname{grad}\Phi)\cdot\mathrm{d}\mathbf{r}$. From the expansion of the vector field in its components

$$\operatorname{grad}\Phi = (\operatorname{grad}\Phi)_r\mathbf{e}_r + (\operatorname{grad}\Phi)_\theta\mathbf{e}_\theta + (\operatorname{grad}\Phi)_\phi\mathbf{e}_\phi \quad (5.9)$$

and (5.4) it follows that

$$\mathrm{d}\Phi(\mathbf{r}) = (\operatorname{grad}\Phi)_r g_r\mathrm{d}r + (\operatorname{grad}\Phi)_\theta g_\theta\mathrm{d}\theta + (\operatorname{grad}\Phi)_\phi g_\phi\mathrm{d}\phi, \quad (5.10)$$

from which we obtain

$$(\operatorname{grad}\Phi)_r = \frac{1}{g_r}\frac{\partial\Phi}{\partial r}, \quad (\operatorname{grad}\Phi)_\theta = \frac{1}{g_\theta}\frac{\partial\Phi}{\partial\theta}, \quad (\operatorname{grad}\Phi)_\phi = \frac{1}{g_\phi}\frac{\partial\Phi}{\partial\phi} \quad (5.11)$$

for the components of the gradient.

5.a.β The Divergence

In order to calculate the divergence we use the divergence theorem (B.59). We integrate the divergence of $\mathbf{A}(\mathbf{r})$ in a volume limited by the coordinates $r, r+\Delta r$, $\theta, \theta+\Delta\theta$, $\phi, \phi+\Delta\phi$. We obtain

$$\begin{aligned}
\int \mathrm{d}^3r \,\mathrm{div}\,\mathbf{A} &= \int g_r g_\theta g_\phi \,\mathrm{div}\,\mathbf{A}\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
= \int \mathbf{A}\cdot\mathrm{d}\mathbf{f} &= \int g_\theta \mathrm{d}\theta g_\phi \mathrm{d}\phi A_r\Big|_r^{r+\Delta r} + \int g_r \mathrm{d}r g_\phi \mathrm{d}\phi A_\theta\Big|_\theta^{\theta+\Delta\theta} + \int g_r \mathrm{d}r g_\theta \mathrm{d}\theta A_\phi\Big|_\phi^{\phi+\Delta\phi} \\
&= \int \left[\frac{\partial}{\partial r}\left(g_\theta g_\phi A_r\right) + \frac{\partial}{\partial\theta}\left(g_r g_\phi A_\theta\right) + \frac{\partial}{\partial\phi}\left(g_r g_\theta A_\phi\right)\right]\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \qquad (5.12)
\end{aligned}$$

Since the identity holds for arbitrarily small volumina the integrands on the right-hand side of the first line and on the third line have to agree which yields

$$\mathrm{div}\,\mathbf{A}(\mathbf{r}) = \frac{1}{g_r g_\theta g_\phi}\left[\frac{\partial}{\partial r}\left(g_\theta g_\phi A_r\right) + \frac{\partial}{\partial\theta}\left(g_r g_\phi A_\theta\right) + \frac{\partial}{\partial\phi}\left(g_r g_\theta A_\phi\right)\right]. \qquad (5.13)$$

5.a.γ The Laplacian

Using $\triangle\Phi = \mathrm{div}\;\mathrm{grad}\,\Phi$ we obtain finally

$$\triangle\Phi(\mathbf{r}) = \frac{1}{g_r g_\theta g_\phi}\left[\frac{\partial}{\partial r}\left(\frac{g_\theta g_\phi}{g_r}\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{g_r g_\phi}{g_\theta}\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{g_r g_\theta}{g_\phi}\frac{\partial\Phi}{\partial\phi}\right)\right]. \qquad (5.14)$$

This equation holds generally for curvilinear orthogonal coordinates (if we denote them by r, θ, ϕ). Substituting the values for g we obtain for spherical coordinates

$$\triangle\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2}\triangle_\Omega\Phi, \qquad (5.15)$$

$$\triangle_\Omega\Phi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}. \qquad (5.16)$$

The operator $\triangle_\Omega$ acts only on the two angels θ and ϕ, but not on the distance r. Therefore it is also called Laplacian on the sphere.

5.b Spherical Harmonics

As will be explained in more detail in appendix C there is a complete set of orthonormal functions $Y_{l,m}(\theta, \phi)$, $l = 0, 1, 2, ..., m = -l, -l+1, ...l$, which obey the equation

$$\triangle_\Omega Y_{l,m}(\theta,\phi) = -l(l+1)Y_{l,m}(\theta,\phi). \qquad (5.17)$$

They are called spherical harmonics. Completeness means: If $f(\theta, \phi)$ is differentiable on the sphere and its derivatives are bounded, then $f(\theta, \phi)$ can be represented as a convergent sum

$$f(\theta,\phi) = \sum_{l,m}\hat{f}_{l,m}Y_{l,m}(\theta,\phi). \qquad (5.18)$$

Therefore we perform the corresponding expansion for $\Phi(\mathbf{r})$ and $\rho(\mathbf{r})$

$$\Phi(\mathbf{r}) = \sum_{l,m}\hat{\Phi}_{l,m}(r)Y_{l,m}(\theta,\phi), \qquad (5.19)$$

$$\rho(\mathbf{r}) = \sum_{l,m}\hat{\rho}_{l,m}(r)Y_{l,m}(\theta,\phi). \qquad (5.20)$$

The spherical harmonics are orthonormal, i.e. the integral over the solid angle yields

$$\int \mathrm{d}\Omega Y^*_{l,m}(\theta,\phi)Y_{l',m'}(\theta,\phi) = \int \mathrm{d}\phi \sin\theta \mathrm{d}\theta Y^*_{l,m}(\theta,\phi)Y_{l',m'}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}. \tag{5.21}$$

This orthogonality relation can be used for the calculation of $\hat{\Phi}$ and $\hat{\rho}$

$$\begin{aligned}\int \mathrm{d}\phi \sin\theta \mathrm{d}\theta Y^*_{l,m}(\theta,\phi)\rho(\mathbf{r}) &= \sum_{l',m'} \hat{\rho}_{l',m'}(r) \int \mathrm{d}\phi \sin\theta \mathrm{d}\theta Y^*_{l,m}(\theta,\phi)Y_{l',m'}(\theta,\phi) \\ = \sum_{l',m'} \hat{\rho}_{l',m'}(r)\delta_{l,l'}\delta_{m,m'} &= \hat{\rho}_{l,m}(r).\end{aligned} \tag{5.22}$$

We list some of the spherical harmonics

$$Y_{0,0}(\theta,\phi) = \sqrt{\frac{1}{4\pi}} \tag{5.23}$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{5.24}$$

$$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{5.25}$$

$$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \tag{5.26}$$

$$Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \tag{5.27}$$

$$Y_{2,\pm2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\phi}. \tag{5.28}$$

In general one has

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^m_l(\cos\theta)e^{im\phi} \tag{5.29}$$

with the associated Legendre functions

$$P^m_l(\xi) = \frac{(-)^m}{2^l l!}(1-\xi^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}\xi^{l+m}}(\xi^2-1)^l. \tag{5.30}$$

Generally $Y_{l,m}$ is a product of $(\sin\theta)^{|m|}e^{im\phi}$ and a polynomial of order $l-|m|$ in $\cos\theta$. If $l-|m|$ is even (odd), then this polynomial is even (odd) in $\cos\theta$. There is the symmetry relation

$$Y_{l,-m}(\theta,\phi) = (-)^m Y^*_{l,m}(\theta,\phi). \tag{5.31}$$

5.c Radial Equation and Multipole Moments

Using the expansion of Φ and ρ in spherical harmonics the Poisson equation reads

$$\triangle\Phi(\mathbf{r}) = \sum_{l,m}\left(\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\hat{\Phi}_{l,m}(r)) - \frac{l(l+1)}{r^2}\hat{\Phi}_{l,m}(r)\right)Y_{l,m}(\theta,\phi) = -4\pi\sum_{l,m}\hat{\rho}_{l,m}(r)Y_{l,m}(\theta,\phi). \tag{5.32}$$

Equating the coefficients of $Y_{l,m}$ we obtain the radial equations

$$\hat{\Phi}''_{l,m}(r) + \frac{2}{r}\hat{\Phi}'_{l,m}(r) - \frac{l(l+1)}{r^2}\hat{\Phi}_{l,m}(r) = -4\pi\hat{\rho}_{l,m}(r). \tag{5.33}$$

The solution of the homogeneous equation reads

$$\hat{\Phi}_{l,m}(r) = a_{l,m}r^l + b_{l,m}r^{-l-1}. \tag{5.34}$$

For the inhomogeneous equation we introduce the conventional ansatz (at present I suppress the indices l and m.)

$$\hat{\Phi} = a(r)r^l + b(r)r^{-l-1}. \tag{5.35}$$

Then one obtains

$$\hat{\Phi}' = a'(r)r^l + b'(r)r^{-l-1} + la(r)r^{l-1} - (l+1)b(r)r^{-l-2}. \tag{5.36}$$

As usual we require

$$a'(r)r^l + b'(r)r^{-l-1} = 0 \tag{5.37}$$

and obtain for the second derivative

$$\hat{\Phi}'' = la'(r)r^{l-1} - (l+1)b'(r)r^{-l-2} + l(l-1)a(r)r^{l-2} + (l+1)(l+2)b(r)r^{-l-3}. \tag{5.38}$$

After substitution into the radial equation the contributions which contain a and b without derivative cancel. We are left with

$$la'(r)r^{l-1} - (l+1)b'(r)r^{-l-2} = -4\pi\hat{\rho}, \tag{5.39}$$

From the equations (5.37) and (5.39) one obtains by solving for a' and b'

$$\frac{\mathrm{d}a_{l,m}(r)}{\mathrm{d}r} = -\frac{4\pi}{2l+1}r^{1-l}\hat{\rho}_{l,m}(r), \tag{5.40}$$

$$\frac{\mathrm{d}b_{l,m}(r)}{\mathrm{d}r} = \frac{4\pi}{2l+1}r^{l+2}\hat{\rho}_{l,m}(r). \tag{5.41}$$

Now we integrate these equations

$$a_{l,m}(r) = \frac{4\pi}{2l+1}\int_r^\infty \mathrm{d}r' r'^{1-l}\hat{\rho}_{l,m}(r') \tag{5.42}$$

$$b_{l,m}(r) = \frac{4\pi}{2l+1}\int_0^r \mathrm{d}r' r'^{l+2}\hat{\rho}_{l,m}(r'). \tag{5.43}$$

If we add a constant to $a_{l,m}(r)$, then this is a solution of the Poisson equation too, since $r^l Y_{l,m}(\theta,\phi)$ is a homogeneous solution of the Poisson equation. We request a solution, which decays for large r. Therefore we choose $a_{l,m}(\infty) = 0$. If we add a constant to $b_{l,m}$, then this is a solution for $r \neq 0$. For $r = 0$ however, one obtains a singularity, which does not fulfil the Poisson equation. Therefore $b_{l,m}(0) = 0$ is required.
We may now insert the expansion coefficients $\hat{\rho}_{l,m}$ and obtain

$$a_{l,m}(r) = \frac{4\pi}{2l+1}\int_{r'>r} \mathrm{d}^3r' r'^{-1-l} Y^*_{l,m}(\theta',\phi')\rho(\mathbf{r}') \tag{5.44}$$

$$b_{l,m}(r) = \frac{4\pi}{2l+1}\int_{r'<r} \mathrm{d}^3r' r'^{l} Y^*_{l,m}(\theta',\phi')\rho(\mathbf{r}'). \tag{5.45}$$

We may now insert the expressions for $a_{l,m}$ und $b_{l,m}$ into (5.19) and (5.35). The r- und r'-dependence is obtained for $r < r'$ from the a-term as r^l/r'^{l+1} and for $r > r'$ from the b-term as r'^l/r^{l+1}. This can be put together, if we denote by $r_>$ the larger, by $r_<$ the smaller of both radii r and r'. Then one has

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\sum_{m=-l}^{l}\int \mathrm{d}^3r' \frac{r_<^l}{r_>^{l+1}}\rho(\mathbf{r}')Y^*_{l,m}(\theta',\phi')Y_{l,m}(\theta,\phi). \tag{5.46}$$

If $\rho(\mathbf{r}') = 0$ for $r' > R$, then one obtains for $r > R$

$$\Phi(\mathbf{r}) = \sum_{l,m}\sqrt{\frac{4\pi}{2l+1}}q_{l,m}\frac{Y_{l,m}(\theta,\phi)}{r^{l+1}} \tag{5.47}$$

with the multipole moments

$$q_{l,m} = \sqrt{\frac{4\pi}{2l+1}}\int \mathrm{d}^3r' r'^l Y^*_{l,m}(\theta',\phi')\rho(\mathbf{r}'). \tag{5.48}$$

For $l = 0$ one obtains the "monopole moment" charge, for $l = 1$ the components of the dipole moment, for $l = 2$ the components of the quadrupole moment. In particular for $m = 0$ one has

$$q_{0,0} = \sqrt{4\pi}\int d^3r' \sqrt{\frac{1}{4\pi}}\rho(\mathbf{r}') = q \tag{5.49}$$

$$q_{1,0} = \sqrt{\frac{4\pi}{3}}\int d^3r' \sqrt{\frac{3}{4\pi}}r'\cos\theta'\rho(\mathbf{r}') = \int d^3r' z'\rho(\mathbf{r}') = p_z \tag{5.50}$$

$$q_{2,0} = \sqrt{\frac{4\pi}{5}}\int d^3r' \sqrt{\frac{5}{4\pi}}r'^2(\frac{3}{2}\cos^2\theta' - \frac{1}{2})\rho(\mathbf{r}') = \int d^3r'(\frac{3}{2}z'^2 - \frac{1}{2}r'^2)\rho(\mathbf{r}') = \frac{3}{2}Q_{zz}. \tag{5.51}$$

5.d Point Charge at r′, Cylindric Charge Distribution

Finally we consider the case of a point charge q located at $\mathbf{r}'$. We start from the potential

$$\Phi(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{r}'|} = \frac{q}{\sqrt{r^2 + r'^2 - 2rr'\cos\psi}}. \tag{5.52}$$

Here ψ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. We expand in $r_</r_>$

$$\Phi(\mathbf{r}) = \frac{q}{r_>\sqrt{1 + (\frac{r_<}{r_>})^2 - 2\frac{r_<}{r_>}\cos\psi}} = q\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\psi). \tag{5.53}$$

The $P_l(\xi)$ are called Legendre polynomials. For $\cos\psi = \pm 1$ one sees immediately from the expansion of $1/(r_> \mp r_<)$, that $P_l(1) = 1$ and $P_l(-1) = (-)^l$ hold.
On the other hand we may work with (5.46) and find

$$\Phi(\mathbf{r}) = q\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{l,m}(\theta,\phi)Y^*_{l,m}(\theta',\phi'). \tag{5.54}$$

By comparison we obtain the addition theorem for spherical harmonics

$$P_l(\cos\psi) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{l,m}(\theta,\phi)Y^*_{l,m}(\theta',\phi'), \tag{5.55}$$

where the angle ψ between $\mathbf{r}$ and $\mathbf{r}'$ can be expressed by $\mathbf{r}\cdot\mathbf{r}' = rr'\cos\psi$ and by use of (5.1-5.3)

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{5.56}$$

We consider now the special case $\theta' = 0$, i.e. $\psi = \theta$. Then all $Y_{l,m}(\theta',\phi')$ vanish because of the factors $\sin\theta'$ with the exception of the term for $m = 0$ and the addition theorem is reduced to

$$P_l(\cos\theta) = \frac{4\pi}{2l+1}Y_{l,0}(\theta)Y_{l,0}(0) = P_l^0(\cos\theta)P_l^0(1). \tag{5.57}$$

From the representation (5.30) $P_l^0(\xi) = 1/(2^l l!)d^l(\xi^2 - 1)^l/d\xi^l$ one obtains for $\xi = 1$ and the decomposition $(\xi^2-1)^l = (\xi+1)^l(\xi-1)^l$ the result $P_l^0(1) = [(\xi+1)^l/2^l]_{\xi=1}[d^l(\xi-1)^l/d\xi^l/l!]_{\xi=1} = 1$. Thus we have

$$P_l^0(\xi) = P_l(\xi). \tag{5.58}$$

In particular for a cylinder symmetric charge distribution $\rho(\mathbf{r})$, which therefore depends only on r and θ, but not on ϕ, one has

$$\Phi(\mathbf{r}) = \sum_l \frac{P_l(\cos\theta)}{r^{l+1}}q_{l,0} \tag{5.59}$$

with the moments

$$q_{l,0} = \int \mathrm{d}^3 r' r'^l P_l(\cos\theta')\rho(\mathbf{r}'). \tag{5.60}$$

All moments with $m \neq 0$ vanish for a cylinder symmetric distribution.
Exercise Calculate the vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ from (5.1) to (5.5) and check that they constitute an orthonormal basis.
Exercise Calculate by means of Stokes' theorem (B.56) the curl in spherical coordinates.
Exercise Calculate for cylindric coordinates $x = \rho\cos\phi$, $y = \rho\sin\phi$ and z the metric factors g_ρ, g_ϕ and g_z, the volume element and gradient and divergence.

6 Electric Field in Matter

6.a Polarization and Dielectric Displacement

The field equations given by now are also valid in matter. In general matter reacts in an external electric field by polarization. The electrons move with respect to the positively charged nuclei, thus generating dipoles, or already existing dipoles of molecules or groups of molecules order against thermal disorder. Thus an electric field displaces the charges q_i from $\mathbf{r}_i$ to $\mathbf{r}_i + \mathbf{a}_i$, i.e. dipoles $\mathbf{p}_i = q_i\mathbf{a}_i$ are induced. One obtains the charge distribution of the polarization charges (4.22)

$$\rho_{\mathrm{P}}(\mathbf{r}) = -\sum_i \mathbf{p}_i \cdot \operatorname{grad}\delta^3(\mathbf{r}-\mathbf{r}_i). \tag{6.1}$$

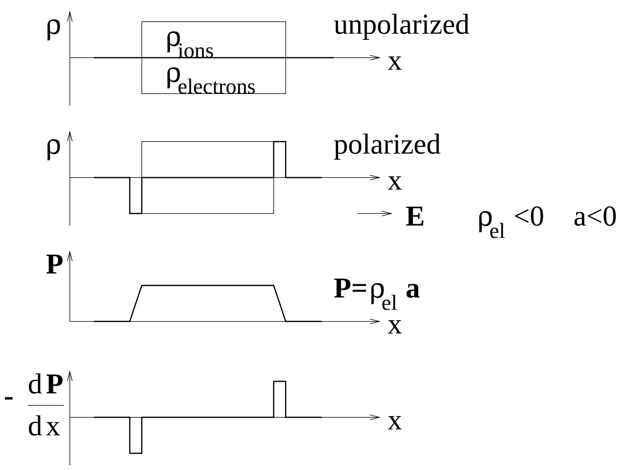
Introducing a density of dipole moments $\mathbf{P}$ called polarization

$$\mathbf{P}(\mathbf{r}) = \frac{\sum \mathbf{p}_i}{\Delta V}, \tag{6.2}$$

where $\sum \mathbf{p}_i$ is the sum of the dipole moments in an infinitesimal volume ΔV, one obtains

$$\rho_{\mathrm{P}}(\mathbf{r}) = -\int \mathrm{d}^3r'\mathbf{P}(\mathbf{r}') \cdot \operatorname{grad}\delta^3(\mathbf{r}-\mathbf{r}') = -\operatorname{div}\left(\int \mathrm{d}^3r'\mathbf{P}(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}')\right) = -\operatorname{div}\mathbf{P}(\mathbf{r}). \tag{6.3}$$

Let us visualize this equation. We start out from a solid body, in which the charges of the ions and electrons (on a scale large in comparison to the distance between the atoms) compensate (upper figure).If one applies a field $\mathbf{E}$ then the electrons move against the ions (second figure). Inside the bulk the charges compensate. Only at the boundaries a net-charge is left. In the third figure the polarization $\mathbf{P} = \rho_{\mathrm{el}}\mathbf{a}$ is shown, which has been continuously smeared at the boundary.The last figure shows the derivative $-\mathrm{d}\mathbf{P}/\mathrm{d}x$. One sees that this charge distribution agrees with that in the second figure.



Thus the charge density ρ consists of the freely moving charge density ρ_{f} and the charge density of the polarization ρ_{P} (the first one may be the charge density on the plates of a condensator)

$$\rho(\mathbf{r}) = \rho_{\mathrm{f}}(\mathbf{r}) + \rho_{\mathrm{P}}(\mathbf{r}) = \rho_{\mathrm{f}}(\mathbf{r}) - \operatorname{div}\mathbf{P}(\mathbf{r}). \tag{6.4}$$

Thus one introduces in MAXWELL's equation

$$\operatorname{div}\mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r}) = 4\pi\rho_{\mathrm{f}}(\mathbf{r}) - 4\pi\operatorname{div}\mathbf{P}(\mathbf{r}) \tag{6.5}$$

the dielectric displacement $\mathbf{D}$

$$\mathbf{D}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) + 4\pi\mathbf{P}(\mathbf{r}), \tag{6.6}$$

so that

$$\operatorname{div}\mathbf{D}(\mathbf{r}) = 4\pi\rho_{\mathrm{f}}(\mathbf{r}) \tag{6.7}$$

holds. The flux of the dielectric displacement through the surface of a volume yields the free charge q_{f} inside this volume

$$\int_{\partial V} \mathrm{d}\mathbf{f} \cdot \mathbf{D}(\mathbf{r}) = 4\pi q_{\mathrm{f}}(V). \tag{6.8}$$

For many substances **P** and **E** are within good approximation proportional as long as the field intensity **E** is not too large

$$\mathbf{P}(\mathbf{r}) = \chi_e \mathbf{E}(\mathbf{r}) \qquad \chi_e \text{ electric susceptibility} \tag{6.9}$$
$$\mathbf{D}(\mathbf{r}) = \epsilon \mathbf{E}(\mathbf{r}) \qquad \epsilon \text{ relative dielectric constant} \tag{6.10}$$
$$\epsilon = 1 + 4\pi\chi_e. \tag{6.11}$$

χ_e and ϵ are tensors for anisotropic matter, otherwise scalars. For ferroelectrica **P** is different from **0** already for $\mathbf{E} = \mathbf{0}$. However, in most cases it is compensated by surface charges. But it is observed, when the polarization is varied by external changes like pressure in the case of quartz (piezo-electricity) or under change of temperature. In Gaussian units the dimensions of **D**, **E** und **P** agree to dyn$^{1/2}$ cm^{-1}. In the SI-system **E** is measured in V/m, **D** and **P** in As/m^2. Since the SI-system is a rational system of units, the Gaussian an irrational one, the conversion factors for **D** and **P** differ by a factor 4π. Consequently the χ_e differ in both systems by a factor 4π. However, the relative dielectric constants ϵ are identical. For more details see appendix A.

6.b Boundaries between Dielectric Media

We now consider the boundary between two dielectric media or a dielectric material and vacuum. From Maxwell's equation $\operatorname{curl}\mathbf{E} = \mathbf{0}$ it follows that the components of the electric field parallel to the boundary coincides in both dielectric media

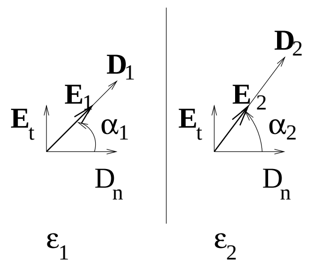
$$\mathbf{E}_{1,t} = \mathbf{E}_{2,t}. \tag{6.12}$$

In order to see this one considers the line integral $\oint \mathrm{d}\mathbf{r}\cdot\mathbf{E}(\mathbf{r})$ along the closed contour which runs tangential to the boundary in one dielectric and returns in the other one, and transforms it into the integral $\int \mathrm{d}\mathbf{f}\cdot\operatorname{curl}\mathbf{E}(\mathbf{r}) = 0$ over the enclosed area. One sees that the integral over the contour vanishes. If the paths of integration in both dielectrica are infinitesimally close to each other, then $\mathbf{E}_t$ vanishes, since the integral over the contour vanishes for arbitrary paths.
On the other hand we may introduce a "pill box" whose covering surface is in one medium, the basal surface in the other one, both infinitesimally separated from the boundary. If there are no free charges at the boundary, then $\int_V \mathrm{d}^3r \operatorname{div}\mathbf{D} = 0$, so that the integral $\int \mathrm{d}\mathbf{f}\cdot\mathbf{D} = 0$ over the surface vanishes. If the surface approaches the boundary, then it follows that the normal component of **D** is continuous

$$D_{1,n} = D_{2,n}. \tag{6.13}$$

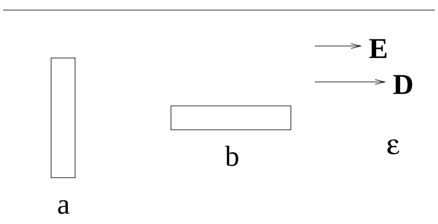
If the angle between the electric field (in an isotropic medium) and the normal to the boundary are α_1 and α_2 then one has

$$E_1 \sin\alpha_1 = E_2 \sin\alpha_2 \tag{6.14}$$
$$D_1 \cos\alpha_1 = D_2 \cos\alpha_2 \tag{6.15}$$
$$\frac{\tan\alpha_1}{\epsilon_1} = \frac{\tan\alpha_2}{\epsilon_2}. \tag{6.16}$$



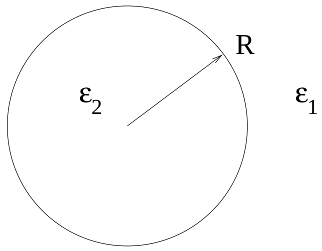
We now consider a cavity in a dielectric medium. If the cavity is very thin in the direction of the field (a) and large in perpendicular direction like a pill box then the displacement **D** agrees in the medium and the cavity.

If on the other hand the cavity has the shape of a slot very long in the direction of the field (b), then the variation of the potential along this direction has to agree, so that inside and outside the cavity **E** coincides. At the edges of the cavities will be scattered fields. It is possible to calculate the field exactly for ellipsoidal cavities. See for example the book by Becker and Sauter. The field is homogeneous inside the ellipsoid. The calculation for a sphere is given below.



6.c Dielectric Sphere in a Homogeneous Electric Field

We consider a dielectric sphere with radius R and dielectric constant ϵ_2 inside a medium with dielectric constant ϵ_1. The electric field in the medium 1 be homogeneous at large distances



$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1 = E_1\mathbf{e}_z \quad r \gg R. \tag{6.17}$$

Thus one obtains for the potential

$$\Phi(\mathbf{r}) = -\mathbf{E}_1 \cdot \mathbf{r} = -E_1 r\cos\theta \quad r \gg R. \tag{6.18}$$

Since $\cos\theta$ is the Legendre polynomial $P_1(\cos\theta)$, the ansatz

$$\Phi(\mathbf{r}) = f(r)\cos\theta \tag{6.19}$$

is successful. The solution of the homogeneous Poisson equation $\triangle(f(r)\cos\theta) = 0$ is a linear combination (5.34) of $f(r) = r$ (homogeneous field) and $f(r) = 1/r^2$ (dipolar field). Since there is no dipole at the origin we may assume

$$\Phi(\mathbf{r}) = \cos\theta \cdot \begin{cases} -E_2 r & r \le R \\ -E_1 r + p/r^2 & r \ge R \end{cases}. \tag{6.20}$$

At the boundary one has $\Phi(R+0) = \phi(R-0)$, which is identical to $\mathbf{E}_{1,\mathrm{t}} = \mathbf{E}_{2,\mathrm{t}}$ and leads to

$$-E_1 R + \frac{p}{R^2} = -E_2 R. \tag{6.21}$$

The condition $D_{1,\mathrm{n}} = D_{2,\mathrm{n}}$ together with $D_\mathrm{n} = -\epsilon\frac{\partial\Phi}{\partial r}$ yields

$$\epsilon_1(E_1 + \frac{2p}{R^3}) = \epsilon_2 E_2. \tag{6.22}$$

From these two equations one obtains

$$E_2 = \frac{3\epsilon_1}{\epsilon_2 + 2\epsilon_1}E_1 \tag{6.23}$$

$$p = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + 2\epsilon_1}R^3 E_1. \tag{6.24}$$

One obtains in particular for the dielectric sphere ($\epsilon_2 = \epsilon$) in the vacuum ($\epsilon_1 = 1$)

$$E_2 = \frac{3}{2+\epsilon}E_1, \quad p = \frac{\epsilon - 1}{\epsilon + 2}R^3 E_1. \tag{6.25}$$

The polarization inside the sphere changes the average field by

$$\mathbf{E}_2 - \mathbf{E}_1 = \frac{1-\epsilon}{2+\epsilon}E_1\mathbf{e}_z = -\frac{4\pi}{3}\mathbf{P}. \tag{6.26}$$

However, for a spherical cavity ($\epsilon_2 = 1$) in a dielectric medium ($\epsilon_1 = \epsilon$) one obtains

$$E_2 = \frac{3\epsilon}{1+2\epsilon}E_1. \tag{6.27}$$

6.d Dielectric Constant according to Clausius and Mossotti

Clausius and Mossotti derive the dielectric constant from the polarizability α of molecules (atoms) as follows: The average dipole moment in the field $\mathbf{E}_\mathrm{eff}$ is

$$\mathbf{p} = \alpha\mathbf{E}_\mathrm{eff}. \tag{6.28}$$

The density n of the dipoles (atoms) yields the polarization

$$\mathbf{P} = n\mathbf{p} = n\alpha\mathbf{E}_{\mathrm{eff}}. \tag{6.29}$$

Therefore we have to determine the effective field $\mathbf{E}_{\mathrm{eff}}$, which acts on the dipole.
For this purpose we cut a sphere of radius R out of the matter around the dipole. These dipoles generate, as we have seen in the example of the dielectric sphere in the vacuum (6.26) an average field

$$\bar{\mathbf{E}}_{\mathrm{P}} = \mathbf{E}_2 - \mathbf{E}_1 = -\frac{4\pi}{3}\mathbf{P}. \tag{6.30}$$

This field is missing after we have cut out the sphere. Instead the rapidly varying field of the dipoles inside the sphere has to be added (with the exception of the field of the dipole at the location, where the field has to be determined)

$$\mathbf{E}_{\mathrm{eff}} = \mathbf{E} - \bar{\mathbf{E}}_{\mathrm{P}} + \sum_i \frac{-\mathbf{p}_i r_i^2 + 3(\mathbf{p}_i\mathbf{r}_i)\mathbf{r}_i}{r_i^5}. \tag{6.31}$$

The sum depends on the location of the dipoles (crystal structure). If the dipoles are located on a cubic lattice, then the sum vanishes, since the contributions from

$$\sum_{\alpha,\beta} \mathbf{e}_\alpha p_\beta \sum_i \frac{-\delta_{\alpha,\beta} r_i^2 + 3x_{i,\alpha}x_{i,\beta}}{r_i^5} \tag{6.32}$$

cancel for $\alpha \neq \beta$, if one adds the contributions for x_α and $-x_\alpha$, those for $\alpha = \beta$, if one adds the three contributions obtained by cyclic permutation of the three components. Thus one obtains for the cubic lattice

$$\chi_{\mathrm{e}}\mathbf{E} = \mathbf{P} = n\alpha\mathbf{E}_{\mathrm{eff}} = n\alpha(\mathbf{E} + \frac{4\pi}{3}\mathbf{P}) = n\alpha(1 + \frac{4\pi}{3}\chi_{\mathrm{e}})\mathbf{E}, \tag{6.33}$$

from which the relation of Clausius (1850) and Mossotti (1879)

$$\chi_{\mathrm{e}} = \frac{n\alpha}{1 - \frac{4\pi n\alpha}{3}} \text{ or } \frac{4\pi}{3}n\alpha = \frac{\epsilon - 1}{\epsilon + 2}. \tag{6.34}$$

follows.

7 Electricity on Conductors

7.a Electric Conductors

The electric field vanishes within a conductor, $\mathbf{E} = \mathbf{0}$, since a nonvanishing field would move the charges. Thus the potential within a conductor is constant. For the conductor #i one has $\Phi(\mathbf{r}) = \Phi_i$.
Outside the conductor the potential is given by Poisson's equation

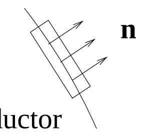
$$\triangle\Phi(\mathbf{r}) = -4\pi\rho(\mathbf{r}) \text{ or } \operatorname{div}(\epsilon(\mathbf{r})\operatorname{grad}\Phi(\mathbf{r})) = -4\pi\rho_{\mathrm{f}}(\mathbf{r}). \tag{7.1}$$

7.a.α Boundary Conditions at the Surface of the Conductor

On the surface of the conductor one has a constant potential (on the side of the dielectric medium, too). Thus the components of $\mathbf{E}$ tangential to the surface vanish

$$\mathbf{E}_{\mathrm{t}}(\mathbf{r}) = \mathbf{0}, \tag{7.2}$$

In general there are charges at the surface of the conductor. We denote its density by $\sigma(\mathbf{r})$.



Integration over a small piece of the surface yields

$$\int \mathrm{d}\mathbf{f} \cdot \mathbf{E}_{\mathrm{a}}(\mathbf{r}) = 4\pi q = 4\pi \int \mathrm{d}f\, \sigma(\mathbf{r}). \tag{7.3}$$

Therefore the field $\mathbf{E}_{\mathrm{a}}$ obeys at the surface in the outside region

$$\mathbf{E}_{\mathrm{a}}(\mathbf{r}) = 4\pi\sigma(\mathbf{r})\mathbf{n}, \quad -\frac{\partial\Phi}{\partial n} = 4\pi\sigma(\mathbf{r}). \tag{7.4}$$

In general the charge density σ at the surface consists of the free charge density σ_{f} at the surface of the conductor and the polarization charge density σ_{P} on the dielectric medium $\sigma(\mathbf{r}) = \sigma_{\mathrm{f}}(\mathbf{r}) + \sigma_{\mathrm{P}}(\mathbf{r})$ with

$$\mathbf{D}_{\mathrm{a}}(\mathbf{r}) = 4\pi\sigma_{\mathrm{f}}(\mathbf{r})\mathbf{n}, \tag{7.5}$$

from which one obtains

$$\sigma_{\mathrm{f}} = \epsilon(\sigma_{\mathrm{f}} + \sigma_{\mathrm{P}}), \quad \sigma_{\mathrm{P}} = (\frac{1}{\epsilon} - 1)\sigma_{\mathrm{f}}. \tag{7.6}$$

7.a.β Force acting on the Conductor (in Vacuo)

Initially one might guess that the force on the conductor is given by $\int \mathrm{d}f \mathbf{E}_{\mathrm{a}}\sigma(\mathbf{r})$. This, however, is wrong. By the same token one could argue that one has to insert the field inside the conductor $\mathbf{E}_{\mathrm{i}} = \mathbf{0}$ into the integral. The truth lies halfway. This becomes clear, if one assumes that the charge is not exactly at the surface but smeared out over a layer of thickness l. If we assume that inside a layer of thickness a one has the charge $s(a)\sigma(\mathbf{r})\mathrm{d}f$ with $s(0) = 0$ and $s(l) = 1$, then the field acting at depth a is $\mathbf{E}_{\mathrm{i}}(\mathbf{r} - a\mathbf{n}) = (1 - s(a))\mathbf{E}_{\mathrm{a}}(\mathbf{r})$, since the fraction $s(a)$ is already screened. With $\rho(\mathbf{r} - a\mathbf{n}) = s'(a)\sigma(\mathbf{r})$ one obtains

$$\mathbf{K} = \int \mathrm{d}f\, \mathrm{d}a\rho(\mathbf{r} - a\mathbf{n})\mathbf{E}(\mathbf{r} - a\mathbf{n}) = \int \mathrm{d}f\sigma(\mathbf{r})\mathbf{E}_{\mathrm{a}}(\mathbf{r}) \int_0^l \mathrm{d}as'(a)(1 - s(a)). \tag{7.7}$$

The integral over a yields $(s(a) - s^2(a)/2)|_0^l = 1/2$, so that finally we obtain the force

$$\mathbf{K} = \frac{1}{2}\int \mathrm{d}f\sigma(\mathbf{r})\mathbf{E}_{\mathrm{a}}(\mathbf{r}). \tag{7.8}$$

7.b Capacities

We now consider several conductors imbedded in the vacuum or in dielectric media. Outside the conductors there should be no free moving charge densities, $\rho_\mathrm{f} = 0$. The electric potentials Φ_i of the conductors #i should be given. We look for the free charges q_i at the conductors. Since MAXWELL's equations are linear (and we assume that there is a linear relation $\mathbf{D} = \epsilon\mathbf{E}$) we may write the potential as a superposition of solutions Ψ_i

$$\Phi(\mathbf{r}) = \sum_i \Phi_i \Psi_i(\mathbf{r}). \tag{7.9}$$

Ψ_i is the solution which assumes the value 1 at the conductor #i, and 0 at all others

$$\Psi_i(\mathbf{r}) = \delta_{i,j} \quad \mathbf{r} \in \text{ conductor } j. \tag{7.10}$$

The charge on conductor #i is then given by

$$q_i = -\frac{1}{4\pi} \int_{F_i} \mathrm{d}f \epsilon \left. \frac{\partial \Phi}{\partial n} \right|_\mathrm{a} = \sum_j C_{i,j} \Phi_j \tag{7.11}$$

with the capacity coefficients

$$C_{i,j} = -\frac{1}{4\pi} \int_{F_i} \mathrm{d}f \epsilon \left. \frac{\partial \Psi_j}{\partial n} \right|_\mathrm{a}. \tag{7.12}$$

The capacity has the dimension charge/(electric potential), which in GAUSSian units is a length. The conversion into the SI-system is by the factor $4\pi\epsilon_0$, so that 1 cm $\hat{=}$ $1/9 \cdot 10^{-11}$ As/V = 10/9 pF (picofarad).
The electrostatic energy is obtained from

$$\mathrm{d}U = \sum_i \Phi_i \mathrm{d}q_i = \sum_{i,j} \Phi_i C_{i,j} \mathrm{d}\Phi_j, \tag{7.13}$$

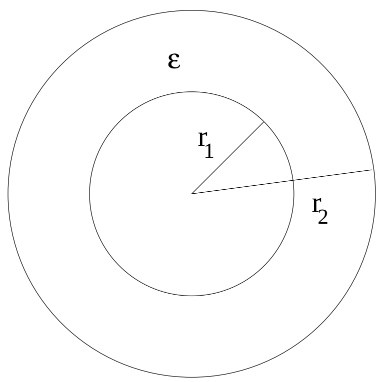
that is

$$\frac{\partial U}{\partial \Phi_j} = \sum_i C_{i,j} \Phi_i, \tag{7.14}$$

$$\frac{\partial^2 U}{\partial \Phi_i \partial \Phi_j} = C_{i,j} = \frac{\partial^2 U}{\partial \Phi_j \partial \Phi_i} = C_{j,i}, \tag{7.15}$$

$$U = \frac{1}{2} \sum_{i,j} C_{i,j} \Phi_i \Phi_j = \frac{1}{2} \sum_i \Phi_i q_i \tag{7.16}$$

As an example we consider a spherical capacitor. Two concentric conducting spheres with radii r_1, r_2 with $r_1 < r_2$ carry the charges q_1 and q_2, resp. Outside be vacuum. Between the two spheres is a medium with dielectric constant ϵ. Then outside the spheres one has



$$\Phi(r) = \frac{q_1 + q_2}{r} \quad r \geq r_2. \tag{7.17}$$

The potential decays in the space between the two spheres like $q_1/(\epsilon r)$. Since the potential is continuous at $r = r_2$, it follows that

$$\Phi(r) = \frac{q_1}{\epsilon r} - \frac{q_1}{\epsilon r_2} + \frac{q_1 + q_2}{r_2} \quad r_1 \leq r \leq r_2. \tag{7.18}$$

Inside the smaller sphere the potential is constant.

$$\Phi(r) = \frac{q_1}{\epsilon r_1} - \frac{q_1}{\epsilon r_2} + \frac{q_1 + q_2}{r_2} \quad r \leq r_1. \tag{7.19}$$

From this one calculates the charges as a function of the potentials $\Phi_i = \Phi(r_i)$

$$q_1 = \frac{\epsilon r_1 r_2}{r_2 - r_1}(\Phi_1 - \Phi_2) \tag{7.20}$$

$$q_2 = \frac{\epsilon r_1 r_2}{r_2 - r_1}(\Phi_2 - \Phi_1) + r_2\Phi_2, \tag{7.21}$$

from which the capacitor coefficients can be read off immediately. If the system is neutral, $q = q_1 = -q_2$, then q can be expressed by the difference of the potential

$$q = C(\Phi_1 - \Phi_2) \tag{7.22}$$

and one calls C the capacity. For the spherical capacitor one obtains $\Phi_2 = 0$ and $\Phi_1 = \frac{q_1}{\epsilon}(\frac{1}{r_1} - \frac{1}{r_2})$, from which the capacity

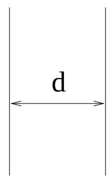
$$C = \frac{\epsilon r_1 r_2}{r_2 - r_1} \tag{7.23}$$

is obtained.
For a single sphere r_2 can go to ∞ and one finds $C = \epsilon r_1$.
We obtain the plate capacitor with a distance d between the plates, by putting $r_2 = r_1 + d$ in the limit of large r_1

$$C = \frac{(r_1^2 + r_1 d)\epsilon}{d} = \frac{4\pi r_1^2 \epsilon}{d}\Big(\frac{1}{4\pi} + \frac{d}{4\pi r_1}\Big), \tag{7.24}$$

which approaches $\frac{\epsilon F}{4\pi d}$ for large r_1 with the area F. Therefore one obtains for the plate capacitor

$$C = \frac{\epsilon F}{4\pi d}. \tag{7.25}$$



A different consideration is the following: The charge q generates the flux $DF = 4\pi q$. Therefore the potential difference between the two plates is $\Phi = \frac{D}{\epsilon}d = \frac{4\pi d}{\epsilon F}q$, from which $C = q/\phi = \frac{\epsilon F}{4\pi d}$ follows. Be aware that here we have denoted the free charge by q.

7.c Influence Charges

If we fix the potentials of all conductors to 0, $\Phi_i = 0$ in the presence of a free charge q' at $\mathbf{r}'$, then we write the potential

$$\Phi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')q' \tag{7.26}$$

with the Green's function G. Apparently this function obeys the equation

$$\nabla(\epsilon(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}')) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}') \tag{7.27}$$

for $\mathbf{r}$ outside the conductor. For $\mathbf{r}$ at the surface of the conductors we have $G(\mathbf{r}, \mathbf{r}') = 0$. The superposition principle yields for a charge density $\rho_{\mathrm{f}}(\mathbf{r}')$ located outside the conductors

$$\Phi(\mathbf{r}) = \int \mathrm{d}^3 r' G(\mathbf{r}, \mathbf{r}')\rho_{\mathrm{f}}(\mathbf{r}') + \sum_i \Phi_i \Psi_i(\mathbf{r}), \tag{7.28}$$

where now we have assumed that the conductors have the potential Φ_i.
We now show that the Green's function is symmetric, $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$. In order to show this we start from the integral over the surfaces of the conductors

$$\int \mathrm{d}\mathbf{f}'' \cdot \{G(\mathbf{r}'', \mathbf{r})\epsilon(\mathbf{r}'')\nabla'' G(\mathbf{r}'', \mathbf{r}') - \epsilon(\mathbf{r}'')[\nabla'' G(\mathbf{r}'', \mathbf{r})]G(\mathbf{r}'', \mathbf{r}')\} = 0, \tag{7.29}$$

since G vanishes at the surface of the conductors. The area element $\mathrm{d}\mathbf{f}''$ is directed into the conductors. We perform the integral also over a sphere of radius R, which includes all conductors. Since $G \sim 1/R$ and since

$\nabla'' G \sim 1/R^2$ the surface integral vanishes for $R \to \infty$. Application of the divergence theorem yields

$$\int \mathrm{d}^3 r'' \{G(\mathbf{r}'', \mathbf{r}) \nabla''[\epsilon(\mathbf{r}'')\nabla'' G(\mathbf{r}'', \mathbf{r}')] - \nabla''[\epsilon(\mathbf{r}'')\nabla'' G(\mathbf{r}'', \mathbf{r})] G(\mathbf{r}'', \mathbf{r}')\} \tag{7.30}$$

$$= -4\pi \int \mathrm{d}^3 r'' \{G(\mathbf{r}'', \mathbf{r})\delta^3(\mathbf{r}'' - \mathbf{r}') - \delta^3(\mathbf{r}'' - \mathbf{r}) G(\mathbf{r}'', \mathbf{r}')\} \tag{7.31}$$

$$= -4\pi(G(\mathbf{r}', \mathbf{r}) - G(\mathbf{r}, \mathbf{r}')) = 0. \tag{7.32}$$

We consider now a few examples:

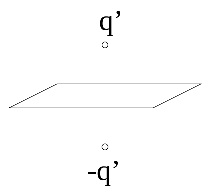
7.c.α Space free of Conductors

In a space with constant dielectric constant ϵ and without conductors one has

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\epsilon|\mathbf{r} - \mathbf{r}'|}. \tag{7.33}$$

7.c.β Conducting Plane

For a conducting plane $z = 0$ ($\epsilon = 1$) one solves the problem by mirror charges. If the given charge q' is located at $\mathbf{r}' = (x', y', z')$, then one should imagine a second charge $-q'$ at $\mathbf{r}'' = (x', y', -z')$. This mirror charge compensates the potential at the surface of the conductor. One obtains



$$G(\mathbf{r}, \mathbf{r}') = \begin{cases} \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}''|} & \text{for } \operatorname{sign} z = \operatorname{sign} z' \\ 0 & \text{for } \operatorname{sign} z = -\operatorname{sign} z'. \end{cases} \tag{7.34}$$

Next we consider the force which acts on the charge q'. The potential is $\Phi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')q'$. The contribution $q'/|\mathbf{r} - \mathbf{r}'|$ is the potential of q' itself that does not exert a force on q' . The second contribution $-q'/|\mathbf{r} - \mathbf{r}''|$ comes, however, from the influence charges on the metal surface and exerts the force

$$\mathbf{K} = -q' \operatorname{grad} \frac{-q'}{|\mathbf{r} - \mathbf{r}''|} = -\frac{q'^2 \mathbf{e}_z}{4z'^2} \operatorname{sign} z'. \tag{7.35}$$

Further one determines the influence charge on the plate. At $z = 0$ one has $4\pi \operatorname{sign} z' \mathbf{e}_z \sigma(\mathbf{r}) = \mathbf{E}(\mathbf{r}) = q' \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - q' \frac{\mathbf{r}-\mathbf{r}''}{|\mathbf{r}-\mathbf{r}''|^3}$. From this one obtains the density of the surface charge per area

$$\sigma(\mathbf{r}) = -\frac{q'}{2\pi} \frac{|z'|}{\sqrt{(x-x')^2 + (y-y')^2 + z'^2}^3} \tag{7.36}$$

With $\mathrm{d}f = \pi \mathrm{d}(x^2 + y^2)$ one obtains

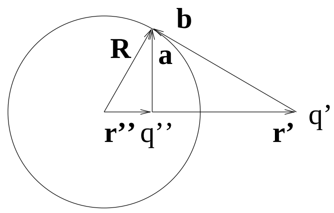
$$\int \mathrm{d}f \sigma(\mathbf{r}) = -\frac{q'|z'|}{2} \int_{z'^2}^{\infty} \frac{\mathrm{d}(x^2 + y^2 + z'^2)}{(x^2 + y^2 + z'^2)^{3/2}} = -q'. \tag{7.37}$$

The force acting on the plate is obtained as

$$\mathbf{K} = \frac{1}{2} \int \mathrm{d}f \mathbf{E}(\mathbf{r})\sigma(\mathbf{r}) = \frac{q'^2 z'|z'|}{2} \mathbf{e}_z \int \frac{\mathrm{d}(x^2 + y^2 + z'^2)}{(x^2 + y^2 + z'^2)^3} = \frac{q'^2 \mathbf{e}_z}{4z'^2} \operatorname{sign} z'. \tag{7.38}$$

7.c.γ Conducting Sphere

We consider a charge q' located at $\mathbf{r}'$ in the presence of a conducting sphere with radius R and center in the origin. Then there is a vector $\mathbf{r}''$, so that the ratio of the distances of all points $\mathbf{R}$ on the surface of the sphere from $\mathbf{r}'$ and $\mathbf{r}''$ is constant. Be



$$a^2 := (\mathbf{R} - \mathbf{r}'')^2 = R^2 + r''^2 - 2\mathbf{R} \cdot \mathbf{r}'' \tag{7.39}$$
$$b^2 := (\mathbf{R} - \mathbf{r}')^2 = R^2 + r'^2 - 2\mathbf{R} \cdot \mathbf{r}' \tag{7.40}$$

This constant ratio of the distances is fulfilled for $\mathbf{r} \parallel \mathbf{r}''$ and

$$\frac{R^2 + r''^2}{R^2 + r'^2} = \frac{r''}{r'}. \tag{7.41}$$

Then one has

$$R^2 = r'r'' \quad \mathbf{r}'' = \frac{R^2}{r'^2}\mathbf{r}' \tag{7.42}$$
$$\frac{a^2}{b^2} = \frac{r''}{r'} = \frac{R^2}{r'^2} = \frac{r''^2}{R^2}. \tag{7.43}$$

Thus one obtains a constant potential on the sphere with the charge q' at $\mathbf{r}'$ and the charge $q'' = -q'R/r'$ at r''

$$G(\mathbf{r}, \mathbf{r}') = \begin{cases} \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{|\mathbf{r}-\mathbf{r}''|} & \text{for } \operatorname{sign}(r-R) = \operatorname{sign}(r'-R), \\ 0 & \text{otherwise}. \end{cases} \tag{7.44}$$

The potential on the sphere vanishes with this Green's function G. For $r' > R$ it carries the charge q'' and for $r' < R$ the charge $-q'$. Thus if the total charge on the sphere vanishes one has to add a potential Φ, which corresponds to a homogeneously distributed charge $-q''$ and q', resp.

8 Energy, Forces and Stress in Dielectric Media

8.a Electrostatic Energy

By displacing the charge densities $\delta\rho = \delta\rho_{\rm f} + \delta\rho_{\rm P}$ the electrostatic energy

$$\delta U = \int d^3r \delta\rho_{\rm f}\Phi + \int d^3r \delta\rho_{\rm P}\Phi \tag{8.1}$$

will be added to the system. Simultaneously there are additional potentials $\Phi_{\rm i}$ in the matter guaranteeing that the polarization is in equilibrium, i. e.

$$\delta U = \int d^3r \delta\rho_{\rm f}\Phi + \int d^3r \delta\rho_{\rm P}(\Phi + \Phi_{\rm i}). \tag{8.2}$$

These potentials are so that $\delta U = 0$ holds for a variation of the polarization, so that the polarizations are in equilibrium

$$\Phi + \Phi_{\rm i} = 0. \tag{8.3}$$

These considerations hold as long as the process is run adiabatically and under the condition that no mechanical energy is added. Thus the matter is in a force-free state (equilibrium $\mathbf{k} = \mathbf{0}$) or it has to be under rigid constraints. Then one obtains with (B.62)

$$\delta U = \int d^3r\, \delta\rho_{\rm f}\, \Phi = \frac{1}{4\pi}\int d^3r\, {\rm div}\, \delta\mathbf{D}\, \Phi = -\frac{1}{4\pi}\int d^3r \delta\mathbf{D}\cdot\, {\rm grad}\, \Phi = \frac{1}{4\pi}\int d^3r \mathbf{E}\cdot\delta\mathbf{D}, \tag{8.4}$$

similarly to the matter-free case (3.25). Then one obtains for the density of the energy at fixed density of matter $\rho_{\rm m}$ (we assume that apart from the electric field only the density of matter determines the energy-density; in general, however, the state of distortion will be essential)

$$du = \frac{1}{4\pi}\mathbf{E}\cdot d\mathbf{D}. \tag{8.5}$$

If $\mathbf{D} = \epsilon\mathbf{E}$, then one obtains

$$u = u_0(\rho_{\rm m}) + \frac{1}{4\pi}\int \epsilon(\rho_{\rm m})\mathbf{E}\cdot d\mathbf{E} = u_0(\rho_{\rm m}) + \frac{1}{8\pi}\epsilon(\rho_{\rm m})E^2 = u_0(\rho_{\rm m}) + \frac{D^2}{8\pi\epsilon(\rho_{\rm m})}, \tag{8.6}$$

since the dielectric constant depends in general on the density of mass.

8.b Force Density in Isotropic Dielectric Matter

We may determine the force density in a dielectric medium by moving the masses and free charges from $\mathbf{r}$ to $\mathbf{r} + \delta\mathbf{s}(\mathbf{r})$ and calculating the change of energy δU. The energy added to the system is

$$\delta U = \int d^3r\, \mathbf{k}_{\rm a}(\mathbf{r})\cdot\delta\mathbf{s}(\mathbf{r}), \tag{8.7}$$

where $\mathbf{k}_{\rm a}$ is an external force density. The internal electric and mechanical force density $\mathbf{k}$ acting against it in equilibrium is

$$\mathbf{k}(\mathbf{r}) = -\mathbf{k}_{\rm a}(\mathbf{r}), \tag{8.8}$$

so that

$$\delta U = -\int d^3r\, \mathbf{k}(\mathbf{r})\cdot\delta\mathbf{s}(\mathbf{r}) \tag{8.9}$$

holds. We bring now δU into this form

$$\delta U = \int d^3r \left(\frac{\partial u}{\partial\mathbf{D}}\cdot\delta\mathbf{D} + \left.\frac{\partial u}{\partial\rho_{\rm m}}\right|_{\mathbf{D}}\delta\rho_{\rm m}\right), \quad u = u(\mathbf{D}, \rho_{\rm m}). \tag{8.10}$$

Since $\partial u/\partial \mathbf{D} = \mathbf{E}/(4\pi)$ we rewrite the first term as in the previous section

$$\delta U = \int d^3r \left(\Phi(\mathbf{r})\delta\rho_f(\mathbf{r}) + \left.\frac{\partial u}{\partial \rho_m}\right|_{\mathbf{D}} \delta\rho_m \right). \tag{8.11}$$

From the equation of continuity $\partial\rho/\partial t = -\operatorname{div}\mathbf{j}$ we derive the relation between $\delta\rho$ and $\delta\mathbf{s}$. The equation has to be multiplied by δt and one has to consider that $\mathbf{j}\delta t = \rho\mathbf{v}\delta t = \rho\delta\mathbf{s}$ holds. With $(\partial\rho/\partial t)\delta t = \delta\rho$ we obtain

$$\delta\rho = -\operatorname{div}(\rho\delta\mathbf{s}). \tag{8.12}$$

Then we obtain

$$\begin{aligned} \delta U &= -\int d^3r \left(\Phi(\mathbf{r}) \operatorname{div}(\rho_f\delta\mathbf{s}) + \frac{\partial u}{\partial\rho_m} \operatorname{div}(\rho_m\delta\mathbf{s}) \right) \\ &= \int d^3r \left(\operatorname{grad}\Phi(\mathbf{r})\rho_f(\mathbf{r}) + \left(\operatorname{grad}\frac{\partial u}{\partial\rho_m} \right)\rho_m(\mathbf{r}) \right) \cdot \delta\mathbf{s}(\mathbf{r}), \end{aligned} \tag{8.13}$$

where the divergence theorem (B.62) has been used by the derivation of the last line. This yields

$$\mathbf{k}(\mathbf{r}) = \rho_f(\mathbf{r})\mathbf{E}(\mathbf{r}) - \rho_m(\mathbf{r}) \operatorname{grad}\left(\frac{\partial u}{\partial\rho_m}\right). \tag{8.14}$$

The first contribution is the Coulomb force on the free charges. The second contribution has to be rewritten. We substitute (8.6) $u = u_0(\rho_m) + D^2/(8\pi\epsilon(\rho_m))$. Then one has

$$\frac{\partial u}{\partial\rho_m} = \frac{du_0}{d\rho_m} + \frac{1}{8\pi}D^2\frac{d(1/\epsilon)}{d\rho_m} = \frac{du_0}{d\rho_m} - \frac{1}{8\pi}E^2\frac{d\epsilon}{d\rho_m}. \tag{8.15}$$

The first term can be written

$$-\rho_m \operatorname{grad}\frac{du_0}{d\rho_m} = -\operatorname{grad}\left(\rho_m\frac{du_0}{d\rho_m} - u_0\right) = -\operatorname{grad}P_0(\rho_m), \tag{8.16}$$

where we use that $(du_0/d\rho_m)\operatorname{grad}\rho_m = \operatorname{grad}u_0$. Here P_0 is the hydrostatic pressure of the liquid without electric field

$$\mathbf{k}_{0,\text{hydro}} = -\operatorname{grad}P_0(\rho_m(\mathbf{r})). \tag{8.17}$$

The hydrostatic force acting on the volume V can be written in terms of a surface integral

$$\mathbf{K}_0 = -\int_V d^3r \operatorname{grad}P_0(\rho_m(\mathbf{r})) = -\int_{\partial V} d\mathbf{f}\, P_0(\rho_m(\mathbf{r})). \tag{8.18}$$

This is a force which acts on the surface ∂V with the pressure P_0. There remains the electrostrictive contribution

$$\frac{1}{8\pi}\rho_m \operatorname{grad}\left(E^2\frac{d\epsilon}{d\rho_m}\right) = \frac{1}{8\pi}\operatorname{grad}\left(E^2\rho_m\frac{d\epsilon}{d\rho_m}\right) - \frac{1}{8\pi}E^2\operatorname{grad}\epsilon, \tag{8.19}$$

where $(d\epsilon/d\rho_m)\operatorname{grad}\rho_m = \operatorname{grad}\epsilon$ has been used. Then the total force density is

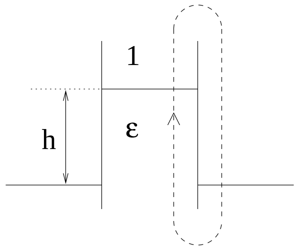
$$\mathbf{k}(\mathbf{r}) = \rho_f(\mathbf{r})\mathbf{E}(\mathbf{r}) + \operatorname{grad}\left(-P_0(\rho_m) + \frac{1}{8\pi}E^2\rho_m\frac{d\epsilon}{d\rho_m}\right) - \frac{1}{8\pi}E^2\operatorname{grad}\epsilon. \tag{8.20}$$

Applications:
Dielectric fluid between two vertical capacitor plates. What is the difference h in height between between the surface of a fluid between the plates of the capacitor and outside the capacitor? For this purpose we introduce the integral along a closed path which goes up between the plates of the capacitor and outside down

$$\oint \mathbf{k}\cdot d\mathbf{r} = \oint \operatorname{grad}\left(-P_0 + \frac{1}{8\pi}E^2\rho_m\frac{\partial\epsilon}{\partial\rho_m}\right)\cdot d\mathbf{r} - \frac{1}{8\pi}\oint E^2\operatorname{grad}\epsilon\cdot d\mathbf{r} = \frac{1}{8\pi}E^2(\epsilon - 1). \tag{8.21}$$

The integral over the gradient along the closed path vanishes, whereas the integral of $E^2 \operatorname{grad} \epsilon$ yields a contribution at the two points where the path of integration intersects the surface. In addition there is the gravitational force. Both have to compensate each other

$$\mathbf{k} + \mathbf{k}_{\mathrm{grav}} = \mathbf{0}, \tag{8.22}$$



that is

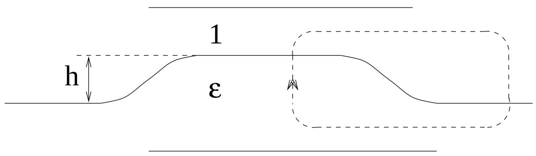
$$\oint \mathrm{d}\mathbf{r} \cdot \mathbf{k}_{\mathrm{grav}} = -\rho_{\mathrm{m}} g h = -\oint \mathrm{d}\mathbf{r} \cdot \mathbf{k}, \tag{8.23}$$

from which one obtains the height

$$h = \frac{E^2(\epsilon - 1)}{8\pi \rho_{\mathrm{m}} g}. \tag{8.24}$$

Dielectric fluid between two horizontal capacitor plates

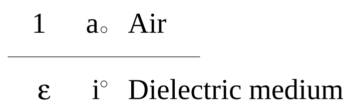
What is the elevation of a dielectric fluid between two horizontal capacitor plates? The problem can be solved in a similar way as between two vertical plates.It is useful, however, to use



$$-\frac{1}{8\pi} E^2 \operatorname{grad} \epsilon = \frac{1}{8\pi} D^2 \operatorname{grad} (\frac{1}{\epsilon}). \tag{8.25}$$

Hydrostatic pressure difference at a boundary

Performing an integration through the boundary from the dielectric medium to air one obtains



$$0 = \int_{\mathrm{i}}^{\mathrm{a}} \mathbf{k} \cdot \mathrm{d}\mathbf{r} = \int \operatorname{grad} (-P_0 + \frac{1}{8\pi} \rho_{\mathrm{m}} E^2 \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_{\mathrm{m}}}) \cdot \mathrm{d}\mathbf{r} - \frac{1}{8\pi} \int E_{\mathrm{t}}^2 \operatorname{grad} \epsilon \cdot \mathrm{d}\mathbf{r} + \frac{1}{8\pi} \int D_{\mathrm{n}}^2 \operatorname{grad} (\frac{1}{\epsilon}) \cdot \mathrm{d}\mathbf{r}. \tag{8.26}$$

This yields the difference in hydrostatic pressure at both sides of the boundary

$$P_{0,\mathrm{i}}(\rho_{\mathrm{m}}) - P_{0,\mathrm{a}} = \frac{1}{8\pi} \left(\rho_{\mathrm{m}} \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_{\mathrm{m}}} E^2 - (\epsilon - 1) E_{\mathrm{t}}^2 + (\frac{1}{\epsilon} - 1) D_{\mathrm{n}}^2 \right) \tag{8.27}$$

Pressure in a practically incompressible dielectric medium
From

$$\mathbf{k} + \mathbf{k}_{\mathrm{grav}} = -\operatorname{grad} (P_0(\rho_{\mathrm{m}})) + \rho_{\mathrm{m}} \operatorname{grad} (\frac{1}{8\pi} E^2 \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_{\mathrm{m}}}) - \rho_{\mathrm{m}} \operatorname{grad} (gz) = \mathbf{0}. \tag{8.28}$$

one obtains for approximately constant ρ_{m}

$$P_0 = \rho_{\mathrm{m}} (\frac{1}{8\pi} E^2 \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_{\mathrm{m}}} - gz) + \text{const}. \tag{8.29}$$

8.c Maxwell's Stress Tensor

Now we represent the force density **k** as divergence of a tensor

$$k_\alpha = \nabla_\beta T_{\alpha,\beta}. \tag{8.30}$$

If one has such a representation, then the force acting on a volume V is given by

$$\mathbf{K} = \int_V \mathrm{d}^3 r \mathbf{k}(\mathbf{r}) = \int \mathrm{d}^3 r \mathbf{e}_\alpha \nabla_\beta T_{\alpha,\beta} = \int_{\partial V} \mathrm{d} f_\beta (\mathbf{e}_\alpha T_{\alpha,\beta}). \tag{8.31}$$

The force acting on the volume is such represented by a force acting on the surface. If it were isotropic $T_{\alpha,\beta} = -P\delta_{\alpha,\beta}$, we would call P the pressure acting on the surface. In the general case we consider here one calls T the stress tensor, since the pressure is anisotropic and there can be shear stress.
In order to calculate T we start from

$$k_\alpha = \rho_\mathrm{f} E_\alpha - \rho_\mathrm{m} \nabla_\alpha \left(\frac{\partial u}{\partial \rho_\mathrm{m}} \right). \tag{8.32}$$

We transform

$$\rho_\mathrm{f} E_\alpha = \frac{1}{4\pi} E_\alpha \nabla_\beta D_\beta = \frac{1}{4\pi} \left(\nabla_\beta (E_\alpha D_\beta) - (\nabla_\beta E_\alpha) D_\beta \right) \tag{8.33}$$

and use $\nabla_\beta E_\alpha = \nabla_\alpha E_\beta$ because of $\operatorname{curl} \mathbf{E} = \mathbf{0}$. This yields

$$k_\alpha = \nabla_\beta \left(\frac{1}{4\pi} E_\alpha D_\beta\right) - \rho_\mathrm{m} \nabla_\alpha \left(\frac{\partial u}{\partial \rho_\mathrm{m}} \right) - \frac{1}{4\pi} D_\beta \nabla_\alpha E_\beta. \tag{8.34}$$

Now there is

$$\nabla_\alpha \left(u - \rho_\mathrm{m} \frac{\partial u}{\partial \rho_\mathrm{m}} - \frac{1}{4\pi} \mathbf{D} \cdot \mathbf{E} \right) = -\rho_\mathrm{m} \nabla_\alpha \frac{\partial u}{\partial \rho_\mathrm{m}} - \frac{1}{4\pi} D_\beta \nabla_\alpha E_\beta, \tag{8.35}$$

since $\partial u / \partial D_\beta = E_\beta / (4\pi)$. This yields the expression for the stress tensor

$$T_{\alpha,\beta} = \frac{1}{4\pi} E_\alpha D_\beta + \delta_{\alpha,\beta} \left(u - \rho_\mathrm{m} \frac{\partial u}{\partial \rho_\mathrm{m}} - \frac{1}{4\pi} \mathbf{D} \cdot \mathbf{E} \right). \tag{8.36}$$

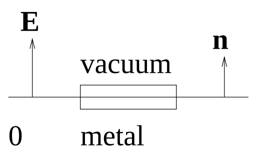
In particular with $u = u_0(\rho_\mathrm{m}) + D^2 / (8\pi\epsilon(\rho_\mathrm{m}))$, (8.6) one obtains

$$T_{\alpha,\beta} = \frac{1}{4\pi} E_\alpha D_\beta + \delta_{\alpha,\beta} \left(-P_0(\rho_\mathrm{m}) - \frac{1}{8\pi} \mathbf{D} \cdot \mathbf{E} + \frac{1}{8\pi} E^2 \rho_\mathrm{m} \frac{d\epsilon}{d\rho_\mathrm{m}} \right). \tag{8.37}$$

Maxwell's stress tensor reads in vacuum

$$T_{\alpha,\beta} = \frac{1}{4\pi} E_\alpha E_\beta - \frac{\delta_{\alpha,\beta}}{8\pi} E^2. \tag{8.38}$$

As an example we consider the electrostatic force on a plane piece of metal of area F. We have to evaluate



$$\mathbf{K} = \int d f_\beta (\mathbf{e}_\alpha T_{\alpha,\beta}) = \left(\frac{1}{4\pi} \mathbf{E}(\mathbf{E}\mathbf{n}) - \frac{1}{8\pi} \mathbf{n} E^2 \right) F = \frac{1}{8\pi} E^2 \mathbf{n} F. \tag{8.39}$$

This is in agreement with the result from (7.8).