E Maxwell's Equations

14 Complete Set of Maxwell's Equations

14.a Consistency of Maxwell's Equations

In section (1) we have introduced the four M 's equations (1.13-1.16)

$$\operatorname{curl}\mathbf{B}(\mathbf{r},t) - \frac{\partial \mathbf{E}(\mathbf{r},t)}{c\partial t} = \frac{4\pi}{c}\mathbf{j}(\mathbf{r},t) \tag{14.1}$$

$$\operatorname{div}\mathbf{E}(\mathbf{r},t) = 4\pi\rho(\mathbf{r},t) \tag{14.2}$$

$$\operatorname{curl}\mathbf{E}(\mathbf{r},t) + \frac{\partial \mathbf{B}(\mathbf{r},t)}{c\partial t} = \mathbf{0} \tag{14.3}$$

$$\operatorname{div}\mathbf{B}(\mathbf{r},t) = 0. \tag{14.4}$$

These are eight component equations for six components B_α and E_α. Thus the equations cannot be independent from each other. Indeed calculating the divergence of the first equation and comparing it with the second equation we find

$$-\frac{1}{c}\operatorname{div}\dot{\mathbf{E}} = \frac{4\pi}{c}\operatorname{div}\mathbf{j} = -\frac{4\pi}{c}\dot{\rho}, \tag{14.5}$$

from which we see that the equation of continuity (1.12) is contained in both equations, and these equations can only be fulfilled if charge is conserved. But it also follows that

$$\frac{\partial}{\partial t}(\operatorname{div}\mathbf{E} - 4\pi\rho) = 0. \tag{14.6}$$

Thus if at a certain time equation (14.2) and at all times the equation of continuity is fulfilled, then equation (14.1) guarantees that (14.2) is fulfilled at all times.
Similarly, it follows from the divergence of (14.3) that

$$\frac{\partial}{\partial t}(\operatorname{div}\mathbf{B}) = 0. \tag{14.7}$$

Thus if (14.4) is fulfilled at a certain time, then due to equation (14.3) it is fulfilled at all times.
Equations (14.1) and (14.3) allow the calculation of $\mathbf{B}$ and $\mathbf{E}$ if $\mathbf{j}$ is given at all times and $\mathbf{B}$ and $\mathbf{E}$ are given at a time t_0 and (14.2) and (14.4) are fulfilled at that time. Then ρ is determined by the equation of continuity.
The only contribution we have not yet considered is the contribution proportional to $\dot{\mathbf{E}}$ in (14.1). It was found by M . He called $\dot{\mathbf{E}}/(4\pi)$ displacement current, since (14.1) may be rewritten

$$\operatorname{curl}\mathbf{B} = \frac{4\pi}{c}(\mathbf{j} + \frac{1}{4\pi}\dot{\mathbf{E}}). \tag{14.8}$$

With the introduction of this term the system of equations (14.1-14.4) became consistent. Simultaneously this system allowed the description of electromagnetic waves.

14.b Maxwell's Equations for Freely Moving Charges and Currents

The density of the charges and currents are separated into (compare sections 6.a and 11)

$$\rho = \rho_{\mathrm{f}} + \rho_{\mathrm{P}} \tag{14.9}$$

$$\mathbf{j} = \mathbf{j}_{\mathrm{f}} + \mathbf{j}_{\mathrm{P}} + \mathbf{j}_{\mathrm{M}}. \tag{14.10}$$

Here ρ_f and $\mathbf{j}_\mathrm{f}$ are the freely moving contributions, whereas ρ_P and the newly introduced $\mathbf{j}_\mathrm{P}$ are the polarization contributions. We expressed the electric dipole moment in the volume ΔV by the dipole moments $\mathbf{p}_i$, and those by the pairs of charges $\pm q_i$ at distance $\mathbf{a}_i$

$$\begin{aligned} \mathbf{P}\Delta V &= \sum \mathbf{p}_i &&= \sum q_i \mathbf{a}_i && (14.11) \\ \mathbf{j}_\mathrm{P}\Delta V &= \sum \dot{\mathbf{p}}_i &&= \sum q_i \dot{\mathbf{a}}_i && (14.12) \end{aligned}$$

with $\mathbf{j}_\mathrm{P} = \dot{\mathbf{P}}$ (in matter at rest). In addition, there is a current density from the magnetization as introduced in section 11

$$\mathbf{j}_\mathrm{M} = c \,\mathrm{curl}\, \mathbf{M}. \tag{14.13}$$

For these charge and current densities one obtains

$$\begin{aligned} \frac{\partial \rho_\mathrm{f}}{\partial t} + \mathrm{div}\, \mathbf{j}_\mathrm{f} &= 0 && (14.14) \\ \frac{\partial \rho_\mathrm{P}}{\partial t} + \mathrm{div}\, \mathbf{j}_\mathrm{P} &= 0 && (14.15) \\ \mathrm{div}\, \mathbf{j}_\mathrm{M} &= 0. && (14.16) \end{aligned}$$

By inserting these charge and current densities into (14.1) one obtains

$$\mathrm{curl}\, \mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}(\mathbf{j}_\mathrm{f} + \dot{\mathbf{P}} + c\,\mathrm{curl}\, \mathbf{M}), \tag{14.17}$$

from which it follows that

$$\mathrm{curl}\,(\mathbf{B} - 4\pi\mathbf{M}) - \frac{\partial}{c\partial t}(\mathbf{E} + 4\pi\mathbf{P}) = \frac{4\pi}{c}\mathbf{j}_\mathrm{f}. \tag{14.18}$$

If we now introduce the magnetic field $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$ and the dielectric displacement $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ in (11.9) and (6.6), eq. (11.10) becomes

$$\mathrm{curl}\, \mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} = \frac{4\pi}{c}\mathbf{j}_\mathrm{f}. \tag{14.19}$$

Similarly, one obtains from (14.2) as in (6.7)

$$\mathrm{div}\, \mathbf{D} = 4\pi\rho_\mathrm{f}. \tag{14.20}$$

M ’s equations (14.3) and (14.4) remain unchanged. Equations (14.19, 14.20) are called M ’s equations in matter.

15 Energy and Momentum Balance

15.a Energy

We consider a volume of a system with freely moving charges and matter at rest. The force density on the freely moving charges is given by $\mathbf{k} = \rho_f(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c)$. If the charges are moved with velocity $\mathbf{v}$, the power $-\int d^3r\mathbf{k} \cdot \mathbf{v} = -\int d^3r\mathbf{j}_f \cdot \mathbf{E}$ has to be fed into the system against the force density. We rewrite this expression by using (14.19), (B.30) and (14.3)

$$\begin{aligned} -\mathbf{j}_f \cdot \mathbf{E} &= -\frac{c}{4\pi}\mathbf{E} \cdot \operatorname{curl}\mathbf{H} + \frac{1}{4\pi}\mathbf{E} \cdot \dot{\mathbf{D}} = \frac{c}{4\pi}\operatorname{div}(\mathbf{E} \times \mathbf{H}) - \frac{c}{4\pi}\mathbf{H} \cdot \operatorname{curl}\mathbf{E} + \frac{1}{4\pi}\mathbf{E} \cdot \dot{\mathbf{D}} \\ &= \frac{c}{4\pi}\operatorname{div}(\mathbf{E} \times \mathbf{H}) + \frac{1}{4\pi}(\mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}}). \end{aligned} \tag{15.1}$$

These contributions are interpreted in the following way: In matter at rest the second contribution is the temporal change of the energy density $u(\rho_m, \mathbf{D}, \mathbf{B})$ with

$$du = \frac{\partial u}{\partial \rho_m}d\rho_m + \frac{1}{4\pi}\mathbf{E} \cdot d\mathbf{D} + \frac{1}{4\pi}\mathbf{H} \cdot d\mathbf{B}. \tag{15.2}$$

For simplicity we assume that the energy of the matter depends on its density ρ_m, but not on the complete state of strain. We have seen earlier that $\partial u/\partial\mathbf{D} = \mathbf{E}/(4\pi)$ holds. Similarly, one can show from the law of induction that $\partial u/\partial\mathbf{B} = \mathbf{H}/(4\pi)$ holds for rigid matter. We give a short account of the derivation

$$\begin{aligned} \delta U_{em} &= -\sum_j V_j^{(ind)}\delta t I_j = \frac{1}{c}\sum_j I_j \delta\Psi_j^m = \frac{1}{c}\sum_j I_j \int d\mathbf{f}_j \cdot \delta\mathbf{B}(\mathbf{r}) = \frac{1}{c}\sum_j I_j \int d\mathbf{r} \cdot \delta\mathbf{A}(\mathbf{r}) \\ &= \frac{1}{c}\int d^3r\mathbf{j}_f(\mathbf{r}) \cdot \delta\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r \operatorname{curl}\mathbf{H}(\mathbf{r}) \cdot \delta\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r\mathbf{H}(\mathbf{r}) \cdot \operatorname{curl}\delta\mathbf{A}(\mathbf{r}) \\ &= \frac{1}{4\pi}\int d^3r\mathbf{H}(\mathbf{r}) \cdot \delta\mathbf{B}(\mathbf{r}). \end{aligned} \tag{15.3}$$

Since the matter is pinned, $\partial u/\partial\rho_m\dot{\rho}_m$ does not contribute. Therefore we write the energy of volume V as

$$U(V) = \int_V d^3r\, u(\rho_m(\mathbf{r}), \mathbf{D}(\mathbf{r}), \mathbf{B}(\mathbf{r})) \tag{15.4}$$

and introduce the P vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \tag{15.5}$$

Then one has

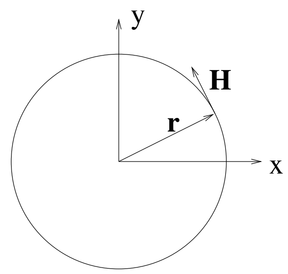
$$-\int_V d^3r\mathbf{j}_f \cdot \mathbf{E} = \dot{U}(V) + \int_V d^3r \operatorname{div}\mathbf{S} = \dot{U}(V) + \int_{\partial V} d\mathbf{f} \cdot \mathbf{S}(\mathbf{r}). \tag{15.6}$$

The energy added to volume V is partially stored in the volume. This stored part is given by ($\dot{U}$). Another part is transported through the surface of the system. This transport of energy is given by the energy current through the surface expressed by the surface integral over $\mathbf{S}$. Similar to the transport of the charge $\int d\mathbf{f} \cdot \mathbf{j}_f$ through a surface per unit time, one has (in matter at rest) the energy transport $\int d\mathbf{f} \cdot \mathbf{S}$ through a surface. Thus the P vector is the density of the electromagnetic energy current.
We note that for $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$ one obtains the energy density

$$u = u^0(\rho_m) + \frac{1}{8\pi}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}). \tag{15.7}$$

Example: Current-carrying straight wire

We consider a straight wire which carries the current I in the direction of the z-axis. Due to A 's law the integral along a concentric circle with radius r around the conductor yields



$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{r} = \frac{4\pi}{c} I, \quad \mathbf{H} = \frac{2I}{cr} \mathbf{e}_\phi. \tag{15.8}$$

There is a voltage drop along the wire due to the ic resistance $V^{(\mathrm{R})}$, which is related to the electric field parallel to the wire, $\mathbf{E} = E_0 \mathbf{e}_z$. This yields the P vector

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} = -\frac{IE_0 \mathbf{e}_\mathrm{r}}{2\pi r} \tag{15.9}$$

with the energy flux

$$\int \mathbf{S} \cdot \mathrm{d}\mathbf{f} = -IE_0 l = -IV^{(\mathrm{R})} \tag{15.10}$$

through the lateral surface of the cylinder of the wire of length l in outward direction. In other words, the power $IV^{(\mathrm{R})}$ flows into the wire. There it is transformed into heat.

15.b Momentum Balance

We consider the momentum balance only for the vacuum with charge densities ρ and current densities $\mathbf{j}$. If we keep the system at rest, a force density $-\mathbf{k}$ has to act against the L force density $\mathbf{k} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B}/c$, so that the momentum $-\int_V \mathrm{d}^3 r\, \mathbf{k}$ is added to the volume V per unit time. We transform by means of (14.1) and (14.3)

$$-\mathbf{k} = -\rho\mathbf{E} - \frac{1}{c}\mathbf{j} \times \mathbf{B} = -\frac{1}{4\pi}\mathbf{E}\,\mathrm{div}\,\mathbf{E} + \frac{1}{4\pi}\mathbf{B} \times \mathrm{curl}\,\mathbf{B} + \frac{1}{4\pi c}\dot{\mathbf{E}} \times \mathbf{B}. \tag{15.11}$$

With (14.3) and (14.4)

$$\begin{aligned} \dot{\mathbf{E}} \times \mathbf{B} &= (\mathbf{E} \times \mathbf{B})^{\cdot} - \mathbf{E} \times \dot{\mathbf{B}} = (\mathbf{E} \times \mathbf{B})^{\cdot} + c\mathbf{E} \times \mathrm{curl}\,\mathbf{E} && (15.12) \\ \mathbf{B}\,\mathrm{div}\,\mathbf{B} &= 0 && (15.13) \end{aligned}$$

one obtains

$$-\mathbf{k} = \frac{1}{4\pi c}(\mathbf{E} \times \mathbf{B})^{\cdot} + \frac{1}{4\pi}(\mathbf{E} \times \mathrm{curl}\,\mathbf{E} - \mathbf{E}\,\mathrm{div}\,\mathbf{E} + \mathbf{B} \times \mathrm{curl}\,\mathbf{B} - \mathbf{B}\,\mathrm{div}\,\mathbf{B}). \tag{15.14}$$

One has

$$\mathbf{E}_\mathrm{c} \times (\nabla \times \mathbf{E}) - \mathbf{E}_\mathrm{c}(\nabla\mathbf{E}) = \nabla(\mathbf{E} \cdot \mathbf{E}_\mathrm{c}) - \mathbf{E}(\nabla\mathbf{E}_\mathrm{c}) - \mathbf{E}_\mathrm{c}(\nabla \cdot \mathbf{E}) = \frac{1}{2}\nabla E^2 - (\nabla\mathbf{E})\mathbf{E}. \tag{15.15}$$

We have indicated quantities on which the ∇-operator does not act with an index $_\mathrm{c}$. The ∇-operator acts on both factors $\mathbf{E}$ in the last term of the expression above. Then we may write

$$-\mathbf{k} = \frac{\partial}{\partial t}\mathbf{g}_\mathrm{s} - \nabla_\beta T_{\alpha,\beta}\mathbf{e}_\alpha, \tag{15.16}$$

with

$$\begin{aligned} \mathbf{g}_\mathrm{s} &= \frac{1}{4\pi c}\mathbf{E} \times \mathbf{B}, && (15.17) \\ T_{\alpha,\beta} &= \frac{1}{4\pi}(E_\alpha E_\beta + B_\alpha B_\beta) - \frac{\delta_{\alpha,\beta}}{8\pi}(E^2 + B^2). && (15.18) \end{aligned}$$

Here $\mathbf{g}_\mathrm{s}$ is called the density of the electromagnetic momentum and $T_{\alpha,\beta}$ are the components of the electromagnetic stress tensor, whose electrostatic part (8.38) we already know. With these quantities we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}^3 r\, \mathbf{g}_\mathrm{s}(\mathbf{r}) = -\int_V \mathrm{d}^3 r\, \mathbf{k} + \int_{\partial V} \mathbf{e}_\alpha T_{\alpha,\beta}\mathrm{d}f_\beta. \tag{15.19}$$

This is the momentum balance for the volume V. The left handside gives the rate of change of momentum in the volume V, the right handside the rate of momentum added to the volume. It consists of two contributions: the first one is the momentum which is added by the action of the reactive force against the L force density $\mathbf{k}$. The second contribution acts by means of stress on the surface. It may also be considered as a flux of momentum through the surface. Thus the stress tensor is apart from its sign the density of momentum flux. It carries two indices. One (α) relates to the components of momentum, the other one (β) to the direction of the flux.

We have only considered the electromagnetic momentum in vacuum, whereas we have considered the electromagnetic energy also in matter. Why is it more difficult to determine momentum in matter? In both cases we consider the system at rest. If one pins the matter, the acting forces do not contribute to the balance of energy, since the power is given by force times velocity. Since velocity vanishes, the forces acting on the matter do not contribute to the balance of the energy. This is different for the balance of momentum. There all forces contribute. One could imagine starting out from a force-free state. Then, however, we have the problem that by moving the free charges, forces will appear which we would have to know. Therefore we can consider here the energy balance in matter, whereas the momentum balance in matter would be more difficult.

In literature there are inconsistent statements: In 1908 M gave $\mathbf{D}\times\mathbf{B}/(4\pi c)$ for the electromagnetic momentum density in matter. This can also be found in the book by S (however with words of caution). On the other hand in 1910 A gave $\mathbf{E}\times\mathbf{H}/(4\pi c)$. This is also found in the textbook by L and L .

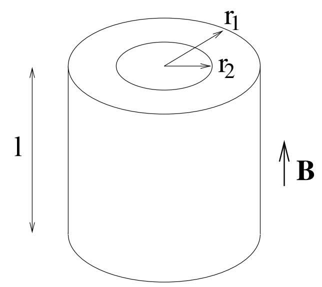
There are two points to be considered, which are often overlooked:

i) The interaction between the electromagnetic field and matter has to be taken into account. Matter cannot be considered rigid.

ii) One has to define precisely what is meant by the electromagnetic momentum, since otherwise any difference can be attributed to the mechanical momentum and the statement is empty. Without derivation it should just be mentioned that a model system can be given which yields the following: The momentum density in the local rest system is $\mathbf{E}\times\mathbf{H}/(4\pi c) = \mathbf{S}/c^2$. However, in homogeneous matter there is a further conserved quantity which in the local rest system is given by $\mathbf{D}\times\mathbf{B}/(4\pi c)$. If one goes through S 's argument, one realizes that it can be carried through only for a space-independent dielectric constant ϵ.

Example: Cylindric capacitor in a magnetic field

We consider a cylindric capacitor of length l with outer radius r_1 and inner radius r_2 with charge q outside and $-q$ inside. We assume that between both cylinders is vacuum. Parallel to the axis be a magnetic field B_0. Then one has in cylinder coordinates



$$\mathbf{E} = -\frac{2q}{lr}\mathbf{e}_\mathrm{r}, \quad \mathbf{B} = B_0\mathbf{e}_z, \quad \mathbf{g}_\mathrm{s} = \frac{1}{4\pi c}\frac{2qB_0}{lr}\mathbf{e}_\phi. \tag{15.20}$$

From this we calculate the angular momentum $\mathbf{L}$ in z-direction

$$L_z = \int \mathrm{d}z\mathrm{d}^2r(\mathbf{r}\times\mathbf{g}_\mathrm{s})_z = \int \mathrm{d}z\mathrm{d}^2r r\frac{2qB_0}{4\pi clr} = \frac{qB_0}{2c}(r_1^2 - r_2^2). \tag{15.21}$$

If the capacitor is decharged, the decharging current flows through the magnetic field. Then the L force acts which gives the system a mechanical torque $\mathbf{M}_\mathrm{mech}$

$$\mathbf{M}_\mathrm{mech} = \int \mathrm{d}^3r\,\mathbf{r}\times(\frac{1}{c}\mathbf{j}\times\mathbf{B}) = \frac{I}{c}\int \mathbf{r}\times(\mathrm{d}\mathbf{r}\times\mathbf{B}) = \frac{I}{c}\int((\mathbf{r}\cdot\mathbf{B})\mathrm{d}\mathbf{r} - (\mathbf{r}\cdot\mathrm{d}\mathbf{r})\mathbf{B}), \tag{15.22}$$

from which one obtains

$$M_{\mathrm{mech},z} = -\frac{IB_0}{c}\int_{r_1}^{r_2} r dr = \frac{IB_0}{2c}(r_1^2 - r_2^2) \tag{15.23}$$

and thus the mechanical angular momentum

$$L_z = \frac{qB_0}{2c}(r_1^2 - r_2^2). \tag{15.24}$$

Thus the electromagnetic angular momentum (15.21) is transformed into a mechanical angular momentum during decharging. Instead of decharging the capacitor one may switch off the magnetic field. Then the electric field

$$\oint \mathbf{E}^{(\mathrm{ind})} \cdot \mathrm{d}\mathbf{r} = -\frac{1}{c} \int \dot{\mathbf{B}} \cdot \mathrm{d}\mathbf{f} = -\frac{1}{c} \pi r^2 \dot{B}_0, \quad \mathbf{E}^{(\mathrm{ind})} = -\frac{1}{2c} r \dot{B}_0 \mathbf{e}_\phi \tag{15.25}$$

is induced, which exerts the torque

$$\mathbf{M}_{\mathrm{mech}} = q\mathbf{r}_1 \times \mathbf{E}^{(\mathrm{ind})}(\mathbf{r}_1) - q\mathbf{r}_2 \times \mathbf{E}^{(\mathrm{ind})}(\mathbf{r}_2) \tag{15.26}$$

$$M_{\mathrm{mech},z} = qr_1(-\frac{1}{2c} r_1 \dot{B}_0) - qr_2(-\frac{1}{2c} r_2 \dot{B}_0) \tag{15.27}$$

so that the capacitor receives the mechanical component of the angular momentum

$$L_z = \frac{qB_0}{2c}(r_1^2 - r_2^2). \tag{15.28}$$

In both cases the electromagnetic angular momentum is transformed into a mechanical one.