G Electrodynamic Potentials

20 Electrodynamic Potentials, Gauge Transformations

We already know the electric potential Φ from electrostatics and the vector potential $\mathbf{A}$ from magnetostatics. Both can also be used for time-dependent problems and allow the determination of $\mathbf{B}$ and $\mathbf{E}$.

20.a Potentials

MAXWELL's third and fourth equations are homogeneous equations, i.e. they do not contain charges and currents explicitly. They allow to express the fields $\mathbf{B}$ and $\mathbf{E}$ by means of potentials. One obtains from $\operatorname{div}\mathbf{B} = 0$

$$\mathbf{B}(\mathbf{r},t) = \operatorname{curl}\mathbf{A}(\mathbf{r},t). \tag{20.1}$$

Proof: Due to $\operatorname{div}\mathbf{B} = 0$ one has $\triangle\mathbf{B} = -\operatorname{curl}\operatorname{curl}\mathbf{B}$ (B.26), from which one concludes similarly as in (9.16) and (9.17)

$$\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r'\Big(\operatorname{curl}'\operatorname{curl}'\mathbf{B}(\mathbf{r}')\Big)\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi}\operatorname{curl}\int d^3r'\,\frac{\operatorname{curl}'\mathbf{B}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{20.2}$$

when the vector potential was introduced in the magnetostatics. An elementary proof is left as exercise.
From $\operatorname{curl}\mathbf{E} + \dot{\mathbf{B}}/c = \mathbf{0}$ one obtains

$$\operatorname{curl}\Big(\mathbf{E} + \frac{1}{c}\dot{\mathbf{A}}\Big) = \mathbf{0}, \tag{20.3}$$

so that the argument under the curl can be expressed as a gradient. Conventionally one calls it $-\operatorname{grad}\Phi$, so that

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \operatorname{grad}\Phi \tag{20.4}$$

follows. The second term is already known from electrostatics. The time derivative of $\mathbf{A}$ contains the law of induction. One sees contrarily that the representations of the potentials in (20.4) and (20.1) fulfill the homogeneous MAXWELL equations.

20.b Gauge Transformations

The potentials $\mathbf{A}$ and Φ are not uniquely determined by the fields $\mathbf{B}$ and $\mathbf{E}$. We may replace $\mathbf{A}$ by

$$\mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \operatorname{grad}\Lambda(\mathbf{r},t) \tag{20.5}$$

without changing $\mathbf{B}$

$$\mathbf{B} = \operatorname{curl}\mathbf{A} = \operatorname{curl}\mathbf{A}', \tag{20.6}$$

since $\operatorname{curl}\operatorname{grad}\Lambda = \mathbf{0}$. It follows that

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}}' - \operatorname{grad}(\Phi - \frac{1}{c}\dot{\Lambda}). \tag{20.7}$$

If we replace simultaneously Φ by

$$\Phi'(\mathbf{r},t) = \Phi(\mathbf{r},t) - \frac{1}{c}\dot{\Lambda}(\mathbf{r},t), \tag{20.8}$$

then $\mathbf{E}$ and $\mathbf{B}$ remain unchanged. One calls the transformations (20.5) and (20.8) gauge transformations.

The arbitrariness in the gauge allows to impose restrictions on the potentials Φ and $\mathbf{A}$

$$\text{Lorenz gauge} \qquad \operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\Phi} = 0, \tag{20.9}$$

$$\text{Coulomb gauge} \qquad \operatorname{div}\mathbf{A} = 0. \tag{20.10}$$

If potentials Φ' and $\mathbf{A}'$ do not obey the desired gauge, the potentials Φ and $\mathbf{A}$ are obtained by an appropriate choice of Λ

$$\text{Lorenz gauge} \qquad \operatorname{div}\mathbf{A}' + \frac{1}{c}\dot{\Phi}' = \Box\Lambda, \tag{20.11}$$

$$\text{Coulomb gauge} \qquad \operatorname{div}\mathbf{A}' = \triangle\Lambda, \tag{20.12}$$

where

$$\Box := \triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{20.13}$$

is D'ALEMBERT's operator. The LORENZ gauge traces back to the Danish physicist LUDVIG V. LORENZ (1867) in contrast to the LORENTZ transformation (section 23) attributed to the Dutch physicist HENDRIK A. LORENTZ. Insertion of the expressions (20.4) and (20.1) for $\mathbf{E}$ and $\mathbf{B}$ into MAXWELL's first equation yields

$$\operatorname{curl}\operatorname{curl}\mathbf{A} + \frac{1}{c^2}\ddot{\mathbf{A}} + \frac{1}{c}\operatorname{grad}\dot{\Phi} = \frac{4\pi}{c}\mathbf{j}, \tag{20.14}$$

that is

$$-\Box\mathbf{A} + \operatorname{grad}\left(\operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\Phi}\right) = \frac{4\pi}{c}\mathbf{j}, \tag{20.15}$$

whereas MAXWELL's second equation reads

$$-\triangle\Phi - \frac{1}{c}\operatorname{div}\dot{\mathbf{A}} = 4\pi\rho. \tag{20.16}$$

From this one obtains for both gauges

$$\text{Lorenz gauge} \qquad \begin{cases} \Box\mathbf{A} = -\frac{4\pi}{c}\mathbf{j} \\ \Box\Phi = -4\pi\rho \end{cases} \tag{20.17}$$

$$\text{Coulomb gauge} \qquad \begin{cases} \Box\mathbf{A} = -\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\operatorname{grad}\dot{\Phi} \\ \triangle\Phi = -4\pi\rho. \end{cases} \tag{20.18}$$

Exercise Show that a vector field $\mathbf{B}(\mathbf{r})$, which obeys $\operatorname{div}\mathbf{B} = 0$ can be represented by $\operatorname{curl}\mathbf{A}(\mathbf{r})$. Therefore put $A_z(\mathbf{r}) = 0$ and express $A_y(\mathbf{r})$ by $A_y(x, y, 0)$ and B_x, similarly $A_x(\mathbf{r})$ by $A_x(x, y, 0)$ and B_y. Insert this in $B_z = (\operatorname{curl}\mathbf{A})_z$ and show by use of $\operatorname{div}\mathbf{B} = 0$ that one can find fitting components of $\mathbf{A}$ at $\mathbf{r} = (x, y, 0)$.

21 Electromagnetic Potentials of a general Charge and Current Distribution

21.a Calculation of the Potentials

Using the Lorenz gauge we had

$$\Box\Phi(\mathbf{r},t) = -4\pi\rho(\mathbf{r},t), \tag{21.1}$$

$$\Box\mathbf{A}(\mathbf{r},t) = -\frac{4\pi}{c}\mathbf{j}(\mathbf{r},t) \tag{21.2}$$

with d'Alembert's operator

$$\Box = \triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{21.3}$$

and the gauge condition

$$\mathrm{div}\,\mathbf{A} + \frac{1}{c}\dot{\Phi} = 0. \tag{21.4}$$

We perform the Fourier transform with respect to time

$$\Phi(\mathbf{r},t) = \int_{-\infty}^{\infty} \mathrm{d}\omega \hat{\Phi}(\mathbf{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{21.5}$$

analogously for $\mathbf{A}$, ρ, $\mathbf{j}$. Then one obtains

$$\Box\Phi(\mathbf{r},t) = \int \mathrm{d}\omega(\triangle + \frac{\omega^2}{c^2})\hat{\Phi}(\mathbf{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t} = \int \mathrm{d}\omega(-4\pi\hat{\rho}(\mathbf{r},\omega))\mathrm{e}^{-\mathrm{i}\omega t}, \tag{21.6}$$

from which by comparison of the integrands

$$\Big(\triangle + \frac{\omega^2}{c^2}\Big)\hat{\Phi}(\mathbf{r},\omega) = -4\pi\hat{\rho}(\mathbf{r},\omega) \tag{21.7}$$

is obtained. We now introduce the Green's function G, i.e. we write the solution of the linear differential equation as

$$\hat{\Phi}(\mathbf{r},\omega) = \int \mathrm{d}^3 r' G(\mathbf{r},\mathbf{r}',\omega)\hat{\rho}(\mathbf{r}',\omega). \tag{21.8}$$

Insertion of this ansatz into the differential equation (21.7) yields

$$\Big(\triangle + \frac{\omega^2}{c^2}\Big)G(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \tag{21.9}$$

Since there is no preferred direction and moreover the equation is invariant against displacement of the vectors $\mathbf{r}$ and $\mathbf{r}'$ by the same constant vector, we may assume that the solution depends only on the distance between $\mathbf{r}$ and $\mathbf{r}'$ and additionally of course on ω

$$G = g(a,\omega), \quad a = |\mathbf{r}-\mathbf{r}'|. \tag{21.10}$$

Then one obtains

$$(\triangle + \frac{\omega^2}{c^2})g(a,\omega) = \frac{1}{a}\frac{d^2(ag)}{da^2} + \frac{\omega^2}{c^2}g = 0 \text{ for } a \neq 0. \tag{21.11}$$

Here we use the Laplacian in the form (5.15), where $\triangle_\Omega g = 0$, since g does not depend on the direction of $\mathbf{a} = \mathbf{r} - \mathbf{r}'$, but on the modulus a. This yields the equation of a harmonic oscillation for ag with the solution

$$G = g(a,\omega) = \frac{1}{a}\Big(c_1\mathrm{e}^{\mathrm{i}\omega a/c} + c_2\mathrm{e}^{-\mathrm{i}\omega a/c}\Big). \tag{21.12}$$

At short distances the solution diverges like $(c_1 + c_2)/a$. In order to obtain the δ-function in (21.9) as an inhomogeneity with the appropriate factor in front, one requires $c_1 + c_2 = 1$. We now insert

$$\begin{aligned}
\Phi(\mathbf{r},t) &= \int d\omega \hat{\Phi}(\mathbf{r},\omega) e^{-i\omega t} \\
&= \int d\omega \int d^3r' e^{-i\omega t} G(\mathbf{r},\mathbf{r}',\omega)\hat{\rho}(\mathbf{r}',\omega) \\
&= \int d\omega \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(c_1 e^{i\omega|\mathbf{r}-\mathbf{r}'|/c} + c_2 e^{-i\omega|\mathbf{r}-\mathbf{r}'|/c}\right) e^{-i\omega t}\hat{\rho}(\mathbf{r}',\omega) \\
&= \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(c_1\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) + c_2\rho(\mathbf{r}', t + \frac{|\mathbf{r}-\mathbf{r}'|}{c})\right). \qquad (21.13)
\end{aligned}$$

Going from the second to the third line we have inserted G. Then we perform the ω-integration, compare (21.5). However, ω in the exponent in (21.13) is not multiplied by t, but by $t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}$. The solution in the last line contains a contribution of Φ at time t which depends on ρ at an earlier time (with factor c_1), and one which depends on ρ at a later time (with factor c_2). The solution which contains only the first contribution ($c_1 = 1$, $c_2 = 0$) is called the retarded solution, and the one which contains only the second contribution ($c_1 = 0$, $c_2 = 1$) the advanced solution.

$$\Phi_{r,a}(\mathbf{r},t) = \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}', t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}). \qquad (21.14)$$

Normally the retarded solution (upper sign) is the physical solution, since the potential is considered to be created by the charges, but not the charges by the potentials. Analogously, one obtains the retarded and advanced solutions for the vector potential

$$\mathbf{A}_{r,a}(\mathbf{r},t) = \frac{1}{c}\int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}', t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}). \qquad (21.15)$$

21.b Gauge Condition

It remains to be shown that the condition for Lorenz gauge (20.9) is fulfilled

$$\dot{\Phi} + c\,\mathrm{div}\,\mathbf{A} = \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\dot{\rho} + \mathrm{div}\,\mathbf{j}) + \int d^3r' \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}. \qquad (21.16)$$

The arguments of ρ and $\mathbf{j}$ are as above $\mathbf{r}'$ and $t' = t \mp |\mathbf{r}-\mathbf{r}'|/c$. In the second integral one can replace ∇ by $-\nabla'$ and perform a partial integration. This yields

$$\dot{\Phi} + c\,\mathrm{div}\,\mathbf{A} = \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\dot{\rho} + (\nabla + \nabla')\mathbf{j}). \qquad (21.17)$$

Since $(\nabla + \nabla')t'(t,\mathbf{r},\mathbf{r}') = \mathbf{0}$, one obtains from the equation of continuity

$$\dot{\rho}(\mathbf{r}', t'(t,\mathbf{r},\mathbf{r}')) + (\nabla + \nabla')\mathbf{j}(\mathbf{r}', t'(t,\mathbf{r},\mathbf{r}')) = \frac{\partial\rho}{\partial t'} + \nabla'\mathbf{j}(\mathbf{r}',t')|_{t'} = 0, \qquad (21.18)$$

so that the gauge condition (20.9) is fulfilled, since the integrand in (21.17) vanishes because of the equation of continuity.

22 Radiation from Harmonic Oscillations

In this section we consider the radiation of oscillating charges and currents.

22.a Radiation Field

We consider harmonic oscillations, i.e. the time dependence of ρ and $\mathbf{j}$ is proportional to $e^{-i\omega t}$

$$\rho(\mathbf{r},t) = \Re\left(\rho_0(\mathbf{r})e^{-i\omega t}\right) \tag{22.1}$$

$$\mathbf{j}(\mathbf{r},t) = \Re\left(\mathbf{j}_0(\mathbf{r})e^{-i\omega t}\right), \tag{22.2}$$

analogously for Φ, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$. One obtains

$$\Phi(\mathbf{r},t) = \Re\int d^3r'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho_0(\mathbf{r}')e^{-i\omega(t-|\mathbf{r}-\mathbf{r}'|/c)}. \tag{22.3}$$

With the notation $k = \omega/c$ it follows that

$$\Phi_0(\mathbf{r}) = \int d^3r'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho_0(\mathbf{r}')e^{ik|\mathbf{r}-\mathbf{r}'|}, \tag{22.4}$$

analogously

$$\mathbf{A}_0(\mathbf{r}) = \frac{1}{c}\int d^3r'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}_0(\mathbf{r}')e^{ik|\mathbf{r}-\mathbf{r}'|}. \tag{22.5}$$

22.a.α Near Zone (Static Zone)

In the near zone, i.e. for $k|\mathbf{r}-\mathbf{r}'| \ll 2\pi$ which is equivalent to $|\mathbf{r}-\mathbf{r}'| \ll \lambda$, where λ is the wave-length of the electromagnetic wave, the expression $e^{ik|\mathbf{r}-\mathbf{r}'|}$ can be approximated by 1. Then the potentials Φ_0, (22.4) and $\mathbf{A}_0$, (22.5) reduce to the potentials of electrostatics (3.14) and of magnetostatics (9.17).

22.a.β Far Zone (Radiation Zone)

At large distances one expands the expression in the exponent

$$|\mathbf{r}-\mathbf{r}'| = r\sqrt{1-2\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}+\frac{r'^2}{r^2}} = r - \mathbf{n}\cdot\mathbf{r}' + O(\frac{r'^2}{r}), \quad \mathbf{n} = \frac{\mathbf{r}}{r}. \tag{22.6}$$

This is justified for $r \gg kR^2$, where R is an estimate of the extension of the charge and current distribution, $r' < R$ for $\rho(\mathbf{r}') \neq 0$ and $\mathbf{j}(\mathbf{r}') \neq \mathbf{0}$, resp. We approximate in the denominator $|\mathbf{r}-\mathbf{r}'| \approx r$ which is reasonable for $r \gg R$. Then one obtains

$$\mathbf{A}_0(\mathbf{r}) = \frac{e^{ikr}}{cr}\mathbf{g}(k\mathbf{n}) + O(\frac{1}{r^2}) \tag{22.7}$$

with the Fourier transform of the current distribution

$$\mathbf{g}(k\mathbf{n}) = \int d^3r'\mathbf{j}_0(\mathbf{r}')e^{-ik\mathbf{n}\cdot\mathbf{r}'}. \tag{22.8}$$

From this one deduces the magnetic field

$$\mathbf{B}_0(\mathbf{r}) = \operatorname{curl}\mathbf{A}_0(\mathbf{r}) = \frac{\operatorname{grad}e^{ikr}}{cr}\times\mathbf{g}(k\mathbf{n}) + O(\frac{1}{r^2}) = ik\frac{e^{ikr}}{cr}\mathbf{n}\times\mathbf{g} + O(\frac{1}{r^2}). \tag{22.9}$$

The electric field is obtained from

$$\operatorname{curl}\mathbf{B} = \frac{1}{c}\dot{\mathbf{E}} \rightarrow \operatorname{curl}\mathbf{B}_0 = -\frac{i\omega}{c}\mathbf{E}_0 \tag{22.10}$$

as

$$\mathbf{E}_0 = \frac{\mathrm{i}}{k}\,\mathrm{curl}\,\mathbf{B}_0 = -\mathbf{n}\times\mathbf{B}_0 + O(\frac{1}{r^2}). \tag{22.11}$$

$\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{n}$ are orthogonal to each other. The moduli of $\mathbf{E}_0$ and $\mathbf{B}_0$ are equal and both decay like $1/r$. The Poynting vector yields in the time average

$$\bar{\mathbf{S}} = \frac{1}{T}\int_0^T \mathbf{S}(t)\mathrm{d}t, \quad T = \frac{2\pi}{\omega} \tag{22.12}$$

$$\begin{aligned}\bar{\mathbf{S}} &= \frac{c}{4\pi}\overline{\Re\mathbf{E}\times\Re\mathbf{B}} = \frac{c}{8\pi}\Re(\mathbf{E}_0^*\times\mathbf{B}_0)\\ &= -\frac{c}{8\pi}\Re\Big((\mathbf{n}\times\mathbf{B}_0^*)\times\mathbf{B}_0\Big) = -\frac{c}{8\pi}\Re(\mathbf{n}\cdot\mathbf{B}_0)\mathbf{B}_0^* + \frac{c}{8\pi}\mathbf{n}(\mathbf{B}_0^*\cdot\mathbf{B}_0).\end{aligned} \tag{22.13}$$

The first term after the last equals sign vanishes, since $\mathbf{B}_0 \perp \mathbf{n}$. Thus there remains

$$\bar{\mathbf{S}}(\mathbf{r}) = \frac{c}{8\pi}\mathbf{n}(\mathbf{B}_0^*\cdot\mathbf{B}_0) = \frac{k^2\mathbf{n}}{8\pi c r^2}|\mathbf{n}\times\mathbf{g}(k\mathbf{n})|^2 + O(\frac{1}{r^3}). \tag{22.14}$$

The average power radiated is

$$\dot{U}_{\mathrm{s}} = \frac{k^2}{8\pi c}\int |\mathbf{n}\times\mathbf{g}(k\mathbf{n})|^2\mathrm{d}\Omega_n, \tag{22.15}$$

where the integral is performed over the solid angle Ω_n of $\mathbf{n}$.

22.b Electric Dipole Radiation (Hertz Dipole)

If the charge and current distribution is within a range R small in comparison to the wave length λ, then it is reasonable to expand $\mathrm{e}^{-\mathrm{i}k\mathbf{n}\cdot\mathbf{r}'}$

$$\mathbf{g}(k\mathbf{n}) = \mathbf{g}^{(0)} - \mathrm{i}k\mathbf{g}^{(1)} + ..., \quad \mathbf{g}^{(0)} = \int \mathrm{d}^3r'\mathbf{j}_0(\mathbf{r}'), \quad \mathbf{g}^{(1)} = \int \mathrm{d}^3r'(\mathbf{n}\cdot\mathbf{r}')\mathbf{j}_0(\mathbf{r}') \tag{22.16}$$

This expansion is sufficient to investigate the radiation field in the far zone. If one is interested to consider it also in the near zone and the intermediate zone, one has to expand

$$\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathrm{e}^{\mathrm{i}kr}}{r} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r}(-\mathrm{i}k + \frac{1}{r})(\mathbf{n}\cdot\mathbf{r}') + O(r'^2) \tag{22.17}$$

in the expression for $\mathbf{A}_0$, which yields

$$\mathbf{A}_0(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{cr}\mathbf{g}^{(0)} + (-\mathrm{i}k + \frac{1}{r})\frac{\mathrm{e}^{\mathrm{i}kr}}{cr}\mathbf{g}^{(1)} + ... \tag{22.18}$$

We first consider the contribution from $\mathbf{g}^{(0)}$. We use that

$$\mathrm{div}'\mathbf{j}(\mathbf{r}') = -\dot{\rho}(\mathbf{r}') = \mathrm{i}\omega\rho(\mathbf{r}') \to \mathrm{div}'\mathbf{j}_0(\mathbf{r}') = \mathrm{i}\omega\rho_0(\mathbf{r}'). \tag{22.19}$$

Then we obtain from

$$\int \mathrm{d}^3r'\,\mathrm{div}'\Big(f(\mathbf{r}')\mathbf{j}_0(\mathbf{r}')\Big) = 0 \tag{22.20}$$

the relation

$$\int \mathrm{d}^3r'\,\mathrm{grad}'f(\mathbf{r}')\cdot\mathbf{j}_0(\mathbf{r}') = -\mathrm{i}\omega\int \mathrm{d}^3r' f(\mathbf{r}')\rho_0(\mathbf{r}'). \tag{22.21}$$

One obtains with $f(\mathbf{r}') = x'_\alpha$

$$g_\alpha^{(0)} = \int \mathrm{d}^3r' j_{0,\alpha}(\mathbf{r}') = -\mathrm{i}\omega\int \mathrm{d}^3r' x'_\alpha\rho_0(\mathbf{r}') = -\mathrm{i}\omega p_{0,\alpha}, \tag{22.22}$$

that is $\mathbf{g}^{(0)}$ can be expressed by the amplitude of the electric dipole moment

$$\mathbf{g}^{(0)} = -\mathrm{i}\omega\mathbf{p}_0. \qquad (22.23)$$

Thus one calls this contribution electric dipole radiation. One finds

$$\mathbf{A}_0(\mathbf{r}) = -\mathrm{i}k\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathbf{p}_0, \qquad (22.24)$$

thus

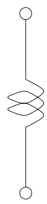
$$\mathbf{B}_0(\mathbf{r}) = \Big(\frac{k^2}{r} + \frac{\mathrm{i}k}{r^2}\Big)\mathrm{e}^{\mathrm{i}kr}\mathbf{n}\times\mathbf{p}_0 \qquad (22.25)$$

$$\mathbf{E}_0(\mathbf{r}) = -\frac{k^2}{r}\mathrm{e}^{\mathrm{i}kr}\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0) + \Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_0) - \mathbf{p}_0\Big)\Big(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\Big)\mathrm{e}^{\mathrm{i}kr}. \qquad (22.26)$$

The first term is leading in the far zone ($1/r \ll k$), the second one in the near zone ($1/r \gg k$). One obtains the time-averaged Poynting vector from the expression for the far zone

$$\bar{\mathbf{S}} = \frac{ck^4\mathbf{n}}{8\pi r^2}|\mathbf{n}\times\mathbf{p}_0|^2 = \frac{ck^4|\mathbf{p}_0|^2\mathbf{n}}{8\pi r^2}\sin^2\theta, \qquad (22.27)$$

In the second expression it is assumed that real and imaginary part of $\mathbf{p}_0$ point into the same direction. Then θ is the angle between $\mathbf{p}_0$ and $\mathbf{n}$. The radiated power is then



$$\dot{U}_{\mathrm{s}} = \frac{ck^4|\mathbf{p}_0|^2}{3}. \qquad (22.28)$$

The radiation increases with the fourth power of the frequency ($\omega = ck$) (Rayleigh radiation). As an example one may consider two capacitor spheres at distance l with $I(t) = \Re(I_0\mathrm{e}^{-\mathrm{i}\omega t})$. Then one has

$$|\mathbf{g}^{(0)}| = |\int \mathrm{d}^3r'\mathbf{j}_0(\mathbf{r}')| = |\int \mathrm{d}lI_0| = |I_0 l|, \quad p_0 = \frac{\mathrm{i}I_0 l}{\omega}, \quad \dot{U}_{\mathrm{s}} = \frac{(klI_0)^2}{3c} \qquad (22.29)$$

This power release yields a radiation resistance R_{s}

$$\dot{U}_{\mathrm{s}} = \frac{1}{2}R_{\mathrm{s}}I_0^2, \quad R_{\mathrm{s}} = \frac{2}{3c}(kl)^2 \hat{=} 20\Omega\cdot(kl)^2 \qquad (22.30)$$

in addition to an Ohmic resistance. Note that $\frac{1}{c}\hat{=}30\Omega$, compare (A.4).

22.c Magnetic Dipole radiation and Electric Quadrupole Radiation

Now we consider the second term in (22.16)

$$g_\alpha^{(1)} = n_\beta \int \mathrm{d}^3r' x'_\beta j_{0,\alpha}(\mathbf{r}')$$
$$= \frac{n_\beta}{2}\int \mathrm{d}^3r'(x'_\beta j_{0,\alpha} - x'_\alpha j_{0,\beta}) + \frac{n_\beta}{2}\int \mathrm{d}^3r'(x'_\beta j_{0,\alpha} + x'_\alpha j_{0,\beta}). \qquad (22.31)$$

The first term yields the magnetic dipole moment (10.7)

$$n_\beta c\epsilon_{\beta,\alpha,\gamma}m_{0,\gamma} = -c(\mathbf{n}\times\mathbf{m}_0)_\alpha. \qquad (22.32)$$

The second term can be expressed by the electric quadrupole moment (4.10). With $f = \frac{1}{2}x'_\alpha x'_\beta$ one obtains from (22.21)

$$-\mathrm{i}\omega\frac{n_\beta}{2}\int \mathrm{d}^3r' x'_\alpha x'_\beta \rho_0(\mathbf{r}') = -\mathrm{i}\omega\frac{n_\beta}{2}\Big(Q_{0,\alpha,\beta} + \frac{1}{3}\delta_{\alpha,\beta}\int \mathrm{d}^3r' r'^2\rho_0(\mathbf{r}')\Big). \qquad (22.33)$$

Thus we have

$$\mathbf{g}^{(1)} = -c\mathbf{n}\times\mathbf{m}_0 - \frac{\mathrm{i}\omega}{2}Q_{0,\alpha,\beta}n_\beta\mathbf{e}_\alpha + \text{ const. }\mathbf{n}. \qquad (22.34)$$

We observe that the third term proportional $\mathbf{n}$ does neither contribute to $\mathbf{B}_0$ (22.9) nor to $\mathbf{E}_0$ (22.11).

22.c.α Magnetic Dipole Radiation

The first contribution in (22.34) yields the magnetic dipole radiation. We obtain

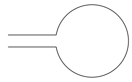
$$\mathbf{A}_0(\mathbf{r}) = \Big(\mathrm{i}k - \frac{1}{r}\Big)\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathbf{n}\times\mathbf{m}_0 \quad (22.35)$$

$$\mathbf{B}_0(\mathbf{r}) = -k^2\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathbf{n}\times(\mathbf{n}\times\mathbf{m}_0) + \Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}_0) - \mathbf{m}_0\Big)\Big(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\Big)\mathrm{e}^{\mathrm{i}kr} \quad (22.36)$$

$$\mathbf{E}_0(\mathbf{r}) = (-\frac{k^2}{r} - \frac{\mathrm{i}k}{r^2}\Big)(\mathbf{n}\times\mathbf{m}_0)\mathrm{e}^{\mathrm{i}kr}. \quad (22.37)$$

As an example we consider a current along a loop which includes the area f,

$$m_0 = I_0 f/c, \quad \dot{U}_\mathrm{s} = \frac{ck^4 m_0^2}{3} = \frac{k^4 I_0^2 f^2}{3c}, \quad (22.38)$$



which corresponds to a radiation resistance

$$R_\mathrm{s} = \frac{2}{3c}k^4 f^2 \mathrel{\hat{=}} 20\Omega\,(k^2 f)^2. \quad (22.39)$$

22.c.β Electric Quadrupole Radiation

We finally consider the second term in (22.34) in the far zone. It yields

$$\mathbf{g} = -\mathrm{i}k\mathbf{g}^{(1)} = -\frac{k^2 c}{2}Q_{0,\alpha,\beta}n_\beta\mathbf{e}_\alpha. \quad (22.40)$$

As special case we investigate the symmetric quadrupole (4.27), $Q_{0,x,x} = Q_{0,y,y} = -\frac{1}{3}Q_0$, $Q_{0,z,z} = \frac{2}{3}Q_0$, whereas the off-diagonal elements vanish. Then one has

$$Q_{0,\alpha,\beta} = -\frac{1}{3}Q_0\delta_{\alpha,\beta} + Q_0\delta_{\alpha,3}\delta_{\beta,3}, \quad (22.41)$$

from which

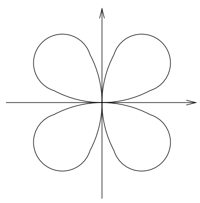
$$\mathbf{g} = -\frac{k^2 c}{2}Q_0 n_3\mathbf{e}_3 + \frac{k^2 c}{6}Q_0\mathbf{n}, \quad n_3 = \cos\theta \quad (22.42)$$

$$\mathbf{B}_0 = -\mathrm{i}k^3\frac{\mathrm{e}^{\mathrm{i}kr}}{2r}Q_0\mathbf{n}\times\mathbf{e}_3\cos\theta \quad (22.43)$$

$$\mathbf{E}_0 = \mathrm{i}k^3\frac{\mathrm{e}^{\mathrm{i}kr}}{2r}Q_0\mathbf{n}\times(\mathbf{n}\times\mathbf{e}_3)\cos\theta \quad (22.44)$$

$$\bar{\mathbf{S}} = \frac{ck^6\mathbf{n}}{32\pi r^2}|Q_0|^2\sin^2\theta\cos^2\theta \quad (22.45)$$

$$\dot{U}_\mathrm{s} = \frac{ck^6}{60}|Q_0|^2 \quad (22.46)$$



follows. The intensity of the quadrupole-radiation is radially sketched as function of the angle θ.