D Law of Induction

12 FARADAY's Law of Induction

The force acting on charges is $q(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c)$. It does not matter for the charges, whether the force is exerted by the electric field or by the magnetic induction. Thus they experience in a time-dependent magnetic field an effective electric field

$$\mathbf{E}^{(\mathrm{ind})} = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \qquad (12.1)$$

with $\operatorname{curl} \mathbf{E} = -\dot{\mathbf{B}}/c$. Therefore the voltage along a loop of a conductor is given by

$$V^{(\mathrm{ind})} = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{r} + \oint (\frac{\mathbf{v}}{c} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r}. \qquad (12.2)$$

The first integral gives a contribution due to the variation of the magnetic induction. For a fixed loop and varying $\mathbf{B}$ one obtains (since $\mathbf{v} \parallel \mathrm{d}\mathbf{r}$)

$$V^{(\mathrm{ind})} = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{r} = \int \operatorname{curl} \mathbf{E} \cdot \mathrm{d}\mathbf{f} = -\frac{1}{c} \int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{f} = -\frac{1}{c} \left.\frac{\mathrm{d}\Psi^{\mathrm{m}}}{\mathrm{d}t}\right|_{\text{loop fixed}}. \qquad (12.3)$$

The second integral in (12.2) gives a contribution due to the motion of the loop. In order to investigate a loop which moves (and is distorted) we use a parameter representation of the loop $\mathbf{r} = \mathbf{r}(t, p)$ with the body-fixed parameter p. For fixed t we have $\mathrm{d}\mathbf{r} = (\partial \mathbf{r}/\partial p)\mathrm{d}p$ and

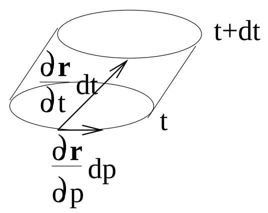
$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} + \lambda(p, t)\frac{\partial \mathbf{r}}{\partial p} \qquad (12.4)$$

with a $\lambda = \mathrm{d}p/\mathrm{d}t$ which depends on the motion of the charges in the conductor. This yields

$$\mathrm{d}t \oint \Big(\frac{\mathbf{v}}{c} \times \mathbf{B}\Big) \cdot \mathrm{d}\mathbf{r} = -\frac{1}{c} \int \Big(\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial p}\Big) \cdot \mathbf{B}\, \mathrm{d}p\, \mathrm{d}t = -\frac{1}{c} \int \mathrm{d}\mathbf{f} \cdot \mathbf{B}, \qquad (12.5)$$

since $\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial p}\, \mathrm{d}p\, \mathrm{d}t$ is the element of the area which in time $\mathrm{d}t$ is swept over by the conductor element $\mathrm{d}p$. Therefore we obtain

$$\oint (\frac{\mathbf{v}}{c} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r} = -\frac{1}{c} \left.\frac{\mathrm{d}\Psi^{m}}{\mathrm{d}t}\right|_{\mathbf{B}\,\text{fixed}}. \qquad (12.6)$$

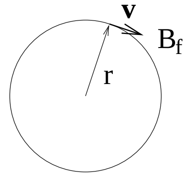


The total induced voltage is composed by the change of the magnetic flux due to the change of the magnetic induction (12.3) and by the motion of the loop (12.6)

$$V^{(\mathrm{ind})} = -\frac{1}{c}\frac{\mathrm{d}\Psi^{\mathrm{m}}}{\mathrm{d}t}, \qquad (12.7)$$

and is thus given by the total change of the magnetic flux through the loop. Thus it does not matter for a generator whether the generating magnetic field rotates or whether the coil is rotating.

The betatron (non-relativistic) The electrons move along circular orbits and are kept on these by the LORENTZ force exerted by the guide field B_{f}. Thus the centrifugal force and the LORENTZ force have to compensate each other



$$\frac{mv^2}{r} = e_0 \frac{v}{c} B_{\mathrm{f}} \quad \rightarrow \quad mv = \frac{e_0}{c} B_{\mathrm{f}} r. \qquad (12.8)$$

The electrons are accelerated by the induction

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = -e_0 E = \frac{e_0}{2\pi r}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{c}\int B\mathrm{d}f = \frac{e_0}{2\pi rc} r^2 \pi \frac{\mathrm{d}\overline{B}}{\mathrm{d}t}. \tag{12.9}$$

Here $\overline{B}$ is the averaged magnetic induction inside the circle. Thus one has

$$mv = \frac{e_0}{2}\overline{B}\frac{r}{c} = \frac{e_0}{c}B_\mathrm{f} r, \tag{12.10}$$

from which the betatron condition $B_\mathrm{f} = \overline{B}/2$ follows.

13 Inductances and Electric Circuits

13.a Inductances

The magnetic flux through a coil and a circuit $\#j$, resp. is given by

$$\Psi_j^{\rm m} = \int {\rm d}\mathbf{f}_j \cdot \mathbf{B}(\mathbf{r}_j) = \int {\rm d}\mathbf{f}_j \cdot \operatorname{curl}\mathbf{A}(\mathbf{r}_j) = \oint {\rm d}\mathbf{r}_j \cdot \mathbf{A}(\mathbf{r}_j). \tag{13.1}$$

Several circuits generate the vector-potential

$$\mathbf{A}(\mathbf{r}) = \sum_k \frac{I_k}{c} \oint \frac{{\rm d}\mathbf{r}_k}{|\mathbf{r}-\mathbf{r}_k|}. \tag{13.2}$$

Therefore the magnetic flux can be expressed by

$$\frac{1}{c}\Psi_j^{\rm m} = \sum_k L_{j,k} I_k \tag{13.3}$$

with

$$L_{j,k} = \frac{1}{c^2} \int \frac{{\rm d}\mathbf{r}_j \cdot {\rm d}\mathbf{r}_k}{|\mathbf{r}_j - \mathbf{r}_k|}. \tag{13.4}$$

Therefore one has $L_{j,k} = L_{k,j}$. For $j \neq k$ they are called mutual inductances, for $j = k$ self-inductances. In calculating the self-inductances according to (13.4) logarithmic divergencies appear, when $\mathbf{r}_j$ approaches $\mathbf{r}_k$, if the current distribution across the cross-section is not taken into account. The contributions $|\mathbf{r}_j - \mathbf{r} - k| < r_0/(2e^{1/4})$ have to be excluded from the integral, where r_0 is the radius of the circular cross-section of the wire (compare Becker-Sauter).
The dimension of the inductances is given by s^2/cm. The conversion into the SI-system is given by 1s^2/cm $\hat{=} 9 \cdot 10^{11}$Vs/A = $9 \cdot 10^{11}$ H (Henry).
If the regions in which the magnetic flux is of appreciable strength is filled with a material of permeability μ, then from $\operatorname{curl}\mathbf{H} = 4\pi\mathbf{j}_{\rm f}/c$ one obtains $\operatorname{curl}(\mathbf{B}/\mu) = 4\pi\mathbf{j}_{\rm f}/c$, so that

$$L_{j,k}^{\rm Mat} = \mu L_{j,k}^{\rm Vak}. \tag{13.5}$$

holds. Thus one obtains large inductances by cores of high permeability $\mu \approx 10^3...10^4$ in the yoke.
Inductance of a long coil If a closed magnetic yoke of length l and cross-section f is surrounded by N windings of wire, through which a current I flows, then from Ampere's law $Hl = 4\pi IN/c$ one obtains the magnetic induction $B = 4\pi IN\mu/(cl)$. The magnetic flux can then be written $Bf = cL_0NI$ with $L_0 = 4\pi\mu f/c^2l$. For N turns the magnetic flux is to be multiplied by N, which yields the self-induction $L = L_0N^2$. For mutual inductances between two circuits with N_1 and N_2 turns one obtains $L_{1,2} = L_0N_1N_2$. Thus we obtain in general

$$L_{i,j} = L_0 N_i N_j, \quad L_0 = \frac{4\pi\mu f}{c^2 l}. \tag{13.6}$$

13.b Elements of Circuits

We consider now circuits, which contain the following elements: voltage sources, ohmic resistors, inductances, and capacitors. Whereas we have already introduced inductances and capacitors, we have to say a few words on the two other elements.
Voltage sources A voltage source or electromotive force with voltage $V^{(\rm e)}(t)$ feeds the power $V^{(\rm e)}I$ into the system. An example is a battery which transforms chemical energy into electromagnetic one. The voltages $V^{(\rm ind)}$ of the inductances are also called electromotive forces.
Ohmic resistors In many materials the current density and the electric field are proportional if the field is not too strong. The coefficient of proportionality σ is called conductivity

$$\mathbf{j} = \sigma\mathbf{E}. \tag{13.7}$$

For a wire of length l and cross-section f one obtains

$$I = jf = \sigma f E = \sigma \frac{f}{l} V^{(R)}. \qquad (13.8)$$

Here $V^{(R)}$ is the OHMic voltage drop along the conductor. Thus one has

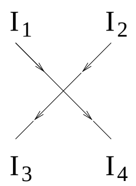
$$V^{(R)} = RI, \quad R = \frac{l}{\sigma f} \qquad (13.9)$$

with the OHMic resistance R. In GAUSSian units the conductivity σ is measured in 1/s and the resistance R in s/cm. The conversion into the SI-system is obtained by $c^{-1} \hat{=} 30\Omega$. The electromagnetic energy is dissipated in an OHMic resistor into heat at the rate $V^{(R)} I$.

13.c KIRCHHOFF's Rules

KIRCHHOFF's first Law (Current Law)

KIRCHHOFF's first law states that at each electrical contact, where several wires are joined, the sum of the incoming currents equals the sum of the outgoing currents

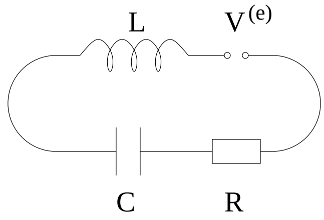


$$\sum I_{\text{incoming}} = \sum I_{\text{outgoing}}. \qquad (13.10)$$

This rule is the macroscopic form of $\operatorname{div} \mathbf{j} = 0$. In the figure aside it implies $I_1 + I_2 = I_3 + I_4$.

KIRCHHOFF's second Law (Voltage Law)

The second law says that along a closed path the sum of electromotive forces equals the sum of the other voltage drops



$$\sum \left(V^{(e)} + V^{(ind)}\right) = \sum \left(V^{(R)} + V^{(C)}\right), \qquad (13.11)$$

where

$$V^{(ind)} = -\mathrm{d}(LI)/\mathrm{d}t, \quad V^{(C)} = q/C, \quad \mathrm{d}V^{(C)}/\mathrm{d}t = I/C. \qquad (13.12)$$

This rule is FARADAY's induction law in macroscopic form.

13.d Energy of Inductances

In order to determine the energies of inductances we consider circuits with electromotive forces, OHMic resistors and inductive couplings

$$V_j^{(e)} + V_j^{(ind)} = R_j I_j. \qquad (13.13)$$

The variation of the electromagnetic energy as a function of time is then given by

$$\dot{U}_{em} = \sum_j I_j V_j^{(e)} - \sum_j R_j I_j^2 + L_{mech} = -\sum_j I_j V_j^{(ind)} + L_{mech} \qquad (13.14)$$

with

$$V_j^{(ind)} = -\frac{1}{c} \dot{\Psi}_j^m = -\frac{\mathrm{d}}{\mathrm{d}t} (\sum_k L_{j,k} I_k). \qquad (13.15)$$

Here L_{mech} is the mechanical power fed into the system.
Now we consider various cases:

13.d.α Constant Inductances

We keep the circuits fixed, then $L_{j,k}$ = const, $L_{\rm mech} = 0$ holds. From this it follows that

$$\dot{U}_{\rm em} = \sum_{j,k} I_j L_{j,k} \dot{I}_k, \tag{13.16}$$

from which we obtain the energies of the inductances

$$U_{\rm em} = \frac{1}{2} \sum_{j,k} I_j L_{j,k} I_k. \tag{13.17}$$

13.d.β Moving Loops of Currents

Now we move the circuits against each other. This yields

$$\begin{aligned} L_{\rm mech} &= \dot{U}_{\rm em} + \sum_j I_j V_j^{\rm (ind)} = \sum_{j,k} (I_j L_{j,k} \dot{I}_k + \frac{1}{2} I_j \dot{L}_{j,k} I_k) - \sum_{j,k} (I_j \dot{L}_{j,k} I_k + I_j L_{j,k} \dot{I}_k) \\ &= -\frac{1}{2} \sum_{j,k} I_j \dot{L}_{j,k} I_k = - \left. \frac{\partial U_{\rm em}}{\partial t} \right|_I . \end{aligned} \tag{13.18}$$

Thus the mechanical work to be done is not given by the change of the electromagnetic energy $U_{\rm em}$ at constant currents I, but by its negative.

13.d.γ Constant Magnetic Fluxes

In case there are no electromotive forces $V_j^{\rm (e)} = 0$ and no resistors $R_j = 0$ in the loops, then according to (13.13) we have $V^{\rm (ind)} = 0$, from which we conclude that the magnetic fluxes $\Psi_j^{\rm m}$ remain unchanged. Thus the induction tries to keep the magnetic fluxes unaltered (example superconducting loop-currents). If we express the energy $U_{\rm em}$ in terms of the fluxes

$$U_{\rm em} = \frac{1}{2c^2} \sum_{j,k} \Psi_j^{\rm m} (L^{-1})_{j,k} \Psi_k^{\rm m}, \tag{13.19}$$

and use the matrix identity $\dot{L}^{-1} = -L^{-1} \dot{L} L^{-1}$ then we obtain (the identity can be obtained by differentiating $LL^{-1} = 1$ and solving for $\dot{L}^{-1}$)

$$\left. \frac{\partial U_{\rm em}}{\partial t} \right|_{\Psi^{\rm m}} = -\frac{1}{2} \sum_{j,k} I_j \dot{L}_{j,k} I_k = L_{\rm mech}. \tag{13.20}$$

The mechanical power is thus the rate by which the electromagnetic energy changes at constant magnetic fluxes.

13.d.δ Force between two Electric Circuits

After these considerations we return to the force between two electric circuits. In section (9.e) we calculated the force from circuit 1 on circuit 2 as (9.21)

$$\mathbf{K}_2 = \frac{1}{c^2} \int {\rm d}^3 r {\rm d}^3 r' (\mathbf{j}_1(\mathbf{r}') \cdot \mathbf{j}_2(\mathbf{r})) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{13.21}$$

Now if we consider two filamentary wires

$$\mathbf{r} = \mathbf{r}_2 + \mathbf{a} \qquad \mathbf{r}' = \mathbf{r}_1 \tag{13.22}$$

$${\rm d}^3 r' \mathbf{j}_1(\mathbf{r}') \to {\rm d}\mathbf{r}_1 I_1, \qquad {\rm d}^3 r \mathbf{j}_2(\mathbf{r}) \to {\rm d}\mathbf{r}_2 I_2, \tag{13.23}$$

we obtain

$$\mathbf{K}_2 = \frac{I_1 I_2}{c^2} \int ({\rm d}\mathbf{r}_1 \cdot {\rm d}\mathbf{r}_2) \nabla_2 \frac{1}{|\mathbf{r}_2 + \mathbf{a} - \mathbf{r}_1|} = I_1 I_2 \nabla_a L_{1,2}(\mathbf{a}). \tag{13.24}$$

Thus

$$L_{\rm mech} = -\mathbf{K}_2 \cdot \dot{\mathbf{a}} = -I_1 I_2 \dot{L}_{1,2} \tag{13.25}$$

is in agreement with (13.18).

13.d.ϵ Energy of a Magnetic Dipole in an External Magnetic Induction

On the other hand we may now write the interaction energy between a magnetic dipole generated by a density of current $\mathbf{j}$ in an external magnetic field $\mathbf{B}_\mathrm{a}$ generated by a density of current $\mathbf{j}_\mathrm{a}$

$$\begin{aligned} U &= \frac{1}{c^2}\int \mathrm{d}^3r\mathrm{d}^3r'(\mathbf{j}(\mathbf{r})\cdot\mathbf{j}_\mathrm{a}(\mathbf{r}'))\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{c^2}\int \mathrm{d}^3r\mathbf{j}(\mathbf{r})\cdot\int \mathrm{d}^3r'\frac{\mathbf{j}_\mathrm{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{c}\int \mathrm{d}^3r\mathbf{j}(\mathbf{r})\cdot\mathbf{A}_\mathrm{a}(\mathbf{r}) \\ &= \frac{1}{c}\int \mathrm{d}^3r\mathbf{j}(\mathbf{r})\cdot(\mathbf{A}_\mathrm{a}(0) + x_\alpha\nabla_\alpha\mathbf{A}_\mathrm{a}|_{r=0} + \ldots) = \frac{1}{c}\int \mathrm{d}^3r x_\alpha j_\beta\nabla_\alpha A_{\mathrm{a},\beta} \\ &= \epsilon_{\alpha,\beta,\gamma}m_\gamma\nabla_\alpha A_{\mathrm{a},\beta} = \mathbf{m}\cdot\mathbf{B}_\mathrm{a}. \end{aligned} \tag{13.26}$$

This is the correct expression for the interaction energy of a magnetic dipole $\mathbf{m}$ in an external magnetic induction $\mathbf{B}_\mathrm{a}$.

13.d.ζ Permanent Magnetic Moments

Permanent magnetic moments may be considered as loop currents with large self inductance $L_{j,j}$ and constant flux Ψ^m_j. For further calculation we first solve (13.3) for I_j

$$I_j = \frac{\Psi^\mathrm{m}_j}{cL_{j,j}} - \sum_{k\neq j}\frac{L_{j,k}I_k}{L_{j,j}}. \tag{13.27}$$

Upon moving the magnetic moments the mutual inductances change, and one obtains

$$\dot{I}_j = -\frac{1}{L_{j,j}}\Big(\sum_{k\neq j}\dot{L}_{j,k}I_k + \sum_{k\neq j}L_{j,k}\dot{I}_k\Big). \tag{13.28}$$

If the self-inductances $L_{j,j}$ are very large in comparison to the mutual inductances, the currents vary only a little bit, and the second sum is negligible. Then one obtains from the self-inductance contribution of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}L_{j,j}I_j^2\right) = L_{j,j}I_j\dot{I}_j = -I_j\sum_{k\neq j}\dot{L}_{j,k}I_k. \tag{13.29}$$

Thus one obtains from a change of $L_{j,k}$ a contribution $\dot{L}_{j,k}I_jI_k$ directly from the interaction between the currents I_j and I_k, which yields a contribution of the form (13.26) to U_em and two contributions with the opposite sign from $\frac{1}{2}L_{j,j}I_j^2$ and $\frac{1}{2}L_{k,k}I_k^2$. This explains the difference between (10.24) and (13.26).