F Electromagnetic Waves

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16 Electromagnetic Waves in Vacuum and in Homogeneous Isotropic Insulators

16.a Wave Equation

We consider electromagnetic waves in a homogeneous isotropic insulator including the vacuum. More precisely we require that the dielectric constant ϵ and the permeability μ are independent of space and time. Further we require that there are no freely moving currents and charges $\rho_{\mathrm{f}} = 0$, $\mathbf{j}_{\mathrm{f}} = \mathbf{0}$. Thus the matter is an insulator. Then MAXWELL's equations read, expressed in terms of $\mathbf{E}$ and $\mathbf{H}$ by means of $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$

$$\operatorname{div}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{H} = 0, \tag{16.1}$$

$$\operatorname{curl}\mathbf{H} = \frac{\epsilon}{c}\dot{\mathbf{E}}, \qquad \operatorname{curl}\mathbf{E} = -\frac{\mu}{c}\dot{\mathbf{H}}. \tag{16.2}$$

From these equations one obtains

$$\operatorname{curl}\operatorname{curl}\mathbf{H} = \frac{\epsilon}{c}\operatorname{curl}\dot{\mathbf{E}} = -\frac{\epsilon\mu}{c^2}\ddot{\mathbf{H}} \tag{16.3}$$

With

$$\operatorname{curl}\operatorname{curl}\mathbf{H} = \nabla\times(\nabla\times\mathbf{H}) = -\triangle\mathbf{H} + \nabla(\nabla\cdot\mathbf{H}) \tag{16.4}$$

one obtains for $\mathbf{H}$ using (16.1) and similarly for $\mathbf{E}$

$$\triangle\mathbf{H} = \frac{1}{c'^2}\ddot{\mathbf{H}}, \tag{16.5}$$

$$\triangle\mathbf{E} = \frac{1}{c'^2}\ddot{\mathbf{E}}, \tag{16.6}$$

$$c' = \frac{c}{\sqrt{\epsilon\mu}}. \tag{16.7}$$

The equations (16.5) and (16.6) are called wave equations.

16.b Plane Waves

Now we look for particular solutions of the wave equations and begin with solutions which depend only on z and t, $\mathbf{E} = \mathbf{E}(z,t)$, $\mathbf{H} = \mathbf{H}(z,t)$. One obtains for the z-components

$$\operatorname{div}\mathbf{E} = 0 \quad\rightarrow\quad \frac{\partial E_z}{\partial z} = 0 \tag{16.8}$$

$$(\operatorname{curl}\mathbf{H})_z = 0 = \frac{\epsilon}{c}\dot{E}_z \quad\rightarrow\quad \frac{\partial E_z}{\partial t} = 0. \tag{16.9}$$

Thus only a static homogeneous field is possible with this ansatz in z-direction, i.e. a constant field E_z. The same is true for H_z. We already see that electromagnetic waves are transversal waves.
For the x- and y-components one obtains

$$(\nabla\times\mathbf{H})_x = \frac{\epsilon}{c}\dot{E}_x \;\rightarrow\; -\nabla_z H_y = \frac{\epsilon}{c}\dot{E}_x \;\rightarrow\; -\nabla_z(\sqrt{\mu}H_y) = \frac{1}{c'}(\sqrt{\epsilon}E_x)^{\cdot} \tag{16.10}$$

$$(\nabla\times\mathbf{E})_y = -\frac{\mu}{c}\dot{H}_y \;\rightarrow\; \nabla_z E_x = -\frac{\mu}{c}\dot{H}_y \;\rightarrow\; \nabla_z(\sqrt{\epsilon}E_x) = -\frac{1}{c'}(\sqrt{\mu}H_y)^{\cdot}. \tag{16.11}$$

E_x is connected with H_y, and in the same way E_y with $-H_x$. We may combine the equations (16.10) and (16.11)

$$\frac{\partial}{\partial t}(\sqrt{\epsilon}E_x \pm \sqrt{\mu}H_y) = \mp c' \frac{\partial}{\partial z}(\sqrt{\epsilon}E_x \pm \sqrt{\mu}H_y). \tag{16.12}$$

The solution of this equation and the corresponding one for E_y with $-H_x$ is

$$\sqrt{\epsilon}E_x \pm \sqrt{\mu}H_y = 2f_\pm(z \mp c't), \tag{16.13}$$
$$\sqrt{\epsilon}E_y \mp \sqrt{\mu}H_x = 2g_\pm(z \mp c't), \tag{16.14}$$

with arbitrary (differentiable) functions $f_\pm$ and $g_\pm$, from which one obtains

$$\sqrt{\epsilon}E_x = f_+(z - c't) + f_-(z + c't) \tag{16.15}$$
$$\sqrt{\mu}H_y = f_+(z - c't) - f_-(z + c't) \tag{16.16}$$
$$\sqrt{\epsilon}E_y = g_+(z - c't) + g_-(z + c't) \tag{16.17}$$
$$\sqrt{\mu}H_x = -g_+(z - c't) + g_-(z + c't). \tag{16.18}$$

This is the superposition of waves of arbitrary shapes, which propagate upward (f_+, g_+) and downward (f_-, g_-), resp, with velocity c'. Thus $c' = c/\sqrt{\epsilon\mu}$ is the velocity of propagation of the electromagnetic wave (light) in the corresponding medium. In particular we find that c is the light velocity in vacuum.
We calculate the density of energy

$$u = \frac{1}{8\pi}(\epsilon E^2 + \mu H^2) = \frac{1}{4\pi}(f_+^2 + g_+^2 + f_-^2 + g_-^2) \tag{16.19}$$

and the density of the energy current by means of the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} = \frac{c'\mathbf{e}_z}{4\pi}(f_+^2 + g_+^2 - f_-^2 - g_-^2), \tag{16.20}$$

where a homogeneous field in z-direction is not considered. Comparing the expressions for u and $\mathbf{S}$ separately for the waves moving up and down, one observes that the energy of the wave is transported with velocity $\pm c'\mathbf{e}_z$, since $\mathbf{S} = \pm c'\mathbf{e}_z u$. We remark that the wave which obeys $E_y = 0$ and $H_x = 0$, that is $g_\pm = 0$, is called linearly polarized in x-direction. For the notation of the direction of polarization one always considers that of the vector $\mathbf{E}$.

16.c Superposition of Plane Periodic Waves

In general one may describe the electric field in terms of a Fourier integral

$$\mathbf{E}(\mathbf{r}, t) = \int d^3k d\omega \mathbf{E}_0(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{16.21}$$

analogously for $\mathbf{H}$. Then the fields are expressed as a superposition of plane periodic waves.

16.c.α Insertion on Fourier Series and Integrals

The Fourier series of a function with period L, $f(x + L) = f(x)$ reads

$$f(x) = \hat{c} \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n x/L}. \tag{16.22}$$

f_n are the Fourier coefficients of f. This representation is possible for square integrable functions with a finite number of points of discontinuity. $\hat{c}$ is an appropriate constant. The back-transformation, that is the calculation of the Fourier coefficients is obtained from

$$\int_{-L/2}^{L/2} dx e^{-2\pi i n x/L} f(x) = \hat{c} L f_n, \tag{16.23}$$

as can be seen easily by inserting in (16.22) and exchanging summation and integration. The Fourier transform for a (normally not-periodic) function defined from $-\infty$ to $+\infty$ can be obtained by performing the limit $L \to \infty$ and introducing

$$k := \frac{2\pi n}{L}, \quad f_n = f_0(k), \quad \hat{c} = \Delta k = \frac{2\pi}{L}. \tag{16.24}$$

Then (16.22) transforms into

$$f(x) = \sum \Delta k f_0(k) e^{ikx} \to \int_{-\infty}^{\infty} dk f_0(k) e^{ikx} \tag{16.25}$$

and the back-transformation (16.23) into

$$\int_{-\infty}^{\infty} dx f(x) e^{-ikx} = 2\pi f_0(k). \tag{16.26}$$

This allows us, e.g., to give the back-transformation from (16.21) to

$$\mathbf{E}_0(\mathbf{k}, \omega) = \frac{1}{(2\pi)^4} \int d^3 r dt e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathbf{E}(\mathbf{r}, t). \tag{16.27}$$

16.c.β Back to Maxwell's Equations

The representation by the Fourier transform has the advantage that the equations become simpler. Applying the operations ∇ and $\partial/\partial t$ on the exponential function

$$\nabla e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = i\mathbf{k} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad \frac{\partial}{\partial t} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = -i\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{16.28}$$

in Maxwell's equations yields for the Fourier components

$$\nabla \cdot \mathbf{E} = 0 \quad \to \quad i\mathbf{k} \cdot \mathbf{E}_0(\mathbf{k}, \omega) = 0 \tag{16.29}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \to \quad i\mathbf{k} \cdot \mathbf{H}_0(\mathbf{k}, \omega) = 0 \tag{16.30}$$

$$\nabla \times \mathbf{H} = \frac{\epsilon}{c} \dot{\mathbf{E}} \quad \to \quad i\mathbf{k} \times \mathbf{H}_0(\mathbf{k}, \omega) = -i\frac{\epsilon}{c} \omega \mathbf{E}_0(\mathbf{k}, \omega) \tag{16.31}$$

$$\nabla \times \mathbf{E} = -\frac{\mu}{c} \dot{\mathbf{H}} \quad \to \quad i\mathbf{k} \times \mathbf{E}_0(\mathbf{k}, \omega) = i\frac{\mu}{c} \omega \mathbf{H}_0(\mathbf{k}, \omega). \tag{16.32}$$

The advantage of this representation is that only Fourier components with the same $\mathbf{k}$ and ω are connected to each other. For $\mathbf{k} = \mathbf{0}$ one obtains $\omega = 0$, where $\mathbf{E}_0$, $\mathbf{H}_0$ are arbitrary. These are the static homogeneous fields. For $\mathbf{k} \neq \mathbf{0}$ one obtains from (16.29) and (16.30)

$$\mathbf{E}_0(\mathbf{k}, \omega) \perp \mathbf{k}, \quad \mathbf{H}_0(\mathbf{k}, \omega) \perp \mathbf{k}. \tag{16.33}$$

From the two other equations (16.31) and (16.32) one obtains

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0(\mathbf{k}, \omega)) = \frac{\mu}{c} \omega \mathbf{k} \times \mathbf{H}_0(\mathbf{k}, \omega) = -\frac{\epsilon\mu}{c^2} \omega^2 \mathbf{E}_0(\mathbf{k}, \omega). \tag{16.34}$$

From this one obtains

$$\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_0(\mathbf{k}, \omega)) - k^2 \mathbf{E}_0(\mathbf{k}, \omega) = -\frac{1}{c'^2} \omega^2 \mathbf{E}_0(\mathbf{k}, \omega), \tag{16.35}$$

analogously for $\mathbf{H}_0$. The first term on the left hand-side of (16.35) vanishes because of (16.29). Thus there are non-vanishing solutions, if the condition $\omega = \pm c' k$ is fulfilled. This is the dispersion relation for electromagnetic waves that is the relation between frequency and wave-vector for electromagnetic waves. Taking these conditions into account we may write

$$\mathbf{E}_0(\mathbf{k}, \omega) = \frac{1}{2} \delta(\omega - c'k) \mathbf{E}_1(\mathbf{k}) + \frac{1}{2} \delta(\omega + c'k) \mathbf{E}_2(\mathbf{k}). \tag{16.36}$$

and thus

$$\mathbf{E}(\mathbf{r},t) = \int d^3k \left(\frac{1}{2}\mathbf{E}_1(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-c'kt)} + \frac{1}{2}\mathbf{E}_2(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}+c'kt)}\right). \tag{16.37}$$

Since the electric field has to be real, it must coincide with its conjugate complex.

$$\begin{aligned}\mathbf{E}^*(\mathbf{r},t) &= \int d^3k \left(\frac{1}{2}\mathbf{E}_1^*(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-c'kt)} + \frac{1}{2}\mathbf{E}_2^*(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}+c'kt)}\right)\\ &= \int d^3k \left(\frac{1}{2}\mathbf{E}_1^*(-\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}+c'kt)} + \frac{1}{2}\mathbf{E}_2^*(-\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-c'kt)}\right).\end{aligned} \tag{16.38}$$

From comparison of the coefficients one obtains

$$\mathbf{E}_2^*(\mathbf{k}) = \mathbf{E}_1(-\mathbf{k}). \tag{16.39}$$

Thus we obtain

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \int d^3k \left(\frac{1}{2}\mathbf{E}_1(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-c'kt)} + \frac{1}{2}\mathbf{E}_1^*(-\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}+c'kt)}\right)\\ &= \int d^3k \left(\frac{1}{2}\mathbf{E}_1(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-c'kt)} + \frac{1}{2}\mathbf{E}_1^*(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-c'kt)}\right)\\ &= \Re\Big(\int d^3k\mathbf{E}_1(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-c'kt)}\Big).\end{aligned} \tag{16.40}$$

Eq. (16.32) yields for $\mathbf{H}_0$

$$\mathbf{H}_0(\mathbf{k},\omega) = \frac{c}{\mu\omega}\mathbf{k}\times\mathbf{E}_0(\mathbf{k},\omega) = \sqrt{\frac{\epsilon}{\mu}}\left(\delta(\omega-c'k)\frac{\mathbf{k}}{2k}\times\mathbf{E}_1(\mathbf{k}) - \delta(\omega+c'k)\frac{\mathbf{k}}{2k}\times\mathbf{E}_2(\mathbf{k})\right) \tag{16.41}$$

and thus for $\mathbf{H}$

$$\mathbf{H}(\mathbf{r},t) = \Re\Big(\int d^3k\sqrt{\frac{\epsilon}{\mu}}\frac{\mathbf{k}}{k}\times\mathbf{E}_1(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-c'kt)}\Big). \tag{16.42}$$

If only one Fourier component $\mathbf{E}_1(\mathbf{k}) = \delta^3(\mathbf{k}-\mathbf{k}_0)\mathbf{E}_{1,0}$ (idealization) contributes then one has a monochromatic wave

$$\mathbf{E}(\mathbf{r},t) = \Re(\mathbf{E}_{1,0}e^{i(\mathbf{k}_0\cdot\mathbf{r}-c'k_0t)}) \tag{16.43}$$

$$\mathbf{H}(\mathbf{r},t) = \sqrt{\frac{\epsilon}{\mu}}\Re\Big(\frac{\mathbf{k}_0}{k_0}\times\mathbf{E}_{1,0}e^{i(\mathbf{k}_0\cdot\mathbf{r}-c'k_0t)}\Big). \tag{16.44}$$

The wave (light) is called linearly polarized, if $\mathbf{E}_{1,0} = \mathbf{e}_1 E_{1,0}$ with a real unit-vector $\mathbf{e}_1$, it is called circularly polarized if $\mathbf{E}_{1,0} = (\mathbf{e}_1 \mp i\mathbf{e}_2)E_{1,0}/\sqrt{2}$ with real unit vectors $\mathbf{e}_1$ und $\mathbf{e}_2$, where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{k}_0$ form an orthogonal right-handed basis. The upper sign applies for a right-, the lower for a left-polarized wave.

16.c.γ Time averages and time integrals

Energy-density and Poynting vector are quantities bilinear in the fields. In case of a monochromatic wave as in (16.43) and (16.44) these quantities oscillate. One is often interested in the averages of these quantities. Thus if we have two quantities

$$a = \Re\Big(a_0e^{-i\omega t}\Big), \quad b = \Re\Big(b_0e^{-i\omega t}\Big), \tag{16.45}$$

then one has

$$ab = \frac{1}{4}a_0b_0e^{-2i\omega t} + \frac{1}{4}(a_0b_0^* + a_0^*b_0) + \frac{1}{4}a_0^*b_0^*e^{2i\omega t}. \tag{16.46}$$

The first and the last term oscillate (we assume $\omega \neq 0$). They cancel in the time average. Thus one obtains in the time average

$$\overline{ab} = \frac{1}{4}(a_0b_0^* + a_0^*b_0) = \frac{1}{2}\Re(a_0^*b_0). \tag{16.47}$$

Please note that a_0 and b_0 are in general complex and that the time average depends essentially on the relative phase between both quantities and not only on the moduli $|a_0|$ and $|b_0|$.
If a and b are given by Fourier integrals

$$a(t) = \Re\Big(\int d\omega a_0(\omega)e^{-i\omega t}\Big) \tag{16.48}$$

and analogously for $b(t)$, then often the time integrals of these quantities and their products over all times will be finite. For this purpose the time integral $\int_{-\infty}^{\infty} dt e^{-i\omega t}$ has to be determined. This integral is not well defined. In practice it has often to be multiplied with a function continuous in ω. Thus it is sufficient to find out how the time-integral of this frequency-integral behaves. For this purpose we go back to the insertion on Fourier series with x und k and find that

$$\int_{-L/2}^{L/2} dx e^{-2\pi i n x/L} = L\delta_{0,n}, \tag{16.49}$$

thus

$$\sum_{n=n_-}^{n_+} \int_{-L/2}^{L/2} dx e^{-2\pi i n x/L} = L, \tag{16.50}$$

if $n_- \le 0$ and $n_+ \ge 0$. Otherwise the sum vanishes. Now we perform again the limit $L \to \infty$ and obtain

$$\sum_{k_-}^{k_+} \Delta k \int_{-L/2}^{L/2} dx e^{-ikx} = \Delta k L \to \int_{k_-}^{k_+} dk \int_{-\infty}^{\infty} dx e^{-ikx} = 2\pi, \tag{16.51}$$

if k_- is negative and k_+ positive, otherwise it vanishes. Thus we obtain

$$\int_{-\infty}^{\infty} dx e^{-ikx} = 2\pi\delta(k). \tag{16.52}$$

With this result we obtain

$$\int_{-\infty}^{\infty} dt a(t)b(t) = \frac{\pi}{2}\int_{-\infty}^{\infty} d\omega\Big(a_0(\omega) + a_0^*(-\omega))(b_0(-\omega) + b_0^*(\omega)\Big). \tag{16.53}$$

If there are only positive frequencies ω under the integral then one obtains

$$\int_{-\infty}^{\infty} dt a(t)b(t) = \frac{\pi}{2}\int_0^{\infty} d\omega\Big(a_0(\omega)b_0^*(\omega) + a_0^*(\omega)b_0(\omega)\Big) = \pi\Re\Big(\int_0^{\infty} d\omega a_0^*(\omega)b_0(\omega)\Big). \tag{16.54}$$

17 Electromagnetic Waves in Homogeneous Conductors

17.a Transverse Oscillations at Low Frequencies

We investigate the transverse oscillations in a homogeneous conductor. We put $\mu = 1$. From

$$\mathbf{j}_{\mathrm{f}} = \sigma \mathbf{E} \tag{17.1}$$

one obtains

$$\mathrm{curl}\,\mathbf{B} - \frac{1}{c}\epsilon\dot{\mathbf{E}} = \frac{4\pi}{c}\sigma\mathbf{E}. \tag{17.2}$$

For periodic fields of frequency ω,

$$\mathbf{E} = \mathbf{E}_0(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}, \quad \mathbf{B} = \mathbf{B}_0(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}, \tag{17.3}$$

and similarly for ρ_{f} and $\mathbf{j}_{\mathrm{f}}$, one obtains

$$\mathrm{curl}\,\mathbf{B}_0 + (\frac{\mathrm{i}\omega}{c}\epsilon - \frac{4\pi}{c}\sigma)\mathbf{E}_0 = \mathbf{0}. \tag{17.4}$$

This can be written

$$\mathrm{curl}\,\mathbf{B}_0 + \frac{\mathrm{i}\omega}{c}\epsilon(\omega)\mathbf{E}_0 = \mathbf{0}, \quad \epsilon(\omega) = \epsilon - \frac{4\pi\sigma}{\mathrm{i}\omega}. \tag{17.5}$$

From the equation of continuity

$$\dot{\rho}_{\mathrm{f}} + \mathrm{div}\,\mathbf{j}_{\mathrm{f}} = 0 \tag{17.6}$$

one obtains

$$-\mathrm{i}\omega\rho_{\mathrm{f},0} + \mathrm{div}\,\mathbf{j}_{\mathrm{f},0} = 0 \tag{17.7}$$

and thus

$$\mathrm{div}\,\mathbf{D}_0 = 4\pi\rho_{\mathrm{f},0} = \frac{4\pi}{\mathrm{i}\omega}\mathrm{div}\,\mathbf{j}_{\mathrm{f},0} = \frac{4\pi\sigma}{\mathrm{i}\omega}\mathrm{div}\,\mathbf{E}_0. \tag{17.8}$$

Thus we have

$$\epsilon(\omega)\,\mathrm{div}\,\mathbf{E}_0 = 0 \tag{17.9}$$

because of $\mathrm{div}\,\mathbf{D}_0 = \epsilon\,\mathrm{div}\,\mathbf{E}_0$. We may thus transfer our results from insulators to conductors, if we replace ϵ by $\epsilon(\omega)$. Thus we obtain

$$k^2 = \epsilon(\omega)\frac{\omega^2}{c^2}. \tag{17.10}$$

Since $\epsilon(\omega)$ is complex, one obtains for real ω a complex wave-vector $\mathbf{k}$. We put

$$\sqrt{\epsilon(\omega)} = n + \mathrm{i}\kappa, \quad k = \frac{\omega}{c}(n + \mathrm{i}\kappa) \tag{17.11}$$

and obtain a damped wave with

$$\mathrm{e}^{\mathrm{i}kz} = \mathrm{e}^{\mathrm{i}\omega nz/c - \omega\kappa z/c}. \tag{17.12}$$

For the fields we obtain

$$\mathbf{E} = \Re(\mathbf{E}_0\mathrm{e}^{\mathrm{i}\omega(nz/c-t)})\mathrm{e}^{-\omega\kappa z/c}, \tag{17.13}$$

$$\mathbf{B} = \Re(\sqrt{\epsilon(\omega)}\mathbf{e}_z \times \mathbf{E}_0\mathrm{e}^{\mathrm{i}\omega(nz/c-t)})\mathrm{e}^{-\omega\kappa z/c}. \tag{17.14}$$

The amplitude decays in a distance $d = \frac{c}{\omega\kappa}$ by a factor 1/e. This distance is called penetration depth or skin depth. For small frequencies one can approximate

$$\sqrt{\epsilon(\omega)} \approx \sqrt{-\frac{4\pi\sigma}{\mathrm{i}\omega}} = (1+\mathrm{i})\sqrt{\frac{2\pi\sigma}{\omega}}, \quad n = \kappa = \sqrt{\frac{2\pi\sigma}{\omega}}, \quad d = \frac{c}{\sqrt{2\pi\sigma\omega}}. \tag{17.15}$$

For copper one has $\sigma = 5.8 \cdot 10^{17}\ \mathrm{s}^{-1}$, for $\omega = 2\pi \cdot 50\ \mathrm{s}^{-1}$ one obtains $d = 9$ mm. This effect is called the skin-effect. The alternating current decays exponentially inside the conductor. For larger frequencies the decay is more rapidly.

17.b Transverse Oscillations at High Frequencies

In reality ϵ and σ depend on ω. We will now consider the frequency dependence of the conductivity within a simple model and start out from the equation of motion of a charge (for example an electron in a metal)

$$m_0\ddot{\mathbf{r}} = e_0\mathbf{E} - \frac{m_0}{\tau}\dot{\mathbf{r}}, \tag{17.16}$$

where m_0 and e_0 are mass and charge of the carrier. The last term is a friction term which takes the collisions with other particles in a rough way into account. There τ is the relaxation time, which describes how fast the velocity decays in the absense of an electric field. One obtains with $\mathbf{j}_\mathrm{f} = \rho_\mathrm{f}\dot{\mathbf{r}} = n_0 e_0\dot{\mathbf{r}}$, where n_0 is the density of the freely moving carriers

$$\frac{m_0}{n_0 e_0}\frac{\partial \mathbf{j}_\mathrm{f}}{\partial t} = e_0\mathbf{E} - \frac{m_0}{n_0\tau e_0}\mathbf{j}_\mathrm{f}. \tag{17.17}$$

In the stationary case $\partial\mathbf{j}_\mathrm{f}/\partial t = \mathbf{0}$ one obtains the static conductivity $\sigma_0 = \frac{n_0\tau e_0^2}{m_0}$. Thus we can write

$$\tau\frac{\partial \mathbf{j}_\mathrm{f}}{\partial t} = \sigma_0\mathbf{E} - \mathbf{j}_\mathrm{f}. \tag{17.18}$$

With the time dependence $\propto \mathrm{e}^{-\mathrm{i}\omega\tau}$ one obtains

$$(1 - \mathrm{i}\omega\tau)\mathbf{j}_{\mathrm{f},0} = \sigma_0\mathbf{E}_0, \tag{17.19}$$

which can be rewritten

$$\mathbf{j}_{\mathrm{f},0} = \sigma(\omega)\mathbf{E}_0 \tag{17.20}$$

$$\sigma(\omega) = \frac{\sigma_0}{1 - \mathrm{i}\omega\tau} \tag{17.21}$$

$$\epsilon(\omega) = \epsilon - \frac{4\pi\sigma_0}{\mathrm{i}\omega(1 - \mathrm{i}\omega\tau)}. \tag{17.22}$$

For large frequencies, $\omega\tau \gg 1$ one obtains

$$\epsilon(\omega) = \epsilon - \frac{4\pi\sigma_0}{\tau\omega^2} = \epsilon - \frac{4\pi n_0 e_0^2}{m_0\omega^2} = \epsilon\Big(1 - \frac{\omega_\mathrm{P}^2}{\omega^2}\Big) \tag{17.23}$$

with the plasma frequency

$$\omega_\mathrm{P} = \sqrt{\frac{4\pi n_0 e_0^2}{\epsilon m_0}}. \tag{17.24}$$

For $\omega < \omega_\mathrm{P}$ one obtains a negative $\epsilon(\omega)$, that is

$$n = 0, \quad \kappa = \sqrt{\epsilon\Big(\frac{\omega_\mathrm{P}^2}{\omega^2} - 1\Big)} \tag{17.25}$$

with an exponential decay of the wave. However, for $\omega > \omega_\mathrm{P}$ one obtains a positive ϵ

$$n = \sqrt{\epsilon\Big(1 - \frac{\omega_\mathrm{P}^2}{\omega^2}\Big)}, \quad \kappa = 0. \tag{17.26}$$

For such large frequencies the conductor becomes transparent. For copper one has $1/\tau = 3.7 \cdot 10^{13}\ \mathrm{s}^{-1}$, $\sigma_0 = 5.8 \cdot 10^{17}\ \mathrm{s}^{-1}$ and $\omega_\mathrm{P} = 1.6 \cdot 10^{16}\ \mathrm{s}^{-1}$. For visible light one has the frequency-region $\omega = 2.4...5.2 \cdot 10^{15}\ \mathrm{s}^{-1}$, so that copper is non-transparent in the visible range. In electrolytes, however, the carrier density is less, the mass is bigger, so that the plasma-frequency is smaller. Thus electrolytes are normally transparent.

17.c Longitudinal = Plasma Oscilations

One has $\epsilon(\omega) = 0$ for $\omega = \omega_\mathrm{P}$. Then (17.9) allows for longitudinal electric waves

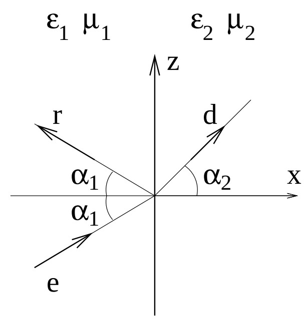
$$\mathbf{E} = E_0\mathbf{e}_z\mathrm{e}^{\mathrm{i}(k_z z - \omega_\mathrm{P} t)}, \quad \mathbf{B} = \mathbf{0}. \tag{17.27}$$

These go along with longitudinal oscillations of the charge carriers, which are obtained by neglecting the friction term in (17.17).

18 Reflection and Refraction at a Planar Surface

18.a Problem and Direction of Propagation

We consider an incident plane wave $\propto$ $e^{i(\mathbf{k}_e\cdot\mathbf{r}-\omega t)}$ for $x < 0$, $\mathbf{k}_e = (k', 0, k_z)$, which hits the plane boundary $x = 0$. For $x < 0$ the dielectric constant and the permeability be ϵ_1 and μ_1, resp., for $x > 0$ these constants are ϵ_2 and μ_2. At the boundary $x = 0$ the wave oscillates $\propto e^{i(k_z z-\omega t)}$. The reflected and the refracted wave show the same behaviour at the boundary, i.e. all three waves have k_z, k_y and ω in common and differ only in k_x. From



$$\mathbf{k}_i^2 = \frac{\epsilon_i\mu_i\omega^2}{c^2} = \frac{n_i^2\omega^2}{c^2}, \quad n_i = \sqrt{\epsilon_i\mu_i} \tag{18.1}$$

one obtains

$$k_1^2 = k_z^2 + k'^2 = \frac{n_1^2\omega^2}{c^2} \qquad \mathbf{k}_r = (-k', 0, k_z) \tag{18.2}$$

$$k_2^2 = k_z^2 + k''^2 = \frac{n_2^2\omega^2}{c^2} \qquad \mathbf{k}_d = (k'', 0, k_z). \tag{18.3}$$

Here $n_{1,2}$ are the indices of refraction of both media. The x-component of the wave-vector of the reflected wave $\mathbf{k}_r$ is the negative of that of the incident wave. Thus the angle of the incident wave α_1 and of the reflected wave are equal. If k'' is real then $k'' > 0$ has to be chosen so that the wave is outgoing and not incoming. If k'' is imaginary then $\Im k'' > 0$ has to be chosen, so that the wave decays exponentially in the medium 2 and does grow exponentially. For real k'' one has

$$k_z = k_1 \sin\alpha_1 = k_2 \sin\alpha_2. \tag{18.4}$$

Thus SNELL's law follows from (18.1)

$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2. \tag{18.5}$$

If $\sin\alpha_2 > 1$ results, then this corresponds to an imaginary k''. We finally remark

$$\frac{k'}{k''} = \frac{k_1\cos\alpha_1}{k_2\cos\alpha_2} = \frac{\tan\alpha_2}{\tan\alpha_1}. \tag{18.6}$$

18.b Boundary Conditions, Amplitudes

In the following we have to distinguish two polarizations. They are referred to the plane of incidence. The plane of incidence is spaned by the direction of the incident wave and by the normal to the boundary (in our coordinates the x-z-plane). The polarization 1 is perpendicular to the plane of incidence, i.e. **E** is polarized in y-direction. The polarization 2 lies in the plane of incidence, **H** points in y-direction. One obtains the following conditions on polarization and continuity

	polarization 1	polarization 2	
E	$\perp$ plane of inc.	in plane of inc.	
H	in plane of inc.	$\perp$ plane of inc.	
$\mathbf{E}_t$	$E_{1,y} = E_{2,y}$	$E_{1,z} = E_{2,z}$	(18.7)
$D_n = \epsilon E_n$		$\epsilon_1 E_{1,x} = \epsilon_2 E_{2,x}$	(18.8)
$\mathbf{H}_t$	$H_{1,z} = H_{2,z}$	$H_{1,y} = H_{2,y}$	(18.9)
$B_n = \mu H_n$	$\mu_1 H_{1,x} = \mu_2 H_{2,x}$		(18.10)

Thus the ansatz for the electric field of polarization 1 is

$$\mathbf{E}(\mathbf{r},t) = \mathrm{e}^{\mathrm{i}(k_z z-\omega t)}\mathbf{e}_y \cdot \begin{cases} (E_\mathrm{e}\mathrm{e}^{\mathrm{i}k'x} + E_\mathrm{r}\mathrm{e}^{-\mathrm{i}k'x}) & x<0 \\ E_\mathrm{d}\mathrm{e}^{\mathrm{i}k''x} & x>0 \end{cases}. \tag{18.11}$$

From MAXWELL's equations one obtains for the magnetic field

$$\mathrm{curl}\,\mathbf{E} = -\frac{\mu}{c}\dot{\mathbf{H}} = \frac{\mathrm{i}\omega\mu}{c}\mathbf{H}, \tag{18.12}$$

$$\mu H_x = \frac{c}{\mathrm{i}\omega}(\mathrm{curl}\,\mathbf{E})_x = -\frac{c}{\mathrm{i}\omega}\frac{\partial E_y}{\partial z} = -\frac{ck_z}{\omega}E_y \tag{18.13}$$

$$H_z = \frac{c}{\mathrm{i}\mu\omega}\frac{\partial E_y}{\partial x} = \mathrm{e}^{\mathrm{i}(k_z z-\omega t)}\frac{c}{\omega}\cdot\begin{cases} \frac{k'}{\mu_1}(E_\mathrm{e}\mathrm{e}^{\mathrm{i}k'x} - E_\mathrm{r}\mathrm{e}^{-\mathrm{i}k'x}) & x<0 \\ \frac{k''}{\mu_2}E_\mathrm{d}\mathrm{e}^{\mathrm{i}k''x} & x>0. \end{cases} \tag{18.14}$$

The boundary conditions come from the continuity of E_y, which is identical to the continuity of μH_x, and from the continuity of H_z,

$$E_\mathrm{e} + E_\mathrm{r} = E_\mathrm{d}, \quad \frac{k'}{\mu_1}(E_\mathrm{e} - E_\mathrm{r}) = \frac{k''}{\mu_2}E_\mathrm{d}, \tag{18.15}$$

from which one obtains the amplitudes

$$E_\mathrm{r} = \frac{\mu_2 k' - \mu_1 k''}{\mu_2 k' + \mu_1 k''}E_\mathrm{e}, \quad E_\mathrm{d} = \frac{2\mu_2 k'}{\mu_2 k' + \mu_1 k''}E_\mathrm{e}. \tag{18.16}$$

One comes from polarization 1 to polarization 2 by the transformation

$$\mathbf{E} \to \mathbf{H}, \quad \mathbf{H} \to -\mathbf{E}, \quad \epsilon \leftrightarrow \mu. \tag{18.17}$$

Thus one obtains for the amplitudes

$$H_\mathrm{r} = \frac{\epsilon_2 k' - \epsilon_1 k''}{\epsilon_2 k' + \epsilon_1 k''}H_\mathrm{e}, \quad H_\mathrm{d} = \frac{2\epsilon_2 k'}{\epsilon_2 k' + \epsilon_1 k''}H_\mathrm{e}. \tag{18.18}$$

18.c Discussion for $\mu_1 = \mu_2$

Now we discuss the results for $\mu = 1 = \mu_2$, since for many media the permeability is practically equal to 1.

18.c.α Insulator, $|\sin\alpha_2| < 1$: Refraction

Now we determine the amplitude of the reflected wave from that of the incident wave. The reflection coefficient R, i.e. the percentage of the incident power which is reflected is given by

$$R = \left(\frac{E_\mathrm{r}}{E_\mathrm{e}}\right)^2 = \left(\frac{H_\mathrm{r}}{H_\mathrm{e}}\right)^2, \tag{18.19}$$

since the modulus of the time averaged POYNTING vector $\mathbf{S} = c\mathbf{E}\times\mathbf{H}/(4\pi)$ for vectors $\mathbf{E}$ and $\mathbf{H}$ which are orthogonal to each other yields

$$\overline{|\mathbf{S}|} = \frac{c}{8\pi}|\mathbf{E}|\cdot|\mathbf{H}| = \frac{c'}{8\pi}\epsilon E^2 = \frac{c'}{8\pi}\mu H^2. \tag{18.20}$$

For the polarisation 1 one obtains with (18.6)

$$E_\mathrm{r} = \frac{k'-k''}{k'+k''}E_\mathrm{e} = \frac{\tan\alpha_2 - \tan\alpha_1}{\tan\alpha_2 + \tan\alpha_1}E_\mathrm{e} = \frac{\sin(\alpha_2-\alpha_1)}{\sin(\alpha_2+\alpha_1)}E_\mathrm{e}. \tag{18.21}$$

For polarization 2 one has

$$H_\mathrm{r} = \frac{n_2^2 k' - n_1^2 k''}{n_2^2 k' + n_1^2 k''}H_\mathrm{e} = \frac{\sin^2\alpha_1\tan\alpha_2 - \sin^2\alpha_2\tan\alpha_1}{\sin^2\alpha_1\tan\alpha_2 + \sin^2\alpha_2\tan\alpha_1}H_\mathrm{e} = \frac{\tan(\alpha_1-\alpha_2)}{\tan(\alpha_1+\alpha_2)}H_\mathrm{e}. \tag{18.22}$$

One finds that the reflection vanishes for polarization 2 for $\alpha_1+\alpha_2 = 90^0$, from which one obtains $\tan\alpha_1 = n_2/n_1$ because of $\sin\alpha_2 = \cos\alpha_1$ and (18.5). This angle is called BREWSTER's angle. By incidence of light under this angle only light of polarization 1 is reflected. This can be used to generate linearly polarized light. In the limit α approaching zero, i.e. by incidence of the light perpendicular to the surface one obtains for both polarizations (which can no longer be distinguished)

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2, \quad \alpha = 0 \tag{18.23}$$

18.c.β Insulator, $|sin\alpha_2| > 1$: Total Reflection

In this case k'' is imaginary. The wave penetrates only exponentially decaying into the second medium. From the first expressions of (18.21) and (18.22) one finds since the numerator of the fraction is the conjugate complex of the denominator that

$$|E_r| = |E_e|, \quad |H_r| = |H_e|. \quad R = 1, \tag{18.24}$$

Thus one has total reflection.

18.c.γ Metallic Reflection, $\alpha = 0$

In the case of metallic reflection we set $n_1 = 1$ (vacuum or air) and $n_2 = n + i\kappa$ (17.11). Then one obtains from (18.23) for $\alpha = 0$ the reflection coefficient

$$R = \left|\frac{n + i\kappa - 1}{n + i\kappa + 1}\right|^2 = \frac{(n-1)^2 + \kappa^2}{(n+1)^2 + \kappa^2} = 1 - \frac{4n}{(n+1)^2 + \kappa^2}. \tag{18.25}$$

For $\omega\epsilon \ll 2\pi\sigma$ one obtains from (17.5) and (17.15)

$$n \approx \kappa \approx \sqrt{\frac{2\pi\sigma}{\omega}}, \quad R \approx 1 - \frac{2}{n} \approx 1 - \sqrt{\frac{2\omega}{\pi\sigma}}, \tag{18.26}$$

a result named after HAGEN and RUBENS.

18.c.δ Surface Waves along a Conductor

Finally we consider waves, which run along the boundary of a conductor with the vacuum. Thus we set $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon(\omega)$ from (17.5). Then we need one wave on each side of the boundary. We obtain this by looking for a solution, where no wave is reflected. Formally this means that we choose a wave of polarization 2 for which H_r in (18.18) vanishes, thus

$$\epsilon(\omega)k' = k'' \tag{18.27}$$

has to hold. With (18.2) and (18.3)

$$k_z^2 + k'^2 = \frac{\omega^2}{c^2}, \quad k_z^2 + k''^2 = \frac{\epsilon(\omega)\omega^2}{c^2} \tag{18.28}$$

one obtains the solution

$$k_z = \frac{\omega}{c}\sqrt{\frac{\epsilon(\omega)}{1+\epsilon(\omega)}}, \quad k' = \frac{k_z}{\sqrt{\epsilon(\omega)}}, \quad k'' = \sqrt{\epsilon(\omega)}k_z. \tag{18.29}$$

Using approximation (17.15) one obtains for frequencies which are not too large

$$k_z = \frac{\omega}{c}\left(1 + \frac{i\omega}{8\pi\sigma}\right) \tag{18.30}$$

$$k' = \frac{(1-i)\omega^{3/2}}{2c\sqrt{2\pi\sigma}} \tag{18.31}$$

$$k'' = \frac{(1+i)\omega^{1/2}\sqrt{2\pi\sigma}}{c}. \tag{18.32}$$

Thus for small frequencies, $\omega < \sigma$, the exponential decay in direction of propagation (k_z) is smallest, into the vacuum it is faster (k') and into the metal it is fastest (k'').

19 Wave Guides

There are various kinds of wave guides. They may consist for example of two conductors, which run in parallel (two wires) or which are coaxial conductors. But one may also guide an electro-magnetic wave in a dielectric wave guide (for light e.g.) or in a hollow metallic cylinder.
In all cases we assume translational invariance in z-direction, so that material properties ϵ, μ, and σ are only functions of x and y. Then the electromagnetic fields can be written

$$\mathbf{E} = \mathbf{E}_0(x,y)\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}, \quad \mathbf{B} = \mathbf{B}_0(x,y)\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}. \tag{19.1}$$

Then the functions $\mathbf{E}_0$, $\mathbf{B}_0$ and $\omega(k_z)$ have to be determined.

19.a Wave Guides

We will carry through this program for a wave guide which is a hollow metallic cylinder (not necessarily with circular cross-section). We start out from the boundary conditions, where we assume that the cylinder surface is an ideal metal $\sigma = \infty$. Then one has at the surface

$$\mathbf{E}_\mathrm{t} = \mathbf{0}, \tag{19.2}$$

since a tangential component would yield an infinite current density at the surface. Further from $\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}/c$ it follows that

$$\mathrm{i}kB_\mathrm{n} = (\operatorname{curl}\mathbf{E})_\mathrm{n} = (\operatorname{curl}\mathbf{E}_\mathrm{t})\cdot\mathbf{e}_\mathrm{n} \quad k = \omega/c, \tag{19.3}$$

from which one obtains

$$B_\mathrm{n} = 0. \tag{19.4}$$

Inside the wave guide one has

$$(\operatorname{curl}\mathbf{E})_y = -\frac{1}{c}\dot{B}_y \quad \to \quad \mathrm{i}k_z E_{0,x} - \nabla_x E_{0,z} = \mathrm{i}kB_{0,y} \tag{19.5}$$

$$(\operatorname{curl}\mathbf{B})_x = \frac{1}{c}\dot{E}_x \quad \to \quad \nabla_y B_{0,z} - \mathrm{i}k_z B_{0,y} = -\mathrm{i}kE_{0,x}. \tag{19.6}$$

By use of

$$k_\perp^2 = k^2 - k_z^2 \tag{19.7}$$

one can express the transverse components by the longitudinal components

$$k_\perp^2 E_{0,x} = \mathrm{i}k_z\nabla_x E_{0,z} + \mathrm{i}k\nabla_y B_{0,z} \tag{19.8}$$

$$k_\perp^2 B_{0,y} = \mathrm{i}k\nabla_x E_{0,z} + \mathrm{i}k_z\nabla_y B_{0,z}. \tag{19.9}$$

Similar equations hold for $E_{0,y}$ and $B_{0,x}$. In order to determine the longitudinal components we use the wave equation

$$\left(\triangle - \frac{\partial^2}{c^2\partial t^2}\right)(E_{0,z}\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}) = 0, \tag{19.10}$$

from which one obtains

$$(\nabla_x^2 + \nabla_y^2 + k_\perp^2)E_{0,z}(x,y) = 0 \tag{19.11}$$

and similarly

$$(\nabla_x^2 + \nabla_y^2 + k_\perp^2)B_{0,z}(x,y) = 0. \tag{19.12}$$

One can show that the other equations of MAXWELL are fulfilled for $k_\perp \neq 0$, since

$$\left.\begin{array}{r} k_\perp^2 \operatorname{div}\mathbf{E} = \mathrm{i}k_z \\ k_\perp^2(\operatorname{curl}\mathbf{B} - \dot{\mathbf{E}}/c)_z = \mathrm{i}k \end{array}\right\}\cdot(\nabla_x^2 + \nabla_y^2 + k_\perp^2)E_{0,z}\mathrm{e}^{\mathrm{i}(k_z z-\omega t)} \tag{19.13}$$

$$\left.\begin{array}{r} k_\perp^2 \operatorname{div}\mathbf{B} = \mathrm{i}k_z \\ k_\perp^2(\operatorname{curl}\mathbf{E} + \dot{\mathbf{B}}/c)_z = -\mathrm{i}k \end{array}\right\}\cdot(\nabla_x^2 + \nabla_y^2 + k_\perp^2)B_{0,z}\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}. \tag{19.14}$$

Thus it is sufficient to fulfill the wave equations. We further note that $E_{0,z}$ and $B_{0,z}$ are independent from each other. Correspondingly one distinguishes TE-modes (transverse electric) with $E_{0,z} = 0$ and TM-modes (transverse magnetic) with $B_{0,z} = 0$.
We return to the boundary conditions. The components perpendicular to the direction of propagation z read

$$k_\perp^2(\mathbf{e}_x E_{0,x} + \mathbf{e}_y E_{0,y}) = \mathrm{i}k_z \operatorname{grad} E_{0,z} - \mathrm{i}k\mathbf{e}_z \times \operatorname{grad} B_{0,z} \tag{19.15}$$
$$k_\perp^2(\mathbf{e}_x B_{0,x} + \mathbf{e}_y B_{0,y}) = \mathrm{i}k_z \operatorname{grad} B_{0,z} + \mathrm{i}k\mathbf{e}_z \times \operatorname{grad} E_{0,z}. \tag{19.16}$$

If we introduce besides the normal vector $\mathbf{e}_n$ and the vector $\mathbf{e}_z$ a third unit vector $\mathbf{e}_c = \mathbf{e}_z \times \mathbf{e}_n$ at the surface of the waveguide then the tangential plain of the surface is spanned by $\mathbf{e}_z$ and $\mathbf{e}_c$. $\mathbf{e}_c$ itself lies in the xy-plain. Since $\mathbf{e}_n$ lies in the xy-plain too, we may transform to n and c components

$$\mathbf{e}_x E_{0,x} + \mathbf{e}_y E_{0,y} = \mathbf{e}_c E_{0,c} + \mathbf{e}_n E_{0,n}. \tag{19.17}$$

Then (19.15, 19.16) can be brought into the form

$$k_\perp^2(\mathbf{e}_n E_{0,n} + \mathbf{e}_c E_{0,c}) = \mathrm{i}k_z(\mathbf{e}_n\partial_n E_{0,z} + \mathbf{e}_c\partial_c E_{0,z}) - \mathrm{i}k(\mathbf{e}_c\partial_n B_{0,z} - \mathbf{e}_n\partial_c B_{0,z}), \tag{19.18}$$
$$k_\perp^2(\mathbf{e}_n B_{0,n} + \mathbf{e}_c B_{0,c}) = \mathrm{i}k_z(\mathbf{e}_n\partial_n B_{0,z} + \mathbf{e}_c\partial_c B_{0,z}) + \mathrm{i}k(\mathbf{e}_c\partial_n E_{0,z} - \mathbf{e}_n\partial_c E_{0,z}). \tag{19.19}$$

At the surface one has

$$E_{0,z} = E_{0,c} = B_{0,\mathrm{n}} = 0 \tag{19.20}$$

according to (19.2, 19.4). From (19.18, 19.19) one obtains

$$k_\perp^2 E_{0,c} = \mathrm{i}k_z\partial_c E_{0,z} - \mathrm{i}k\partial_n B_{0,z}, \tag{19.21}$$
$$k_\perp^2 B_{0,\mathrm{n}} = \mathrm{i}k_z\partial_n B_{0,z} + \mathrm{i}k\partial_c E_{0,z}. \tag{19.22}$$

Since $E_{0,z} = 0$ holds at the surface one has $\partial_c E_{0,z} = 0$ at the surface too. Apparently the second condition is $\partial_n B_{0,z} = 0$.
Then the following eigenvalue problem has to be solved

$$\text{TM-Mode:} \quad (\nabla_x^2 + \nabla_y^2 + k_\perp^2)E_{0,z} = 0, \quad E_{0,z} = 0 \text{ at the surface}, \tag{19.23}$$
$$\text{TE-Mode:} \quad (\nabla_x^2 + \nabla_y^2 + k_\perp^2)B_{0,z} = 0, \quad (\operatorname{grad} B_{0,z})_\mathrm{n} = 0 \text{ at the surface}. \tag{19.24}$$

Then one obtains the dispersion law

$$\omega = c\sqrt{k_z^2 + k_\perp^2}. \tag{19.25}$$

TEM-modes By now we did not discuss the case $k_\perp = 0$. We will not do this in all details. One can show that for these modes both longitudinal components vanish, $E_{0,z} = B_{0,z} = 0$. Thus one calls them TEM-modes. Using $k_z = \pm k$ from (19.5) and similarly after a rotation of $\mathbf{E}$ and $\mathbf{B}$ around the z-axis by 90^0 $E_{0,x} \to E_{0,y}$, $B_{0,y} \to -B_{0,x}$ one obtains

$$B_{0,y} = \pm E_{0,x}, \quad B_{0,x} = \mp E_{0,y}. \tag{19.26}$$

From $(\operatorname{curl}\mathbf{E})_z = 0$ it follows that $\mathbf{E}_0$ can be expressed by the gradient of a potential

$$\mathbf{E}_0 = -\operatorname{grad}\Phi(x,y), \tag{19.27}$$

which due to $\operatorname{div}\mathbf{E}_0 = 0$ fulfills Laplace's equation

$$(\nabla_x^2 + \nabla_y^2)\Phi(x,y) = 0. \tag{19.28}$$

Thus Laplace's homogeneous equation in two dimensions has to be solved. Because of $\mathbf{E}_{0,\mathrm{t}} = \mathbf{0}$ the potential on the surface has to be constant. Thus one obtains a non-trivial solution only in multiply connected regions, i.e. not inside a circular or rectangular cross-section, but outside such a region or in a coaxial wire or outside two wires.

19.b Solution for a Rectangular Cross Section

We determine the waves in a wave guide of rectangular cross-section with sides a and b. For the TM-wave we start with the factorization ansatz

$$E_{0,z}(x,y) = f(x)g(y) \tag{19.29}$$

Insertion into (19.11) yields

$$f''g + fg'' + k_\perp^2 fg = 0 \tag{19.30}$$

and equivalently

$$\frac{f''}{f} + \frac{g''}{g} = -k_\perp^2, \tag{19.31}$$

from which one concludes that f''/f and g''/g have to be constant. Since $E_{0,z}$ has to vanish at the boundary, one obtains

$$E_{0,z}(x,y) = E_0 \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{b}), \quad k_\perp^2 = \left(\frac{nu\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2, \quad n \geq 1, m \geq 1. \tag{19.32}$$

For the TE-wave one obtains with the corresponding ansatz

$$B_{0,z}(x,y) = f(x)g(y) \tag{19.33}$$

and the boundary condition $(\operatorname{grad} B_{0,z})_{\rm n} = 0$ the solutions

$$B_{0,z}(x,y) = B_0 \cos(\frac{n\pi x}{a}) \cos(\frac{m\pi y}{b}), \quad k_\perp^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2, \quad n \geq 0, \quad m \geq 0, \quad n+m \geq 1. \tag{19.34}$$

19.c Wave Packets

Often one does not deal with monochromatic waves, but with wave packets, which consist of Fourier components with $k_z \approx k_{z,0}$

$$\mathbf{E} = \mathbf{E}_0(x,y) \int dk_z f_0(k_z) e^{i(k_z z - \omega(k_z)t)}, \tag{19.35}$$

where $f_0(k_z)$ has a maximum at $k_z = k_{z,0}$ and decays rapidly for other values of k_z. Then one expands $\omega(k_z)$ around $k_{z,0}$

$$\omega(k_z) = \omega(k_{z,0}) + v_{\rm gr}(k_z - k_{z,0}) + ... \tag{19.36}$$

$$v_{\rm gr} = \left.\frac{d\omega(k_z)}{dk_z}\right|_{k_z=k_{z,0}}. \tag{19.37}$$

In linear approximation of this expansion one obtains

$$\mathbf{E} = \mathbf{E}_0(x,y) e^{i(k_{z,0}z-\omega(k_{z,0})t)} f(z - v_{\rm gr}t), \quad f(z - v_{\rm gr}t) = \int {\rm d}k_z f_0(k_z) e^{i(k_z - k_{z,0})(z - v_{\rm gr}t)}. \tag{19.38}$$

The factor in front contains the phase $\phi = k_{z,0}z - \omega(k_{z,0})t$. Thus the wave packet oscillates with the phase velocity

$$v_{\rm ph} = \left.\frac{\partial z}{\partial t}\right|_\phi = \frac{\omega(k_{z,0})}{k_{z,0}}. \tag{19.39}$$

On the other hand the local dependence of the amplitude is contained in the function $f(z - v_{\rm gr}t)$. Thus the wave packet moves with the group velocity (signal velocity) $v_{\rm gr}$, (19.37).
For the waves in the wave-guide we obtain from (19.25)

$$v_{\rm ph} = c\frac{\sqrt{k_\perp^2 + k_{z,0}^2}}{k_{z,0}}, \tag{19.40}$$

$$v_{\rm gr} = c\frac{k_{z,0}}{\sqrt{k_\perp^2 + k_{z,0}^2}}. \tag{19.41}$$

The phase velocity is larger than the velocity of light in vacuum c, the group velocity (velocity of a signal) less than c. If one performs the expansion (19.36) beyond the linear term, then one finds that the wave packets spread in time.
Exercise Determine $\omega(k)$ for transverse oscillations in a conductor above the plasma frequency (section 17.b) for $\epsilon = 1$ and the resulting phase- and group-velocities, resp.