Classical Electrodynamics

Theoretical Physics II   Manuscript

English Edition

Franz Wegner
Institut für Theoretische Physik
Ruprecht-Karls-Universität Heidelberg
2003

I appreciate being informed of misprints.
I am grateful to Jörg Raufeisen, Andreas Haier, Stephan Frank, and Bastian Engeser for informing me of a number of misprints in the first German edition. Similarly I thank Björn Feuerbacher, Sebastian Diehl, Karsten Frese, Markus Gabrysch, and Jan Tomczak for informing me of misprints in the second edition.
I am indebted to Cornelia Merkel, Melanie Steiert, and Sonja Bartsch for carefully reading and correcting the text of the bilingual edition.

Books:

BECKER, SAUTER: Theorie der Elektrizität I
JACKSON, Classical Electrodynamics
LANDAU, LIFSCHITZ: Lehrbuch der Theoretischen Physik II: Klassische Feldtheorie
PANOFSKY, PHILLIPS, Classical Electricity and Magnetism
SOMMERFELD: Vorlesungen über Theoretische Physik III: Elektrodynamik
STRATTON, Electromagnetic Theory
STUMPF, SCHULER: Elektrodynamik

A Basic Equations

Introductory Remarks

I assume that the student is already somewhat familiar with classical electrodynamics from an introductory course. Therefore I start with the complete set of equations and from this set I spezialize to various cases of interest.

In this manuscript I will use Gaussian units instead of the SI-units. The connection between both systems and the motivation for using Gaussian units will be given in the next section and in appendix A.

Formulae for vector algebra and vector analysis are given in appendix B. A warning to the reader: Sometimes (B.11, B.15, B.34-B.50 and exercise after B.71) he/she should insert the result by him/herself. He/She is requested to perform the calculations by him/herself or should at least insert the results given in this script.

1 Basic Equations of Electrodynamics

Electrodynamics describes electric and magnetic fields, their generation by charges and electric currents, their propagation (electromagnetic waves), and their reaction on matter (forces).

1.a Charges and Currents

1.a.α Charge Density

The charge density ρ is defined as the charge Δq per volume element ΔV

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{\mathrm{d}q}{\mathrm{d}V}. \tag{1.1}$$

Therefore the charge q in the volume V is given by

$$q = \int_V \mathrm{d}^3 r \rho(\mathbf{r}). \tag{1.2}$$

If the charge distribution consists of point charges q_i at points $\mathbf{r}_i$, then the charge density is given by the sum

$$\rho(\mathbf{r}) = \sum_i q_i \delta^3(\mathbf{r}_i - \mathbf{r}), \tag{1.3}$$

where Dirac's delta-function (correctly delta-distribution) has the property

$$\int_V \mathrm{d}^3 r f(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{r}_0) = \left\{ \begin{array}{cl} f(\mathbf{r}_0) & \text{if } \mathbf{r}_0 \in V \\ 0 & \text{if } \mathbf{r}_0 \notin V \end{array} \right. . \tag{1.4}$$

Similarly one defines the charge density per area $\sigma(\mathbf{r})$ at boundaries and surfaces as charge per area

$$\sigma(\mathbf{r}) = \frac{\mathrm{d}q}{\mathrm{d}f}, \tag{1.5}$$

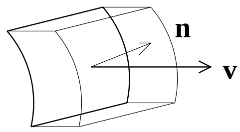
similarly the charge density on a line.

1.a.β Current and Current Density

The current I is the charge dq that flows through a certain area F per time dt,

$$I = \frac{\mathrm{d}q}{\mathrm{d}t}. \quad (1.6)$$

Be $\mathbf{v}(\mathbf{r}, t)$ the average velocity of the charge carriers and $\mathbf{n}$ the unit vector normal to the area element. Then $\mathbf{v}\mathrm{d}t$ is the distance vector traversed during time dt. Multiplied by $\mathbf{n}$ one obtains the thickness of the layer $\mathbf{v}\cdot\mathbf{n}\mathrm{d}t$ of the carriers which passed the surface during time dt.Multiplied by the surface element df one obtains the volume of the charge, which flows through the area. Additional multiplication by ρ yields the charge dq which passes during time dt the surface df



$$\mathrm{d}q = \int_F \mathbf{v}\mathrm{d}t \cdot \mathbf{n}\mathrm{d}f\rho \quad (1.7)$$

$$I = \mathrm{d}q/\mathrm{d}t = \int_F \mathbf{v}(\mathbf{r}, t)\rho(\mathbf{r}, t)\cdot\mathbf{n}(\mathbf{r})\mathrm{d}f = \int_F \mathbf{j}(\mathbf{r}, t)\cdot\mathrm{d}\mathbf{f} \quad (1.8)$$

with the current density $\mathbf{j} = \rho\mathbf{v}$ and the oriented area element $\mathrm{d}\mathbf{f} = \mathbf{n}\mathrm{d}f$.

1.a.γ Conservation of Charge and Equation of Continuity

The charge q in a fixed volume V

$$q(t) = \int_V \mathrm{d}^3r\rho(\mathbf{r}, t) \quad (1.9)$$

changes as a function of time by

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \int_V \mathrm{d}^3r\frac{\partial\rho(\mathbf{r}, t)}{\partial t}. \quad (1.10)$$

This charge can only change, if some charge flows through the surface ∂V of the volume, since charge is conserved. We denote the current which flows outward by I. Then

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = -I(t) = -\int_{\partial V}\mathbf{j}(\mathbf{r}, t)\cdot\mathrm{d}\mathbf{f} = -\int_V \mathrm{d}^3r\,\mathrm{div}\,\mathbf{j}(\mathbf{r}, t), \quad (1.11)$$

where we make use of the divergence theorem (B.59). Since (1.10) and (1.11) hold for any volume and volume element, the integrands in the volume integrals have to be equal

$$\frac{\partial\rho(\mathbf{r}, t)}{\partial t} + \mathrm{div}\,\mathbf{j}(\mathbf{r}, t) = 0. \quad (1.12)$$

This equation is called the equation of continuity. It expresses in differential form the conservation of charge.

1.b MAXWELL's Equations

The electric charges and currents generate the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic induction $\mathbf{B}(\mathbf{r}, t)$. This relation is described by the four MAXWELL Equations

$$\mathrm{curl}\,\mathbf{B}(\mathbf{r}, t) - \frac{\partial\mathbf{E}(\mathbf{r}, t)}{c\partial t} = \frac{4\pi}{c}\mathbf{j}(\mathbf{r}, t) \quad (1.13)$$

$$\mathrm{div}\,\mathbf{E}(\mathbf{r}, t) = 4\pi\rho(\mathbf{r}, t) \quad (1.14)$$

$$\mathrm{curl}\,\mathbf{E}(\mathbf{r}, t) + \frac{\partial\mathbf{B}(\mathbf{r}, t)}{c\partial t} = \mathbf{0} \quad (1.15)$$

$$\mathrm{div}\,\mathbf{B}(\mathbf{r}, t) = 0. \quad (1.16)$$

These equations named after MAXWELL are often called MAXWELL's Equations in the vacuum. However, they are also valid in matter. The charge density and the current density contain all contributions, the densities of free charges and polarization charges, and of free currents and polarization- and magnetization currents.
Often one requires as a boundary condition that the electric and the magnetic fields vanish at infinity.

1.c Coulomb and Lorentz Force

The electric field **E** and the magnetic induction **B** exert a force **K** on a charge q located at **r**, moving with a velocity **v**

$$\mathbf{K} = q\mathbf{E}(\mathbf{r}) + q\frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{r}). \tag{1.17}$$

Here **E** and **B** are the contributions which do not come from q itself. The fields generated by q itself exert the reaction force which we will not consider further.
The first contribution in (1.17) is the Coulomb force, the second one the Lorentz force. One has c= 299 792 458 m/s. Later we will see that this is the velocity of light in vacuum. (It has been defined with the value given above in order to introduce a factor between time and length.) The force acting on a small volume ΔV can be written as

$$\Delta\mathbf{K} = \mathbf{k}(\mathbf{r})\Delta V \tag{1.18}$$

$$\mathbf{k}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) + \frac{1}{c}\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}). \tag{1.19}$$

k is called the density of force. The electromagnetic force acting on the volume V is given by

$$\mathbf{K} = \int_V \mathrm{d}^3r\,\mathbf{k}(\mathbf{r}). \tag{1.20}$$

2 Dimensions and Units

2.a Gaussian Units

In this course we use Gaussian units. We consider the dimensions of the various quantities. From the equation of continuity (1.12) and Maxwell's equations (1.13 to 1.16) one obtains

$$[\rho]/[t] = [j]/[x] \tag{2.1}$$
$$[B]/[x] = [E]/([c][t]) = [j]/[c] \tag{2.2}$$
$$[E]/[x] = [B]/([c][t]) = [\rho]. \tag{2.3}$$

From this one obtains

$$[j] = [\rho][x]/[t] \tag{2.4}$$
$$[E] = [\rho][x] \tag{2.5}$$
$$[B] = [\rho][c][t] = [\rho][x]^2/([c][t]), \tag{2.6}$$

and

$$[c] = [x]/[t] \tag{2.7}$$
$$[B] = [\rho][x]. \tag{2.8}$$

From (2.7) one sees that c really has the dimension of a velocity. In order to determine the dimensions of the other quantities we still have to use expression (1.19) for the force density k

$$[k] = [\rho][E] = [\rho]^2[x]. \tag{2.9}$$

From this one obtains

$$[\rho]^2 = [k]/[x] = \text{dyn cm}^{-4} \tag{2.10}$$
$$[\rho] = \text{dyn}^{1/2}\ \text{cm}^{-2} \tag{2.11}$$
$$[E] = [B] = \text{dyn}^{1/2}\ \text{cm}^{-1} \tag{2.12}$$
$$[j] = \text{dyn}^{1/2}\ \text{cm}^{-1}\ \text{s}^{-1} \tag{2.13}$$
$$[q] = [\rho][x]^3 = \text{dyn}^{1/2}\ \text{cm} \tag{2.14}$$
$$[I] = [j][x]^2 = \text{dyn}^{1/2}\ \text{cm s}^{-1}. \tag{2.15}$$

2.b Other Systems of Units

The unit for each quantity can be defined independently. Fortunately, this is not used extensively.
Besides the Gaussian system of units a number of other cgs-systems is used as well as the SI-system (international system of units, Giorgi-system). The last one is the legal system in many countries (e.g. in the US since 1894, in Germany since 1898) and is used for technical purposes.
Whereas all electromagnetic quantities in the Gaussian system are expressed in cm, g und s, the Giorgi-system uses besides the mechanical units m, kg and s two other units, A (ampere) und V (volt). They are not independent, but related by the unit of energy

$$1\ \text{kg m}^2\ \text{s}^{-2} = 1\ \text{J} = 1\ \text{W s} = 1\ \text{A V s}. \tag{2.16}$$

The conversion of the conventional systems of units can be described by three conversion factors ϵ_0, μ_0 and ψ. The factors ϵ_0 and μ_0 (known as the dielectric constant and permeability constant of the vacuum in the SI-system) and the interlinking factor

$$\gamma = c\sqrt{\epsilon_0\mu_0} \tag{2.17}$$

can carry dimensions whereas ψ is a dimensionless number. One distinguishes between rational systems $\psi = 4\pi$) and non-rational systems ($\psi = 1$) of units. The conversion factors of some conventional systems of units are

System of Units	ϵ_0	μ_0	γ	ψ
GAUSSian	1	1	c	1
electrostatic (esu)	1	c^{-2}	1	1
electromagnetic (emu)	c^{-2}	1	1	1
HEAVISIDE-LORENTZ	1	1	c	4π
GIORGI (SI)	$(c^2\mu_0)^{-1}$	$\frac{4\pi}{10^7}\frac{\mathrm{Vs}}{\mathrm{Am}}$	1	4π

The quantities introduced until now are expressed in GAUSSian units by those of other systems of units (indicated by an asterisk) in the following way

$$\mathbf{E} = \sqrt{\psi\epsilon_0}\mathbf{E}^* \qquad 1\ \mathrm{dyn}^{1/2}\ \mathrm{cm}^{-1} \hat{=} 3\cdot 10^4 \mathrm{V/m} \tag{2.18}$$

$$\mathbf{B} = \sqrt{\psi/\mu_0}\mathbf{B}^* \qquad 1\ \mathrm{dyn}^{1/2}\ \mathrm{cm}^{-1} \hat{=} 10^{-4} \mathrm{Vs/m}^2 \tag{2.19}$$

$$q = \frac{1}{\sqrt{\psi\epsilon_0}} q^* \qquad 1\ \mathrm{dyn}^{1/2}\ \mathrm{cm} \hat{=} 10^{-9}/3\mathrm{As, similarly}\ \rho, \sigma, I, j. \tag{2.20}$$

An example of conversion: The COULOMB-LORENTZ-force can be written

$$\mathbf{K} = q(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}) = \frac{q^*}{\sqrt{\psi\epsilon_0}}(\sqrt{\psi\epsilon_0}\mathbf{E}^* + \frac{\sqrt{\psi}}{c\sqrt{\mu_0}}\mathbf{v}\times\mathbf{B}^*) = q^*(\mathbf{E}^* + \frac{1}{c\sqrt{\epsilon_0\mu_0}}\mathbf{v}\times\mathbf{B}^*) = q^*(\mathbf{E}^* + \frac{1}{\gamma}\mathbf{v}\times\mathbf{B}^*). \tag{2.21}$$

The elementary charge e_0 is $4.803 \cdot 10^{-10}$ $\mathrm{dyn}^{1/2}$ cm in GAUSSian units and $1.602 \cdot 10^{-19}$ As in SI-units. The electron carries charge $-e_0$, the proton e_0, a nucleus with Z protons the charge Ze_0, quarks the charges $\pm e_0/3$ and $\pm 2e_0/3$.
The conversion of other quantities is given where they are introduced. A summary is given in Appendix A.

2.c Motivation for GAUSSian Units

In the SI-system the electrical field **E** and the dielectric displacement **D** as well as the magnetic induction **B** and the magnetic field **H** carry different dimensions. This leads easily to the misleading impression that these are independent fields. On a microscopic level one deals only with two fields, **E** and **B**, (1.13-1.16) (LORENTZ 1892). However, the second set of fields is introduced only in order to extract the polarization and magnetization contributions of charges and currents in matter from the total charges and currents, and to add them to the fields. (Section 6 and 11).
This close relation is better expressed in cgs-units, where **E** and **D** have the same dimension, as well as **B** and **H**.
Unfortunately, the GAUSSian system belongs to the irrational ones, whereas the SI-system is a rational one, so that in conversions factors 4π appear. I would have preferred to use a rational system like that of HEAVISIDE and LORENTZ. However, in the usual textbooks only the SI-system and the GAUSSian one are used. I do not wish to offer the electrodynamics in a system which in practice is not used in other textbooks.

B Electrostatics

3 Electric Field, Potential, Energy of the Field

3.a Statics

First we consider the time-independent problem: Statics. This means, the quantities depend only on their location, $\rho = \rho(\mathbf{r})$, $\mathbf{j} = \mathbf{j}(\mathbf{r})$, $\mathbf{E} = \mathbf{E}(\mathbf{r})$, $\mathbf{B} = \mathbf{B}(\mathbf{r})$. Then the equation of continuity (1.12) and MAXWELL's equations (1.13-1.16) separate into two groups

$$\begin{array}{cc} \operatorname{div}\mathbf{j}(\mathbf{r}) = 0 & \\ \operatorname{curl}\mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r}) & \operatorname{div}\mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r}) \\ \operatorname{div}\mathbf{B}(\mathbf{r}) = 0 & \operatorname{curl}\mathbf{E}(\mathbf{r}) = \mathbf{0} \\ \text{magnetostatics} & \text{electrostatics} \\ \mathbf{k}_{\mathrm{ma}} = \frac{1}{c}\mathbf{j}(\mathbf{r})\times\mathbf{B}(\mathbf{r}) & \mathbf{k}_{\mathrm{el}} = \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) \end{array} \tag{3.1}$$

The first group of equations contains only the magnetic induction **B** and the current density **j**. It describes magnetostatics. The second group of equations contains only the electric field **E** and the charge density ρ. It is the basis of electrostatics. The expressions for the corresponding parts of the force density **k** is given in the last line.

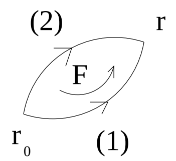
3.b Electric Field and Potential

3.b.α Electric Potential

Now we introduce the electric Potential $\Phi(\mathbf{r})$. For this purpose we consider the path integral over **E** along to different paths (1) and (2) from $\mathbf{r}_0$ to **r**

$$\int_{\mathbf{r}_0 \atop (1)}^{\mathbf{r}} \mathrm{d}\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) = \int_{\mathbf{r}_0 \atop (2)}^{\mathbf{r}} \mathrm{d}\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) + \oint \mathrm{d}\mathbf{r}\cdot\mathbf{E}(\mathbf{r}), \tag{3.2}$$

where the last integral has to be performed along the closed path from $\mathbf{r}_0$ along (1) to **r** and from there in opposite direction along (2) to $\mathbf{r}_0$.This later integral can be transformed by means of STOKES' theorem (B.56) into the integral $\int \mathrm{d}\mathbf{f}\cdot\operatorname{curl}\mathbf{E}(\mathbf{r})$ over the open surface bounded by (1) and (2), which vanishes due to MAXWELL's equation $\operatorname{curl}\mathbf{E}(\mathbf{r}) = \mathbf{0}$ (3.1).



Therefore the integral (3.2) is independent of the path and one defines the electric potential

$$\Phi(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathrm{d}\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) + \Phi(\mathbf{r}_0). \tag{3.3}$$

The choice of $\mathbf{r}_0$ and of $\Phi(\mathbf{r}_0)$ is arbitrary, but fixed. Therefore $\Phi(\mathbf{r})$ is defined apart from an arbitrary additive constant. From the definition (3.3) we have

$$\mathrm{d}\Phi(\mathbf{r}) = -\mathrm{d}\mathbf{r}\cdot\mathbf{E}(\mathbf{r}), \quad \mathbf{E}(\mathbf{r}) = -\operatorname{grad}\Phi(\mathbf{r}). \tag{3.4}$$

3.b.β Electric Flux and Charge

From $\operatorname{div}\mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r})$, (3.1) one obtains

$$\int_V d^3r \operatorname{div}\mathbf{E}(\mathbf{r}) = 4\pi \int_V d^3r\rho(\mathbf{r}) \tag{3.5}$$

and therefore with the divergence theorem (B.59)

$$\int_{\partial V} d\mathbf{f}\cdot\mathbf{E}(\mathbf{r}) = 4\pi q(V), \tag{3.6}$$

id est the electric flux of the field $\mathbf{E}$ through the surface equals 4π times the charge q in the volume V.
This has a simple application for the electric field of a rotational invariant charge distribution $\rho(\mathbf{r}) = \rho(r)$ with $r = |\mathbf{r}|$. For reasons of symmetry the electric field points in radial direction, $\mathbf{E} = E(r)\mathbf{r}/r$

$$4\pi r^2 E(r) = 4\pi \int_0^r \rho(r')r'^2 dr' d\Omega = (4\pi)^2 \int_0^r \rho(r')r'^2 dr', \tag{3.7}$$

so that one obtains

$$E(r) = \frac{4\pi}{r^2}\int_0^r \rho(r')r'^2 dr' \tag{3.8}$$

for the field.
As a special case we consider a point charge in the origin. Then one has

$$4\pi r^2 E(r) = 4\pi q, \quad E(r) = \frac{q}{r^2}, \quad \mathbf{E}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}q. \tag{3.9}$$

The potential depends only on r for reasons of symmetry. Then one obtains

$$\operatorname{grad}\Phi(r) = \frac{\mathbf{r}}{r}\frac{d\Phi(r)}{dr} = -\mathbf{E}(\mathbf{r}), \tag{3.10}$$

which after integration yields

$$\Phi(r) = \frac{q}{r} + \text{const.} \tag{3.11}$$

3.b.γ Potential of a Charge Distribution

We start out from point charges q_i at locations $\mathbf{r}_i$. The corresponding potential and the field is obtained from (3.11) und (3.10) by shifting $\mathbf{r}$ by $\mathbf{r}_i$

$$\Phi(\mathbf{r}) = \sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|} \tag{3.12}$$

$$\mathbf{E}(\mathbf{r}) = -\operatorname{grad}\Phi(\mathbf{r}) = \sum_i \frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3}. \tag{3.13}$$

We change now from point charges to the charge density $\rho(\mathbf{r})$. To do this we perform the transition from $\sum_i q_i f(\mathbf{r}_i) = \sum_i \Delta V\rho(\mathbf{r}_i) f(\mathbf{r}_i)$ to $\int d^3r'\rho(\mathbf{r}') f(\mathbf{r}')$, which yields

$$\Phi(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{3.14}$$

From $\mathbf{E} = -\operatorname{grad}\Phi$ and $\operatorname{div}\mathbf{E} = 4\pi\rho$ one obtains POISSON's equation

$$\triangle\Phi(\mathbf{r}) = -4\pi\rho(\mathbf{r}). \tag{3.15}$$

Please distinguish $\triangle = \nabla\cdot\nabla$ and Δ =Delta. We check eq. (3.15). First we determine

$$\nabla\Phi(\mathbf{r}) = \int d^3r'\rho(\mathbf{r}')\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} = \int d^3a\,\rho(\mathbf{r}+\mathbf{a})\frac{\mathbf{a}}{a^3} \tag{3.16}$$

and

$$\triangle\Phi(\mathbf{r}) = \int \mathrm{d}^3a(\nabla\rho(\mathbf{r}+\mathbf{a}))\cdot\frac{\mathbf{a}}{a^3} = \int_0^\infty \mathrm{d}a \int d\Omega_a \frac{\partial\rho(\mathbf{r}+\mathbf{a})}{\partial a} = \int \mathrm{d}\Omega_a(\rho(\mathbf{r}+\infty\mathbf{e}_a)-\rho(\mathbf{r})) = -4\pi\rho(\mathbf{r}), \quad (3.17)$$

assuming that ρ vanishes at infinity. The three-dimensional integral over a has been separated by the integral over the radius a and the solid angle Ω_a, $\mathrm{d}^3a = a^2\mathrm{d}a\mathrm{d}\Omega$ (compare section 5).
From POISSON's equation one obtains

$$\triangle\Phi(\mathbf{r}) = \int \mathrm{d}^3r'\rho(\mathbf{r}')\triangle\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\rho(\mathbf{r}) = -4\pi\int \mathrm{d}^3r'\rho(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}') \quad (3.18)$$

and from the equality of the integrands

$$\triangle\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \quad (3.19)$$

3.c COULOMB Force and Field Energy

The force acting on the charge q_i at $\mathbf{r}_i$ is

$$\mathbf{K}_i = q_i\mathbf{E}_i(\mathbf{r}_i). \quad (3.20)$$

Here $\mathbf{E}_i$ is the electric field without that generated by the charge q_i itself. Then one obtains the COULOMB force

$$\mathbf{K}_i = q_i\sum_{j\neq i}\frac{q_j(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^3}. \quad (3.21)$$

From this equation one realizes the definition of the unit of charge in GAUSS's units, 1 $\mathrm{dyn}^{1/2}$ cm is the charge, which exerts on the same amount of charge in the distance of 1 cm the force 1 dyn.
The potential energy is

$$U = \frac{1}{2}\sum_i\sum_{j\neq i}\frac{q_iq_j}{|\mathbf{r}_i-\mathbf{r}_j|} = \frac{1}{2}\sum_i q_i\Phi_i(\mathbf{r}_i). \quad (3.22)$$

The factor 1/2 is introduced since each pair of charges appears twice in the sum. E.g., the interaction energy between charge 1 and charge 2 is contained both in $i = 1, j = 2$ and $i = 2, j = 1$. Thus we have to divide by 2. The contribution from q_i is excluded from the potential Φ_i. The force is then as usually

$$\mathbf{K}_i = -\,\mathrm{grad}_{\mathbf{r}_i}U. \quad (3.23)$$

In the continuum one obtains by use of (B.62)

$$U = \frac{1}{2}\int \mathrm{d}^3r\rho(\mathbf{r})\Phi(\mathbf{r}) = \frac{1}{8\pi}\int \mathrm{d}^3r\,\mathrm{div}\,\mathbf{E}(\mathbf{r})\Phi(\mathbf{r}) = \frac{1}{8\pi}\int_F \mathrm{d}\mathbf{f}\cdot\mathbf{E}(\mathbf{r})\Phi(\mathbf{r}) - \frac{1}{8\pi}\int \mathrm{d}^3r\mathbf{E}(\mathbf{r})\cdot\,\mathrm{grad}\,\Phi(\mathbf{r}), \quad (3.24)$$

where no longer the contribution from the charge density at the same location has to be excluded from Φ, since it is negligible for a continuous distribution. F should include all charges and may be a sphere of radius R. In the limit $R\to\infty$ one obtains $\Phi \propto 1/R$, $E \propto 1/R^2$, $\int_F \propto 1/R \to 0$. Then one obtains the electrostatic energy

$$U = \frac{1}{8\pi}\int \mathrm{d}^3rE^2(\mathbf{r}) = \int \mathrm{d}^3r\,u(\mathbf{r}) \quad (3.25)$$

with the energy density

$$u(\mathbf{r}) = \frac{1}{8\pi}E^2(\mathbf{r}). \quad (3.26)$$

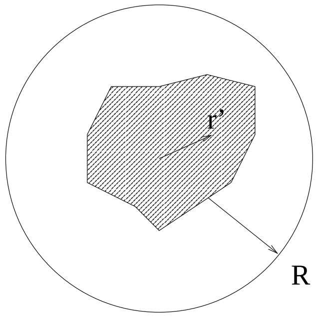
Classical Radius of the Electron As an example we consider the "classical radius of an electron" R_0: One assumes that the charge is homogeneously distributed on the surface of the sphere of radius R. The electric field energy should equal the energy m_0c^2, where m_0 is the mass of the electron.

$$\frac{1}{8\pi}\int_{R_0}^\infty\left(\frac{e_0}{r^2}\right)^2 r^2\mathrm{d}r\mathrm{d}\Omega = \frac{e_0^2}{2R_0} = m_0c^2 \quad (3.27)$$

yields $R_0 = 1.4\cdot10^{-13}$ cm. The assumption of a homogeneous distribution of the charge inside the sphere yields a slightly different result.
From scattering experiments at high energies one knows that the extension of the electron is at least smaller by a factor of 100, thus the assumption made above does not apply.

4 Electric Dipole and Quadrupole

A charge distribution $\rho(\mathbf{r}')$ inside a sphere of radius R around the origin is given. We assume $\rho(\mathbf{r}') = 0$ outside the sphere.



4.a The Field for $r > R$

The potential of the charge distribution is

$$\Phi(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.1}$$

We perform a TAYLOR-expansion in $\mathbf{r}'$, i.e. in the three variables x_1', x_2' und x_3'

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{(-\mathbf{r}'\nabla)^l}{l!}\frac{1}{r} = \frac{1}{r} - (\mathbf{r}'\nabla)\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\nabla)(\mathbf{r}'\nabla)\frac{1}{r} - ... \tag{4.2}$$

At first we have to calculate the gradient of $1/r$

$$\nabla\frac{1}{r} = -\frac{\mathbf{r}}{r^3}, \text{ since } \nabla f(r) = \frac{\mathbf{r}}{r}f'(r), \tag{4.3}$$

solve (B.39, B.42). Then one obtains

$$(\mathbf{r}'\nabla)\frac{1}{r} = -\frac{\mathbf{r}'\cdot\mathbf{r}}{r^3}. \tag{4.4}$$

Next we calculate (B.47)

$$\nabla\frac{\mathbf{c}\cdot\mathbf{r}}{r^3} = \frac{1}{r^3}\,\mathrm{grad}\,(\mathbf{c}\cdot\mathbf{r}) + (\mathbf{c}\cdot\mathbf{r})\,\mathrm{grad}\left(\frac{1}{r^3}\right) = \frac{\mathbf{c}}{r^3} - \frac{3(\mathbf{c}\cdot\mathbf{r})\mathbf{r}}{r^5} \tag{4.5}$$

using (B.27) and the solutions of (B.37, B.39). Then we obtain the TAYLOR-expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2r'^2}{2r^5} + ... \tag{4.6}$$

At first we transform $3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2r'^2$

$$3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2r'^2 = x_\alpha' x_\beta'(3x_\alpha x_\beta - r^2\delta_{\alpha,\beta}) = (x_\alpha' x_\beta' - \frac{1}{3}r'^2\delta_{\alpha,\beta})(3x_\alpha x_\beta - r^2\delta_{\alpha,\beta}) \tag{4.7}$$

because of $\delta_{\alpha,\beta}(3x_\alpha x_\beta - \delta_{\alpha,\beta}r^2) = 3x_\alpha x_\alpha - r^2\delta_{\alpha,\alpha} = 0$. Here and in the following we use the summation convention, i.e. we sum over all indices (of components), which appear twice in a product in (4.7), that is over α and β.

We now introduce the quantities

$$q = \int d^3r' \rho(\mathbf{r}') \qquad \text{charge} \tag{4.8}$$

$$\mathbf{p} = \int d^3r' \mathbf{r}' \rho(\mathbf{r}') \qquad \text{dipolar moment} \tag{4.9}$$

$$Q_{\alpha,\beta} = \int d^3r' (x_\alpha' x_\beta' - \frac{1}{3}\delta_{\alpha,\beta}r'^2)\rho(\mathbf{r}') \qquad \text{components of the quadrupolar moment} \tag{4.10}$$

and obtain the expansion for the potential and the electric field

$$\Phi(\mathbf{r}) = \frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + Q_{\alpha,\beta}\frac{3x_\alpha x_\beta - r^2\delta_{\alpha,\beta}}{2r^5} + O(\frac{1}{r^4}) \tag{4.11}$$

$$\mathbf{E}(\mathbf{r}) = -\,\mathrm{grad}\,\Phi(\mathbf{r}) = \frac{q\mathbf{r}}{r^3} + \frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - \mathbf{p}r^2}{r^5} + O(\frac{1}{r^4}) \tag{4.12}$$

4.b Transformation Properties

The multipole moments are defined with respect to a given point, for example with respect to the origin. If one shifts the point of reference by $\mathbf{a}$, i.e. $\mathbf{r}'_1 = \mathbf{r}' - \mathbf{a}$, then one finds with $\rho_1(\mathbf{r}'_1) = \rho(\mathbf{r}')$

$$q_1 = \int d^3r'_1 \rho_1(\mathbf{r}'_1) = \int d^3r' \rho(\mathbf{r}') = q \tag{4.13}$$

$$\mathbf{p}_1 = \int d^3r'_1 \mathbf{r}'_1 \rho_1(\mathbf{r}'_1) = \int d^3r' (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}') = \mathbf{p} - \mathbf{a}q. \tag{4.14}$$

The total charge is independent of the point of reference. The dipolar moment is independent of the point of reference if $q = 0$ (pure dipol), otherwise it depends on the point of reference. Similarly one finds that the quadrupolar moment is independent of the point of reference, if $q = 0$ and $\mathbf{p} = 0$ (pure quadrupole).
The charge q is invariant under rotation (scalar) $x'_{1,\alpha} = D_{\alpha,\beta} x'_\beta$, where D is a rotation matrix, which describes an orthogonal transformation. The dipole $\mathbf{p}$ transforms like a vector

$$p_{1,\alpha} = \int d^3r' D_{\alpha,\beta} x'_\beta \rho(\mathbf{r}') = D_{\alpha,\beta} p_\beta \tag{4.15}$$

and the quadrupole Q like a tensor of rank 2

$$Q_{1,\alpha,\beta} = \int d^3r' (D_{\alpha,\gamma} x'_\gamma D_{\beta,\delta} x'_\delta - \frac{1}{3}\delta_{\alpha,\beta} r'^2)\rho(\mathbf{r}'). \tag{4.16}$$

Taking into account that due to the orthogonality of D

$$\delta_{\alpha,\beta} = D_{\alpha,\gamma} D_{\beta,\gamma} = D_{\alpha,\gamma}\delta_{\gamma,\delta} D_{\beta,\delta}, \tag{4.17}$$

it follows that

$$Q_{1,\alpha,\beta} = D_{\alpha,\gamma} D_{\beta,\delta} Q_{\gamma,\delta}, \tag{4.18}$$

that is the transformation law for tensors of second rank.

4.c Dipole

The prototype of a dipole consists of two charges of opposite sign, q at $\mathbf{r}_0 + \mathbf{a}$ and $-q$ at $\mathbf{r}_0$.

$$\mathbf{p} = q\mathbf{a}. \tag{4.19}$$

Therefore the corresponding charge distribution is

$$\rho(\mathbf{r}) = q(\delta^3(\mathbf{r} - \mathbf{r}_0 - \mathbf{a}) - \delta^3(\mathbf{r} - \mathbf{r}_0)). \tag{4.20}$$

We perform now the TAYLOR expansion in $\mathbf{a}$

$$\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}_0) - q\mathbf{a}\cdot\nabla\delta^3(\mathbf{r} - \mathbf{r}_0) + \frac{q}{2}(\mathbf{a}\cdot\nabla)^2\delta^3(\mathbf{r} - \mathbf{r}_0) + ... - q\delta^3(\mathbf{r} - \mathbf{r}_0), \tag{4.21}$$

where the first and the last term cancel. We consider now the limit $\mathbf{a} \to 0$, where the product $q\mathbf{a} = \mathbf{p}$ is kept fixed. Then we obtain the charge distribution of a dipole $\mathbf{p}$ at location $\mathbf{r}_0$

$$\rho(\mathbf{r}) = -\mathbf{p}\cdot\nabla\delta^3(\mathbf{r} - \mathbf{r}_0) \tag{4.22}$$

and its potential

$$\begin{aligned}\Phi(\mathbf{r}) &= \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = -\mathbf{p}\cdot\int d^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|}\,\mathrm{grad}'\delta^3(\mathbf{r}' - \mathbf{r}_0) = \mathbf{p}\cdot\int d^3r'\,\mathrm{grad}'\frac{1}{|\mathbf{r} - \mathbf{r}'|}\delta^3(\mathbf{r}' - \mathbf{r}_0)\\ &= \mathbf{p}\cdot\int d^3r' \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\delta^3(\mathbf{r}' - \mathbf{r}_0) = \frac{\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3},\end{aligned} \tag{4.23}$$

where equation (B.61) is used and (B.50) has to be solved.

4.d Quadrupole

The quadrupole is described by the second moment of the charge distribution.

4.d.α Symmetries

Q is a symmetric tensor

$$Q_{\alpha,\beta} = Q_{\beta,\alpha}. \tag{4.24}$$

It can be diagonalized by an orthogonal transformation similarly as the tensor of inertia. Further from definition (4.10) it follows that

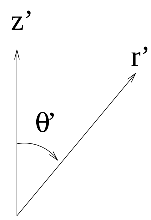
$$Q_{\alpha,\alpha} = 0, \tag{4.25}$$

that is the trace of the quadrupole tensor vanishes. Thus the tensor does not have six, but only five independent components.

4.d.β Symmetric Quadrupole

A special case is the symmetric quadrupole. Its charge distribution depends only on z and on the distance from the z-axis, $\rho = \rho(z, \sqrt{x^2+y^2})$. It obeys

$$Q_{x,y} = Q_{x,z} = Q_{y,z} = 0, \tag{4.26}$$



because $\rho(x,y,z) = \rho(-x,y,z) = \rho(x,-y,z)$. Furthermore one has

$$Q_{x,x} = Q_{y,y} = -\frac{1}{2}Q_{z,z} =: -\frac{1}{3}\hat{Q}. \tag{4.27}$$

The first equality follows from $\rho(x,y,z) = \rho(y,x,z)$, the second one from the vanishing of the trace of Q. The last equality-sign gives the definition of $\hat{Q}$.

One finds

$$\hat{Q} = \frac{3}{2}Q_{z,z} = \int d^3r'(\frac{3}{2}z'^2 - \frac{1}{2}r'^2)\rho(\mathbf{r}') = \int d^3r'r'^2P_2(\cos\theta')\rho(\mathbf{r}') \tag{4.28}$$

with the Legendre polynomial $P_2(\xi) = \frac{3}{2}\xi^2 - \frac{1}{2}$. We will return to the Legendre polynomials in the next section and in appendix C.

As an example we consider the stretched quadrupole with two charges q at $\pm a\mathbf{e}_z$ and a charge $-2q$ in the origin. Then we obtain $\hat{Q} = 2qa^2$. The different charges contribute to the potential of the quadrupole

$$\Phi(\mathbf{r}) = -\frac{1}{3}\hat{Q}\frac{3x^2-r^2}{2r^5} - \frac{1}{3}\hat{Q}\frac{3y^2-r^2}{2r^5} + \frac{2}{3}\hat{Q}\frac{3z^2-r^2}{2r^5} = \frac{\hat{Q}P_2(\cos\theta)}{r^3}. \tag{4.29}$$

4.e Energy, Force and Torque on a Multipole in an external Field

A charge distribution $\rho(\mathbf{r})$ localized around the origin is considered in an external electric potential $\Phi_a(\mathbf{r})$, which may be generated by an external charge distribution ρ_a. The interaction energy is then given by

$$U = \int d^3r\rho(\mathbf{r})\Phi_a(\mathbf{r}). \tag{4.30}$$

No factor 1/2 appears in front of the integral, which might be expected in view of this factor in (3.24), since besides the integral over $\rho(\mathbf{r})\Phi_a(\mathbf{r})$ there is a second one over $\rho_a(\mathbf{r})\Phi(\mathbf{r})$, which yields the same contribution. We now expand the external potential and obtain for the interaction energy

$$\begin{aligned} U &= \int d^3r\rho(\mathbf{r})\left\{\Phi_a(0) + \mathbf{r}\nabla\Phi_a|_{r=0} + \frac{1}{2}x_\alpha x_\beta\ \nabla_\alpha\nabla_\beta\Phi_a\Big|_{r=0} + ...\right\} \\ &= q\Phi_a(0) + \mathbf{p}\cdot\nabla\Phi_a|_{r=0} + \frac{1}{2}\left(Q_{\alpha,\beta} + \frac{1}{3}\delta_{\alpha,\beta}\int d^3r\rho(\mathbf{r})r^2\right)\nabla_\alpha\nabla_\beta\Phi_a\Big|_{r=0} + ... \end{aligned} \tag{4.31}$$

The contribution proportional to the integral over $\rho(\mathbf{r})r^2$ vanishes, since $\nabla_\alpha\nabla_\alpha\Phi_a = \triangle\Phi_a = -4\pi\rho_a(\mathbf{r}) = 0$, since there are no charges at the origin, which generate Φ_a. Therefore we are left with the potential of interaction

$$U = q\Phi_a(0) - \mathbf{p}\cdot\mathbf{E}_a(0) + \frac{1}{2}Q_{\alpha,\beta}\nabla_\alpha\nabla_\beta\Phi_a + ... \tag{4.32}$$

For example we can now determine the potential energy between two dipoles, $\mathbf{p}_b$ in the origin and $\mathbf{p}_a$ at $\mathbf{r}_0$. The dipole $\mathbf{p}_a$ generates the potential

$$\Phi_a(\mathbf{r}) = \frac{\mathbf{p}_a\cdot(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}. \tag{4.33}$$

Then the interaction energy yields (compare B.47)

$$U_{a,b} = \mathbf{p}_b\cdot\nabla\Phi_a|_{r=0} = \frac{\mathbf{p}_a\cdot\mathbf{p}_b}{r_0^3} - \frac{3(\mathbf{p}_a\cdot\mathbf{r}_0)(\mathbf{p}_b\cdot\mathbf{r}_0)}{r_0^5}. \tag{4.34}$$

The force on the dipole in the origin is then given by

$$\mathbf{K} = \int d^3r\rho(\mathbf{r})\mathbf{E}_a(\mathbf{r}) = \int d^3r\rho(\mathbf{r})(\mathbf{E}_a(0) + x_\alpha\nabla_\alpha\mathbf{E}_a|_{r=0} + ...) = q\mathbf{E}_a(0) + (\mathbf{p}\cdot\ \mathrm{grad}\,)\mathbf{E}_a(0) + ... \tag{4.35}$$

The torque on a dipole in the origin is given by

$$\mathbf{M}_{\mathrm{mech}} = \int d^3r'\rho(\mathbf{r}')\mathbf{r}'\times\mathbf{E}_a(\mathbf{r}') = \mathbf{p}\times\mathbf{E}_a(0) + ... \tag{4.36}$$

5 Multipole Expansion in Spherical Coordinates

5.a Poisson Equation in Spherical Coordinates

We first derive the expression for the Laplacian operator in spherical coordinates

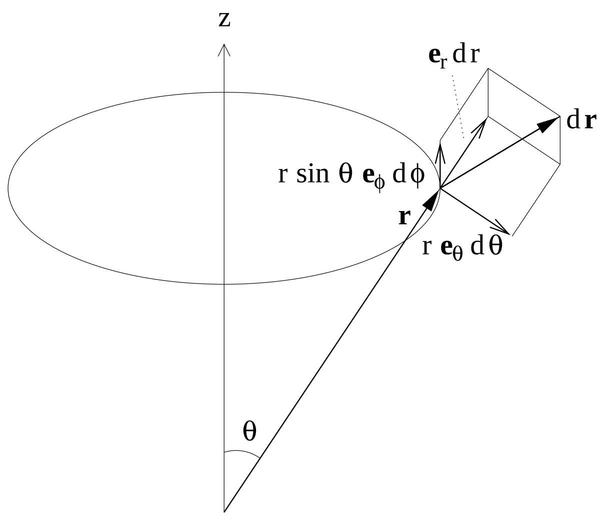
$$x = r\sin\theta\cos\phi \qquad (5.1)$$
$$y = r\sin\theta\sin\phi \qquad (5.2)$$
$$z = r\cos\theta. \qquad (5.3)$$

Initially we use only that we deal with curvilinear coordinates which intersect at right angles, so that we may write

$$\mathrm{d}\mathbf{r} = g_r\mathbf{e}_r\mathrm{d}r + g_\theta\mathbf{e}_\theta\mathrm{d}\theta + g_\phi\mathbf{e}_\phi\mathrm{d}\phi \qquad (5.4)$$

where the $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ constitute an orthonormal space dependent basis. Easily one finds

$$g_r = 1, \quad g_\theta = r, \quad g_\phi = r\sin\theta. \qquad (5.5)$$



The volume element is given by

$$\mathrm{d}^3r = g_r\mathrm{d}r g_\theta\mathrm{d}\theta g_\phi\mathrm{d}\phi = r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\phi = r^2\mathrm{d}r\mathrm{d}\Omega \qquad (5.6)$$

with the element of the solid angle

$$\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi. \qquad (5.7)$$

5.a.α The Gradient

In order to determine the gradient we consider the differential of the function $\Phi(\mathbf{r})$

$$\mathrm{d}\Phi(\mathbf{r}) = \frac{\partial\Phi}{\partial r}\mathrm{d}r + \frac{\partial\Phi}{\partial\theta}\mathrm{d}\theta + \frac{\partial\Phi}{\partial\phi}\mathrm{d}\phi, \qquad (5.8)$$

which coincides with $(\,\mathrm{grad}\,\Phi)\cdot\mathrm{d}\mathbf{r}$. From the expansion of the vector field in its components

$$\mathrm{grad}\,\Phi = (\,\mathrm{grad}\,\Phi)_r\mathbf{e}_r + (\,\mathrm{grad}\,\Phi)_\theta\mathbf{e}_\theta + (\,\mathrm{grad}\,\Phi)_\phi\mathbf{e}_\phi \qquad (5.9)$$

and (5.4) it follows that

$$\mathrm{d}\Phi(\mathbf{r}) = (\,\mathrm{grad}\,\Phi)_r g_r\mathrm{d}r + (\,\mathrm{grad}\,\Phi)_\theta g_\theta\mathrm{d}\theta + (\,\mathrm{grad}\,\Phi)_\phi g_\phi\mathrm{d}\phi, \qquad (5.10)$$

from which we obtain

$$(\,\mathrm{grad}\,\Phi)_r = \frac{1}{g_r}\frac{\partial\Phi}{\partial r}, \quad (\,\mathrm{grad}\,\Phi)_\theta = \frac{1}{g_\theta}\frac{\partial\Phi}{\partial\theta}, \quad (\,\mathrm{grad}\,\Phi)_\phi = \frac{1}{g_\phi}\frac{\partial\Phi}{\partial\phi} \qquad (5.11)$$

for the components of the gradient.

5.a.β The Divergence

In order to calculate the divergence we use the divergence theorem (B.59). We integrate the divergence of $\mathbf{A}(\mathbf{r})$ in a volume limited by the coordinates $r, r+\Delta r,\ \theta, \theta+\Delta\theta,\ \phi, \phi+\Delta\phi$. We obtain

$$\begin{aligned}\int \mathrm{d}^3r\,\mathrm{div}\,\mathbf{A} &= \int g_r g_\theta g_\phi\,\mathrm{div}\,\mathbf{A}\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\ = \int \mathbf{A}\cdot\mathrm{d}\mathbf{f} &= \int g_\theta\mathrm{d}\theta g_\phi\mathrm{d}\phi A_r\Big|_r^{r+\Delta r} + \int g_r\mathrm{d}r g_\phi\mathrm{d}\phi A_\theta\Big|_\theta^{\theta+\Delta\theta} + \int g_r\mathrm{d}r g_\theta\mathrm{d}\theta A_\phi\Big|_\phi^{\phi+\Delta\phi} \\ &= \int\left[\frac{\partial}{\partial r}\left(g_\theta g_\phi A_r\right) + \frac{\partial}{\partial\theta}\left(g_r g_\phi A_\theta\right) + \frac{\partial}{\partial\phi}\left(g_r g_\theta A_\phi\right)\right]\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \qquad (5.12)\end{aligned}$$

Since the identity holds for arbitrarily small volumina the integrands on the right-hand side of the first line and on the third line have to agree which yields

$$\mathrm{div}\,\mathbf{A}(\mathbf{r}) = \frac{1}{g_r g_\theta g_\phi}\left[\frac{\partial}{\partial r}\left(g_\theta g_\phi A_r\right) + \frac{\partial}{\partial\theta}\left(g_r g_\phi A_\theta\right) + \frac{\partial}{\partial\phi}\left(g_r g_\theta A_\phi\right)\right]. \qquad (5.13)$$

5.a.γ The Laplacian

Using $\triangle\Phi = \mathrm{div}\ \mathrm{grad}\,\Phi$ we obtain finally

$$\triangle\Phi(\mathbf{r}) = \frac{1}{g_r g_\theta g_\phi}\left[\frac{\partial}{\partial r}\left(\frac{g_\theta g_\phi}{g_r}\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{g_r g_\phi}{g_\theta}\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{g_r g_\theta}{g_\phi}\frac{\partial\Phi}{\partial\phi}\right)\right]. \qquad (5.14)$$

This equation holds generally for curvilinear orthogonal coordinates (if we denote them by r, θ, ϕ). Substituting the values for g we obtain for spherical coordinates

$$\triangle\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2}\triangle_\Omega\Phi, \qquad (5.15)$$

$$\triangle_\Omega\Phi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}. \qquad (5.16)$$

The operator $\triangle_\Omega$ acts only on the two angels θ and ϕ, but not on the distance r. Therefore it is also called Laplacian on the sphere.

5.b Spherical Harmonics

As will be explained in more detail in appendix C there is a complete set of orthonormal functions $Y_{l,m}(\theta,\phi)$, $l = 0, 1, 2, ..., m = -l, -l+1, ...l$, which obey the equation

$$\triangle_\Omega Y_{l,m}(\theta,\phi) = -l(l+1)Y_{l,m}(\theta,\phi). \qquad (5.17)$$

They are called spherical harmonics. Completeness means: If $f(\theta,\phi)$ is differentiable on the sphere and its derivatives are bounded, then $f(\theta,\phi)$ can be represented as a convergent sum

$$f(\theta,\phi) = \sum_{l,m}\hat{f}_{l,m}Y_{l,m}(\theta,\phi). \qquad (5.18)$$

Therefore we perform the corresponding expansion for $\Phi(\mathbf{r})$ and $\rho(\mathbf{r})$

$$\Phi(\mathbf{r}) = \sum_{l,m}\hat{\Phi}_{l,m}(r)Y_{l,m}(\theta,\phi), \qquad (5.19)$$

$$\rho(\mathbf{r}) = \sum_{l,m}\hat{\rho}_{l,m}(r)Y_{l,m}(\theta,\phi). \qquad (5.20)$$

The spherical harmonics are orthonormal, i.e. the integral over the solid angle yields

$$\int d\Omega Y^*_{l,m}(\theta,\phi)Y_{l',m'}(\theta,\phi) = \int d\phi \sin\theta d\theta Y^*_{l,m}(\theta,\phi)Y_{l',m'}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}. \tag{5.21}$$

This orthogonality relation can be used for the calculation of $\hat{\Phi}$ and $\hat{\rho}$

$$\begin{aligned} \int d\phi \sin\theta d\theta Y^*_{l,m}(\theta,\phi)\rho(\mathbf{r}) &= \sum_{l',m'}\hat{\rho}_{l',m'}(r)\int d\phi \sin\theta d\theta Y^*_{l,m}(\theta,\phi)Y_{l',m'}(\theta,\phi) \\ = \sum_{l',m'}\hat{\rho}_{l',m'}(r)\delta_{l,l'}\delta_{m,m'} &= \hat{\rho}_{l,m}(r). \end{aligned} \tag{5.22}$$

We list some of the spherical harmonics

$$Y_{0,0}(\theta,\phi) = \sqrt{\frac{1}{4\pi}} \tag{5.23}$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{5.24}$$

$$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{5.25}$$

$$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \tag{5.26}$$

$$Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \tag{5.27}$$

$$Y_{2,\pm2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\phi}. \tag{5.28}$$

In general one has

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^m_l(\cos\theta)e^{im\phi} \tag{5.29}$$

with the associated Legendre functions

$$P^m_l(\xi) = \frac{(-)^m}{2^l l!}(1-\xi^2)^{m/2}\frac{d^{l+m}}{d\xi^{l+m}}(\xi^2-1)^l. \tag{5.30}$$

Generally $Y_{l,m}$ is a product of $(\sin\theta)^{|m|}e^{im\phi}$ and a polynomial of order $l-|m|$ in $\cos\theta$. If $l-|m|$ is even (odd), then this polynomial is even (odd) in $\cos\theta$. There is the symmetry relation

$$Y_{l,-m}(\theta,\phi) = (-)^m Y^*_{l,m}(\theta,\phi). \tag{5.31}$$

5.c Radial Equation and Multipole Moments

Using the expansion of Φ and ρ in spherical harmonics the Poisson equation reads

$$\triangle\Phi(\mathbf{r}) = \sum_{l,m}\left(\frac{1}{r}\frac{d^2}{dr^2}(r\hat{\Phi}_{l,m}(r)) - \frac{l(l+1)}{r^2}\hat{\Phi}_{l,m}(r)\right)Y_{l,m}(\theta,\phi) = -4\pi\sum_{l,m}\hat{\rho}_{l,m}(r)Y_{l,m}(\theta,\phi). \tag{5.32}$$

Equating the coefficients of $Y_{l,m}$ we obtain the radial equations

$$\hat{\Phi}''_{l,m}(r) + \frac{2}{r}\hat{\Phi}'_{l,m}(r) - \frac{l(l+1)}{r^2}\hat{\Phi}_{l,m}(r) = -4\pi\hat{\rho}_{l,m}(r). \tag{5.33}$$

The solution of the homogeneous equation reads

$$\hat{\Phi}_{l,m}(r) = a_{l,m}r^l + b_{l,m}r^{-l-1}. \tag{5.34}$$

For the inhomogeneous equation we introduce the conventional ansatz (at present I suppress the indices l and m.)

$$\hat{\Phi} = a(r)r^l + b(r)r^{-l-1}. \tag{5.35}$$

Then one obtains

$$\hat{\Phi}' = a'(r)r^l + b'(r)r^{-l-1} + la(r)r^{l-1} - (l+1)b(r)r^{-l-2}. \tag{5.36}$$

As usual we require

$$a'(r)r^l + b'(r)r^{-l-1} = 0 \tag{5.37}$$

and obtain for the second derivative

$$\hat{\Phi}'' = la'(r)r^{l-1} - (l+1)b'(r)r^{-l-2} + l(l-1)a(r)r^{l-2} + (l+1)(l+2)b(r)r^{-l-3}. \tag{5.38}$$

After substitution into the radial equation the contributions which contain a and b without derivative cancel. We are left with

$$la'(r)r^{l-1} - (l+1)b'(r)r^{-l-2} = -4\pi\hat{\rho}, \tag{5.39}$$

From the equations (5.37) and (5.39) one obtains by solving for a' and b'

$$\frac{\mathrm{d}a_{l,m}(r)}{\mathrm{d}r} = -\frac{4\pi}{2l+1} r^{1-l}\hat{\rho}_{l,m}(r), \tag{5.40}$$

$$\frac{\mathrm{d}b_{l,m}(r)}{\mathrm{d}r} = \frac{4\pi}{2l+1} r^{l+2}\hat{\rho}_{l,m}(r). \tag{5.41}$$

Now we integrate these equations

$$a_{l,m}(r) = \frac{4\pi}{2l+1}\int_r^\infty \mathrm{d}r' r'^{1-l}\hat{\rho}_{l,m}(r') \tag{5.42}$$

$$b_{l,m}(r) = \frac{4\pi}{2l+1}\int_0^r \mathrm{d}r' r'^{l+2}\hat{\rho}_{l,m}(r'). \tag{5.43}$$

If we add a constant to $a_{l,m}(r)$, then this is a solution of the Poisson equation too, since $r^l Y_{l,m}(\theta,\phi)$ is a homogeneous solution of the Poisson equation. We request a solution, which decays for large r. Therefore we choose $a_{l,m}(\infty) = 0$. If we add a constant to $b_{l,m}$, then this is a solution for $r \neq 0$. For $r = 0$ however, one obtains a singularity, which does not fulfil the Poisson equation. Therefore $b_{l,m}(0) = 0$ is required.
We may now insert the expansion coefficients $\hat{\rho}_{l,m}$ and obtain

$$a_{l,m}(r) = \frac{4\pi}{2l+1}\int_{r'>r} \mathrm{d}^3r' r'^{-1-l} Y^*_{l,m}(\theta',\phi')\rho(\mathbf{r}') \tag{5.44}$$

$$b_{l,m}(r) = \frac{4\pi}{2l+1}\int_{r'<r} \mathrm{d}^3r' r'^l Y^*_{l,m}(\theta',\phi')\rho(\mathbf{r}'). \tag{5.45}$$

We may now insert the expressions for $a_{l,m}$ und $b_{l,m}$ into (5.19) and (5.35). The r- und r'-dependence is obtained for $r < r'$ from the a-term as r^l/r'^{l+1} and for $r > r'$ from the b-term as r'^l/r^{l+1}. This can be put together, if we denote by $r_>$ the larger, by $r_<$ the smaller of both radii r and r'. Then one has

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \int \mathrm{d}^3r' \frac{r_<^l}{r_>^{l+1}} \rho(\mathbf{r}') Y^*_{l,m}(\theta',\phi') Y_{l,m}(\theta,\phi). \tag{5.46}$$

If $\rho(\mathbf{r}') = 0$ for $r' > R$, then one obtains for $r > R$

$$\Phi(\mathbf{r}) = \sum_{l,m} \sqrt{\frac{4\pi}{2l+1}} q_{l,m} \frac{Y_{l,m}(\theta,\phi)}{r^{l+1}} \tag{5.47}$$

with the multipole moments

$$q_{l,m} = \sqrt{\frac{4\pi}{2l+1}} \int \mathrm{d}^3r' r'^l Y^*_{l,m}(\theta',\phi')\rho(\mathbf{r}'). \tag{5.48}$$

For $l = 0$ one obtains the "monopole moment" charge, for $l = 1$ the components of the dipole moment, for $l = 2$ the components of the quadrupole moment. In particular for $m = 0$ one has

$$q_{0,0} = \sqrt{4\pi}\int d^3r' \sqrt{\frac{1}{4\pi}}\rho(\mathbf{r}') = q \tag{5.49}$$

$$q_{1,0} = \sqrt{\frac{4\pi}{3}}\int d^3r' \sqrt{\frac{3}{4\pi}} r' \cos\theta' \rho(\mathbf{r}') = \int d^3r' z' \rho(\mathbf{r}') = p_z \tag{5.50}$$

$$q_{2,0} = \sqrt{\frac{4\pi}{5}}\int d^3r' \sqrt{\frac{5}{4\pi}} r'^2 (\frac{3}{2}\cos^2\theta' - \frac{1}{2})\rho(\mathbf{r}') = \int d^3r' (\frac{3}{2}z'^2 - \frac{1}{2}r'^2)\rho(\mathbf{r}') = \frac{3}{2}Q_{zz}. \tag{5.51}$$

5.d Point Charge at r′, Cylindric Charge Distribution

Finally we consider the case of a point charge q located at $\mathbf{r}'$. We start from the potential

$$\Phi(\mathbf{r}) = \frac{q}{|\mathbf{r} - \mathbf{r}'|} = \frac{q}{\sqrt{r^2 + r'^2 - 2rr'\cos\psi}}. \tag{5.52}$$

Here ψ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. We expand in $r_</r_>$

$$\Phi(\mathbf{r}) = \frac{q}{r_> \sqrt{1 + (\frac{r_<}{r_>})^2 - 2\frac{r_<}{r_>}\cos\psi}} = q\sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\psi). \tag{5.53}$$

The $P_l(\xi)$ are called Legendre polynomials. For $\cos\psi = \pm 1$ one sees immediately from the expansion of $1/(r_> \mp r_<)$, that $P_l(1) = 1$ and $P_l(-1) = (-)^l$ hold.
On the other hand we may work with (5.46) and find

$$\Phi(\mathbf{r}) = q\sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{l,m}(\theta,\phi) Y^*_{l,m}(\theta',\phi'). \tag{5.54}$$

By comparison we obtain the addition theorem for spherical harmonics

$$P_l(\cos\psi) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{l,m}(\theta,\phi)Y^*_{l,m}(\theta',\phi'), \tag{5.55}$$

where the angle ψ between $\mathbf{r}$ and $\mathbf{r}'$ can be expressed by $\mathbf{r}\cdot\mathbf{r}' = rr'\cos\psi$ and by use of (5.1-5.3)

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{5.56}$$

We consider now the special case $\theta' = 0$, i.e. $\psi = \theta$. Then all $Y_{l,m}(\theta',\phi')$ vanish because of the factors $\sin\theta'$ with the exception of the term for $m = 0$ and the addition theorem is reduced to

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} Y_{l,0}(\theta)Y_{l,0}(0) = P_l^0(\cos\theta)P_l^0(1). \tag{5.57}$$

From the representation (5.30) $P_l^0(\xi) = 1/(2^l l!) d^l(\xi^2 - 1)^l/d\xi^l$ one obtains for $\xi = 1$ and the decomposition $(\xi^2 - 1)^l = (\xi + 1)^l(\xi - 1)^l$ the result $P_l^0(1) = [(\xi+1)^l/2^l]_{\xi=1}[d^l(\xi - 1)^l/d\xi^l/l!]_{\xi=1} = 1$. Thus we have

$$P_l^0(\xi) = P_l(\xi). \tag{5.58}$$

In particular for a cylinder symmetric charge distribution $\rho(\mathbf{r})$, which therefore depends only on r and θ, but not on ϕ, one has

$$\Phi(\mathbf{r}) = \sum_l \frac{P_l(\cos\theta)}{r^{l+1}} q_{l,0} \tag{5.59}$$

with the moments

$$q_{l,0} = \int \mathrm{d}^3 r' r'^l P_l(\cos\theta')\rho(\mathbf{r}'). \tag{5.60}$$

All moments with $m \neq 0$ vanish for a cylinder symmetric distribution.

Exercise Calculate the vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ from (5.1) to (5.5) and check that they constitute an orthonormal basis.

Exercise Calculate by means of Stokes' theorem (B.56) the curl in spherical coordinates.

Exercise Calculate for cylindric coordinates $x = \rho\cos\phi$, $y = \rho\sin\phi$ and z the metric factors g_ρ, g_ϕ and g_z, the volume element and gradient and divergence.

6 Electric Field in Matter

6.a Polarization and Dielectric Displacement

The field equations given by now are also valid in matter. In general matter reacts in an external electric field by polarization. The electrons move with respect to the positively charged nuclei, thus generating dipoles, or already existing dipoles of molecules or groups of molecules order against thermal disorder. Thus an electric field displaces the charges q_i from $\mathbf{r}_i$ to $\mathbf{r}_i + \mathbf{a}_i$, i.e. dipoles $\mathbf{p}_i = q_i\mathbf{a}_i$ are induced. One obtains the charge distribution of the polarization charges (4.22)

$$\rho_{\mathrm{P}}(\mathbf{r}) = -\sum_i \mathbf{p}_i \cdot \operatorname{grad}\delta^3(\mathbf{r}-\mathbf{r}_i). \tag{6.1}$$

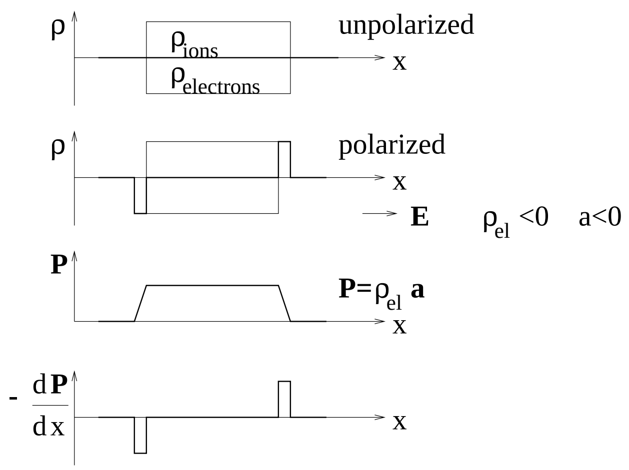
Introducing a density of dipole moments $\mathbf{P}$ called polarization

$$\mathbf{P}(\mathbf{r}) = \frac{\sum \mathbf{p}_i}{\Delta V}, \tag{6.2}$$

where $\sum \mathbf{p}_i$ is the sum of the dipole moments in an infinitesimal volume ΔV, one obtains

$$\rho_{\mathrm{P}}(\mathbf{r}) = -\int \mathrm{d}^3r'\mathbf{P}(\mathbf{r}')\cdot \operatorname{grad}\delta^3(\mathbf{r}-\mathbf{r}') = -\operatorname{div}\left(\int \mathrm{d}^3r'\mathbf{P}(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}')\right) = -\operatorname{div}\mathbf{P}(\mathbf{r}). \tag{6.3}$$

Let us visualize this equation. We start out from a solid body, in which the charges of the ions and electrons (on a scale large in comparison to the distance between the atoms) compensate (upper figure).If one applies a field $\mathbf{E}$ then the electrons move against the ions (second figure). Inside the bulk the charges compensate. Only at the boundaries a net-charge is left. In the third figure the polarization $\mathbf{P} = \rho_{\mathrm{el}}\mathbf{a}$ is shown, which has been continuously smeared at the boundary.The last figure shows the derivative $-\mathrm{d}\mathbf{P}/\mathrm{d}x$. One sees that this charge distribution agrees with that in the second figure.



Thus the charge density ρ consists of the freely moving charge density ρ_{f} and the charge density of the polarization ρ_{P} (the first one may be the charge density on the plates of a condensator)

$$\rho(\mathbf{r}) = \rho_{\mathrm{f}}(\mathbf{r}) + \rho_{\mathrm{P}}(\mathbf{r}) = \rho_{\mathrm{f}}(\mathbf{r}) - \operatorname{div}\mathbf{P}(\mathbf{r}). \tag{6.4}$$

Thus one introduces in MAXWELL's equation

$$\operatorname{div}\mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r}) = 4\pi\rho_{\mathrm{f}}(\mathbf{r}) - 4\pi\operatorname{div}\mathbf{P}(\mathbf{r}) \tag{6.5}$$

the dielectric displacement $\mathbf{D}$

$$\mathbf{D}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) + 4\pi\mathbf{P}(\mathbf{r}), \tag{6.6}$$

so that

$$\operatorname{div}\mathbf{D}(\mathbf{r}) = 4\pi\rho_{\mathrm{f}}(\mathbf{r}) \tag{6.7}$$

holds. The flux of the dielectric displacement through the surface of a volume yields the free charge q_{f} inside this volume

$$\int_{\partial V} \mathrm{d}\mathbf{f}\cdot\mathbf{D}(\mathbf{r}) = 4\pi q_{\mathrm{f}}(V). \tag{6.8}$$

For many substances **P** and **E** are within good approximation proportional as long as the field intensity **E** is not too large

$$\mathbf{P}(\mathbf{r}) = \chi_e \mathbf{E}(\mathbf{r}) \qquad \chi_e \text{ electric susceptibility} \tag{6.9}$$
$$\mathbf{D}(\mathbf{r}) = \epsilon \mathbf{E}(\mathbf{r}) \qquad \epsilon \text{ relative dielectric constant} \tag{6.10}$$
$$\epsilon = 1 + 4\pi\chi_e. \tag{6.11}$$

χ_e and ϵ are tensors for anisotropic matter, otherwise scalars. For ferroelectrica **P** is different from **0** already for $\mathbf{E} = \mathbf{0}$. However, in most cases it is compensated by surface charges. But it is observed, when the polarization is varied by external changes like pressure in the case of quartz (piezo-electricity) or under change of temperature. In Gaussian units the dimensions of **D**, **E** und **P** agree to dyn$^{1/2}$ cm^{-1}. In the SI-system **E** is measured in V/m, **D** and **P** in As/m^2. Since the SI-system is a rational system of units, the Gaussian an irrational one, the conversion factors for **D** and **P** differ by a factor 4π. Consequently the χ_e differ in both systems by a factor 4π. However, the relative dielectric constants ϵ are identical. For more details see appendix A.

6.b Boundaries between Dielectric Media

We now consider the boundary between two dielectric media or a dielectric material and vacuum. From Maxwell's equation $\text{curl}\,\mathbf{E} = \mathbf{0}$ it follows that the components of the electric field parallel to the boundary coincides in both dielectric media

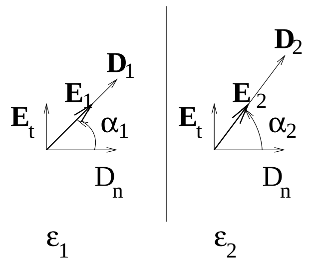
$$\mathbf{E}_{1,t} = \mathbf{E}_{2,t}. \tag{6.12}$$

In order to see this one considers the line integral $\oint d\mathbf{r}\cdot\mathbf{E}(\mathbf{r})$ along the closed contour which runs tangential to the boundary in one dielectric and returns in the other one, and transforms it into the integral $\int d\mathbf{f}\cdot\text{curl}\,\mathbf{E}(\mathbf{r}) = 0$ over the enclosed area. One sees that the integral over the contour vanishes. If the paths of integration in both dielectrica are infinitesimally close to each other, then $\mathbf{E}_t$ vanishes, since the integral over the contour vanishes for arbitrary paths.
On the other hand we may introduce a "pill box" whose covering surface is in one medium, the basal surface in the other one, both infinitesimally separated from the boundary. If there are no free charges at the boundary, then $\int_V d^3r\,\text{div}\,\mathbf{D} = 0$, so that the integral $\int d\mathbf{f}\cdot\mathbf{D} = 0$ over the surface vanishes. If the surface approaches the boundary, then it follows that the normal component of **D** is continuous

$$D_{1,n} = D_{2,n}. \tag{6.13}$$

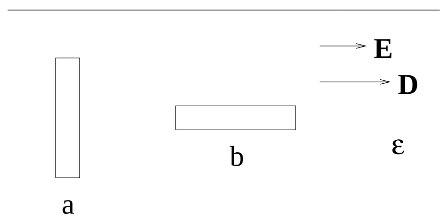
If the angle between the electric field (in an isotropic medium) and the normal to the boundary are α_1 and α_2 then one has

$$E_1 \sin\alpha_1 = E_2 \sin\alpha_2 \tag{6.14}$$
$$D_1 \cos\alpha_1 = D_2 \cos\alpha_2 \tag{6.15}$$
$$\frac{\tan\alpha_1}{\epsilon_1} = \frac{\tan\alpha_2}{\epsilon_2}. \tag{6.16}$$



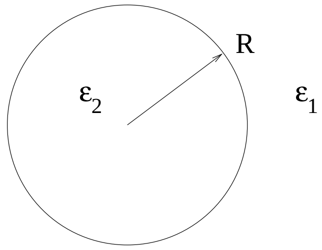
We now consider a cavity in a dielectric medium. If the cavity is very thin in the direction of the field (a) and large in perpendicular direction like a pill box then the displacement **D** agrees in the medium and the cavity.

If on the other hand the cavity has the shape of a slot very long in the direction of the field (b), then the variation of the potential along this direction has to agree, so that inside and outside the cavity **E** coincides. At the edges of the cavities will be scattered fields. It is possible to calculate the field exactly for ellipsoidal cavities. See for example the book by Becker and Sauter. The field is homogeneous inside the ellipsoid. The calculation for a sphere is given below.



6.c Dielectric Sphere in a Homogeneous Electric Field

We consider a dielectric sphere with radius R and dielectric constant ϵ_2 inside a medium with dielectric constant ϵ_1. The electric field in the medium 1 be homogeneous at large distances



$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1 = E_1\mathbf{e}_z \quad r \gg R. \tag{6.17}$$

Thus one obtains for the potential

$$\Phi(\mathbf{r}) = -\mathbf{E}_1 \cdot \mathbf{r} = -E_1 r\cos\theta \quad r \gg R. \tag{6.18}$$

Since $\cos\theta$ is the Legendre polynomial $P_1(\cos\theta)$, the ansatz

$$\Phi(\mathbf{r}) = f(r)\cos\theta \tag{6.19}$$

is successful. The solution of the homogeneous Poisson equation $\triangle(f(r)\cos\theta) = 0$ is a linear combination (5.34) of $f(r) = r$ (homogeneous field) and $f(r) = 1/r^2$ (dipolar field). Since there is no dipole at the origin we may assume

$$\Phi(\mathbf{r}) = \cos\theta \cdot \begin{cases} -E_2 r & r \le R \\ -E_1 r + p/r^2 & r \ge R \end{cases}. \tag{6.20}$$

At the boundary one has $\Phi(R+0) = \phi(R-0)$, which is identical to $\mathbf{E}_{1,\mathrm{t}} = \mathbf{E}_{2,\mathrm{t}}$ and leads to

$$-E_1 R + \frac{p}{R^2} = -E_2 R. \tag{6.21}$$

The condition $D_{1,\mathrm{n}} = D_{2,\mathrm{n}}$ together with $D_\mathrm{n} = -\epsilon\frac{\partial\Phi}{\partial r}$ yields

$$\epsilon_1(E_1 + \frac{2p}{R^3}) = \epsilon_2 E_2. \tag{6.22}$$

From these two equations one obtains

$$E_2 = \frac{3\epsilon_1}{\epsilon_2 + 2\epsilon_1}E_1 \tag{6.23}$$

$$p = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + 2\epsilon_1}R^3 E_1. \tag{6.24}$$

One obtains in particular for the dielectric sphere ($\epsilon_2 = \epsilon$) in the vacuum ($\epsilon_1 = 1$)

$$E_2 = \frac{3}{2+\epsilon}E_1, \quad p = \frac{\epsilon - 1}{\epsilon + 2}R^3 E_1. \tag{6.25}$$

The polarization inside the sphere changes the average field by

$$\mathbf{E}_2 - \mathbf{E}_1 = \frac{1-\epsilon}{2+\epsilon}E_1\mathbf{e}_z = -\frac{4\pi}{3}\mathbf{P}. \tag{6.26}$$

However, for a spherical cavity ($\epsilon_2 = 1$) in a dielectric medium ($\epsilon_1 = \epsilon$) one obtains

$$E_2 = \frac{3\epsilon}{1+2\epsilon}E_1. \tag{6.27}$$

6.d Dielectric Constant according to Clausius and Mossotti

Clausius and Mossotti derive the dielectric constant from the polarizability α of molecules (atoms) as follows: The average dipole moment in the field $\mathbf{E}_\mathrm{eff}$ is

$$\mathbf{p} = \alpha\mathbf{E}_\mathrm{eff}. \tag{6.28}$$

The density n of the dipoles (atoms) yields the polarization

$$\mathbf{P} = n\mathbf{p} = n\alpha\mathbf{E}_{\text{eff}}. \tag{6.29}$$

Therefore we have to determine the effective field $\mathbf{E}_{\text{eff}}$, which acts on the dipole.
For this purpose we cut a sphere of radius R out of the matter around the dipole. These dipoles generate, as we have seen in the example of the dielectric sphere in the vacuum (6.26) an average field

$$\bar{\mathbf{E}}_{\mathrm{P}} = \mathbf{E}_2 - \mathbf{E}_1 = -\frac{4\pi}{3}\mathbf{P}. \tag{6.30}$$

This field is missing after we have cut out the sphere. Instead the rapidly varying field of the dipoles inside the sphere has to be added (with the exception of the field of the dipole at the location, where the field has to be determined)

$$\mathbf{E}_{\text{eff}} = \mathbf{E} - \bar{\mathbf{E}}_{\mathrm{P}} + \sum_i \frac{-\mathbf{p}_i r_i^2 + 3(\mathbf{p}_i\mathbf{r}_i)\mathbf{r}_i}{r_i^5}. \tag{6.31}$$

The sum depends on the location of the dipoles (crystal structure). If the dipoles are located on a cubic lattice, then the sum vanishes, since the contributions from

$$\sum_{\alpha,\beta} \mathbf{e}_\alpha p_\beta \sum_i \frac{-\delta_{\alpha,\beta} r_i^2 + 3x_{i,\alpha}x_{i,\beta}}{r_i^5} \tag{6.32}$$

cancel for $\alpha \neq \beta$, if one adds the contributions for x_α and $-x_\alpha$, those for $\alpha = \beta$, if one adds the three contributions obtained by cyclic permutation of the three components. Thus one obtains for the cubic lattice

$$\chi_{\mathrm{e}}\mathbf{E} = \mathbf{P} = n\alpha\mathbf{E}_{\text{eff}} = n\alpha(\mathbf{E} + \frac{4\pi}{3}\mathbf{P}) = n\alpha(1 + \frac{4\pi}{3}\chi_{\mathrm{e}})\mathbf{E}, \tag{6.33}$$

from which the relation of Clausius (1850) and Mossotti (1879)

$$\chi_{\mathrm{e}} = \frac{n\alpha}{1 - \frac{4\pi n\alpha}{3}} \text{ or } \frac{4\pi}{3}n\alpha = \frac{\epsilon - 1}{\epsilon + 2}. \tag{6.34}$$

follows.

7 Electricity on Conductors

7.a Electric Conductors

The electric field vanishes within a conductor, $\mathbf{E} = \mathbf{0}$, since a nonvanishing field would move the charges. Thus the potential within a conductor is constant. For the conductor $\#i$ one has $\Phi(\mathbf{r}) = \Phi_i$.
Outside the conductor the potential is given by Poisson's equation

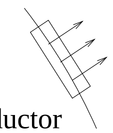
$$\triangle\Phi(\mathbf{r}) = -4\pi\rho(\mathbf{r}) \text{ or } \operatorname{div}(\epsilon(\mathbf{r}) \operatorname{grad}\Phi(\mathbf{r})) = -4\pi\rho_{\mathrm{f}}(\mathbf{r}). \tag{7.1}$$

7.a.α Boundary Conditions at the Surface of the Conductor

On the surface of the conductor one has a constant potential (on the side of the dielectric medium, too). Thus the components of $\mathbf{E}$ tangential to the surface vanish

$$\mathbf{E}_{\mathrm{t}}(\mathbf{r}) = \mathbf{0}, \tag{7.2}$$

In general there are charges at the surface of the conductor. We denote its density by $\sigma(\mathbf{r})$.



Integration over a small piece of the surface yields

$$\int \mathrm{d}\mathbf{f} \cdot \mathbf{E}_{\mathrm{a}}(\mathbf{r}) = 4\pi q = 4\pi \int \mathrm{d}f\, \sigma(\mathbf{r}). \tag{7.3}$$

Therefore the field $\mathbf{E}_{\mathrm{a}}$ obeys at the surface in the outside region

$$\mathbf{E}_{\mathrm{a}}(\mathbf{r}) = 4\pi\sigma(\mathbf{r})\mathbf{n}, \quad -\frac{\partial\Phi}{\partial n} = 4\pi\sigma(\mathbf{r}). \tag{7.4}$$

In general the charge density σ at the surface consists of the free charge density σ_{f} at the surface of the conductor and the polarization charge density σ_{P} on the dielectric medium $\sigma(\mathbf{r}) = \sigma_{\mathrm{f}}(\mathbf{r}) + \sigma_{\mathrm{P}}(\mathbf{r})$ with

$$\mathbf{D}_{\mathrm{a}}(\mathbf{r}) = 4\pi\sigma_{\mathrm{f}}(\mathbf{r})\mathbf{n}, \tag{7.5}$$

from which one obtains

$$\sigma_{\mathrm{f}} = \epsilon(\sigma_{\mathrm{f}} + \sigma_{\mathrm{P}}), \quad \sigma_{\mathrm{P}} = (\frac{1}{\epsilon} - 1)\sigma_{\mathrm{f}}. \tag{7.6}$$

7.a.β Force acting on the Conductor (in Vacuo)

Initially one might guess that the force on the conductor is given by $\int \mathrm{d}f \mathbf{E}_{\mathrm{a}}\sigma(\mathbf{r})$. This, however, is wrong. By the same token one could argue that one has to insert the field inside the conductor $\mathbf{E}_{\mathrm{i}} = \mathbf{0}$ into the integral. The truth lies halfway. This becomes clear, if one assumes that the charge is not exactly at the surface but smeared out over a layer of thickness l. If we assume that inside a layer of thickness a one has the charge $s(a)\sigma(\mathbf{r})\mathrm{d}f$ with $s(0) = 0$ and $s(l) = 1$, then the field acting at depth a is $\mathbf{E}_{\mathrm{i}}(\mathbf{r} - a\mathbf{n}) = (1 - s(a))\mathbf{E}_{\mathrm{a}}(\mathbf{r})$, since the fraction $s(a)$ is already screened. With $\rho(\mathbf{r} - a\mathbf{n}) = s'(a)\sigma(\mathbf{r})$ one obtains

$$\mathbf{K} = \int \mathrm{d}f\, \mathrm{d}a\rho(\mathbf{r} - a\mathbf{n})\mathbf{E}(\mathbf{r} - a\mathbf{n}) = \int \mathrm{d}f\sigma(\mathbf{r})\mathbf{E}_{\mathrm{a}}(\mathbf{r}) \int_0^l \mathrm{d}as'(a)(1 - s(a)). \tag{7.7}$$

The integral over a yields $(s(a) - s^2(a)/2)|_0^l = 1/2$, so that finally we obtain the force

$$\mathbf{K} = \frac{1}{2} \int \mathrm{d}f\sigma(\mathbf{r})\mathbf{E}_{\mathrm{a}}(\mathbf{r}). \tag{7.8}$$

7.b Capacities

We now consider several conductors imbedded in the vacuum or in dielectric media. Outside the conductors there should be no free moving charge densities, $\rho_{\mathrm{f}} = 0$. The electric potentials Φ_i of the conductors #i should be given. We look for the free charges q_i at the conductors. Since MAXWELL's equations are linear (and we assume that there is a linear relation $\mathbf{D} = \epsilon\mathbf{E}$) we may write the potential as a superposition of solutions Ψ_i

$$\Phi(\mathbf{r}) = \sum_i \Phi_i \Psi_i(\mathbf{r}). \tag{7.9}$$

Ψ_i is the solution which assumes the value 1 at the conductor #i, and 0 at all others

$$\Psi_i(\mathbf{r}) = \delta_{i,j} \quad \mathbf{r} \in \text{ conductor } j. \tag{7.10}$$

The charge on conductor #i is then given by

$$q_i = -\frac{1}{4\pi} \int_{F_i} \mathrm{d}f \epsilon \left.\frac{\partial \Phi}{\partial n}\right|_{\mathrm{a}} = \sum_j C_{i,j} \Phi_j \tag{7.11}$$

with the capacity coefficients

$$C_{i,j} = -\frac{1}{4\pi} \int_{F_i} \mathrm{d}f \epsilon \left.\frac{\partial \Psi_j}{\partial n}\right|_{\mathrm{a}}. \tag{7.12}$$

The capacity has the dimension charge/(electric potential), which in GAUSSian units is a length. The conversion into the SI-system is by the factor $4\pi\epsilon_0$, so that 1 cm $\hat{=}$ $1/9 \cdot 10^{-11}$ As/V = 10/9 pF (picofarad).
The electrostatic energy is obtained from

$$\mathrm{d}U = \sum_i \Phi_i \mathrm{d}q_i = \sum_{i,j} \Phi_i C_{i,j} \mathrm{d}\Phi_j, \tag{7.13}$$

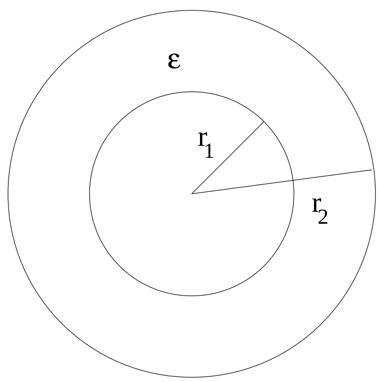
that is

$$\frac{\partial U}{\partial \Phi_j} = \sum_i C_{i,j} \Phi_i, \tag{7.14}$$

$$\frac{\partial^2 U}{\partial \Phi_i \partial \Phi_j} = C_{i,j} = \frac{\partial^2 U}{\partial \Phi_j \partial \Phi_i} = C_{j,i}, \tag{7.15}$$

$$U = \frac{1}{2} \sum_{i,j} C_{i,j} \Phi_i \Phi_j = \frac{1}{2} \sum_i \Phi_i q_i \tag{7.16}$$

As an example we consider a spherical capacitor. Two concentric conducting spheres with radii r_1, r_2 with $r_1 < r_2$ carry the charges q_1 and q_2, resp. Outside be vacuum. Between the two spheres is a medium with dielectric constant ϵ. Then outside the spheres one has



$$\Phi(r) = \frac{q_1 + q_2}{r} \quad r \geq r_2. \tag{7.17}$$

The potential decays in the space between the two spheres like $q_1/(\epsilon r)$. Since the potential is continuous at $r = r_2$, it follows that

$$\Phi(r) = \frac{q_1}{\epsilon r} - \frac{q_1}{\epsilon r_2} + \frac{q_1 + q_2}{r_2} \quad r_1 \leq r \leq r_2. \tag{7.18}$$

Inside the smaller sphere the potential is constant.

$$\Phi(r) = \frac{q_1}{\epsilon r_1} - \frac{q_1}{\epsilon r_2} + \frac{q_1 + q_2}{r_2} \quad r \leq r_1. \tag{7.19}$$

From this one calculates the charges as a function of the potentials $\Phi_i = \Phi(r_i)$

$$q_1 = \frac{\epsilon r_1 r_2}{r_2 - r_1}(\Phi_1 - \Phi_2) \tag{7.20}$$

$$q_2 = \frac{\epsilon r_1 r_2}{r_2 - r_1}(\Phi_2 - \Phi_1) + r_2\Phi_2, \tag{7.21}$$

from which the capacitor coefficients can be read off immediately. If the system is neutral, $q = q_1 = -q_2$, then q can be expressed by the difference of the potential

$$q = C(\Phi_1 - \Phi_2) \tag{7.22}$$

and one calls C the capacity. For the spherical capacitor one obtains $\Phi_2 = 0$ and $\Phi_1 = \frac{q_1}{\epsilon}(\frac{1}{r_1} - \frac{1}{r_2})$, from which the capacity

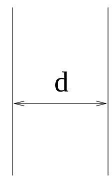
$$C = \frac{\epsilon r_1 r_2}{r_2 - r_1} \tag{7.23}$$

is obtained.
For a single sphere r_2 can go to ∞ and one finds $C = \epsilon r_1$.
We obtain the plate capacitor with a distance d between the plates, by putting $r_2 = r_1 + d$ in the limit of large r_1

$$C = \frac{(r_1^2 + r_1 d)\epsilon}{d} = \frac{4\pi r_1^2 \epsilon}{d}\Big(\frac{1}{4\pi} + \frac{d}{4\pi r_1}\Big), \tag{7.24}$$

which approaches $\frac{\epsilon F}{4\pi d}$ for large r_1 with the area F. Therefore one obtains for the plate capacitor

$$C = \frac{\epsilon F}{4\pi d}. \tag{7.25}$$



A different consideration is the following: The charge q generates the flux $DF = 4\pi q$. Therefore the potential difference between the two plates is $\Phi = \frac{D}{\epsilon}d = \frac{4\pi d}{\epsilon F}q$, from which $C = q/\phi = \frac{\epsilon F}{4\pi d}$ follows. Be aware that here we have denoted the free charge by q.

7.c Influence Charges

If we fix the potentials of all conductors to 0, $\Phi_i = 0$ in the presence of a free charge q' at $\mathbf{r}'$, then we write the potential

$$\Phi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')q' \tag{7.26}$$

with the Green's function G. Apparently this function obeys the equation

$$\nabla(\epsilon(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}')) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}') \tag{7.27}$$

for $\mathbf{r}$ outside the conductor. For $\mathbf{r}$ at the surface of the conductors we have $G(\mathbf{r}, \mathbf{r}') = 0$. The superposition principle yields for a charge density $\rho_f(\mathbf{r}')$ located outside the conductors

$$\Phi(\mathbf{r}) = \int d^3r' G(\mathbf{r}, \mathbf{r}')\rho_f(\mathbf{r}') + \sum_i \Phi_i \Psi_i(\mathbf{r}), \tag{7.28}$$

where now we have assumed that the conductors have the potential Φ_i.
We now show that the Green's function is symmetric, $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$. In order to show this we start from the integral over the surfaces of the conductors

$$\int d\mathbf{f}'' \cdot \{G(\mathbf{r}'', \mathbf{r})\epsilon(\mathbf{r}'')\nabla'' G(\mathbf{r}'', \mathbf{r}') - \epsilon(\mathbf{r}'')[\nabla'' G(\mathbf{r}'', \mathbf{r})]G(\mathbf{r}'', \mathbf{r}')\} = 0, \tag{7.29}$$

since G vanishes at the surface of the conductors. The area element $d\mathbf{f}''$ is directed into the conductors. We perform the integral also over a sphere of radius R, which includes all conductors. Since $G \sim 1/R$ and since

$\nabla'' G \sim 1/R^2$ the surface integral vanishes for $R \to \infty$. Application of the divergence theorem yields

$$\int \mathrm{d}^3 r'' \{G(\mathbf{r}'', \mathbf{r}) \nabla'' [\epsilon(\mathbf{r}'') \nabla'' G(\mathbf{r}'', \mathbf{r}')] - \nabla'' [\epsilon(\mathbf{r}'') \nabla'' G(\mathbf{r}'', \mathbf{r})] G(\mathbf{r}'', \mathbf{r}')\} \tag{7.30}$$

$$= -4\pi \int \mathrm{d}^3 r'' \{G(\mathbf{r}'', \mathbf{r}) \delta^3(\mathbf{r}'' - \mathbf{r}') - \delta^3(\mathbf{r}'' - \mathbf{r}) G(\mathbf{r}'', \mathbf{r}')\} \tag{7.31}$$

$$= -4\pi (G(\mathbf{r}', \mathbf{r}) - G(\mathbf{r}, \mathbf{r}')) = 0. \tag{7.32}$$

We consider now a few examples:

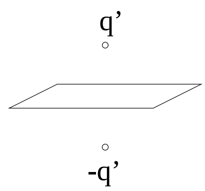
7.c.α Space free of Conductors

In a space with constant dielectric constant ϵ and without conductors one has

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\epsilon |\mathbf{r} - \mathbf{r}'|}. \tag{7.33}$$

7.c.β Conducting Plane

For a conducting plane $z = 0$ ($\epsilon = 1$) one solves the problem by mirror charges. If the given charge q' is located at $\mathbf{r}' = (x', y', z')$, then one should imagine a second charge $-q'$ at $\mathbf{r}'' = (x', y', -z')$. This mirror charge compensates the potential at the surface of the conductor. One obtains



$$G(\mathbf{r}, \mathbf{r}') = \begin{cases} \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}''|} & \text{for } \operatorname{sign} z = \operatorname{sign} z' \\ 0 & \text{for } \operatorname{sign} z = -\operatorname{sign} z'. \end{cases} \tag{7.34}$$

Next we consider the force which acts on the charge q'. The potential is $\Phi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}') q'$. The contribution $q'/|\mathbf{r} - \mathbf{r}'|$ is the potential of q' itself that does not exert a force on q' . The second contribution $-q'/|\mathbf{r} - \mathbf{r}''|$ comes, however, from the influence charges on the metal surface and exerts the force

$$\mathbf{K} = -q' \operatorname{grad} \frac{-q'}{|\mathbf{r} - \mathbf{r}''|} = -\frac{q'^2 \mathbf{e}_z}{4z'^2} \operatorname{sign} z'. \tag{7.35}$$

Further one determines the influence charge on the plate. At $z = 0$ one has $4\pi \operatorname{sign} z' \mathbf{e}_z \sigma(\mathbf{r}) = \mathbf{E}(\mathbf{r}) = q' \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - q' \frac{\mathbf{r}-\mathbf{r}''}{|\mathbf{r}-\mathbf{r}''|^3}$. From this one obtains the density of the surface charge per area

$$\sigma(\mathbf{r}) = -\frac{q'}{2\pi} \frac{|z'|}{\sqrt{(x - x')^2 + (y - y')^2 + z'^2}^3} \tag{7.36}$$

With $\mathrm{d}f = \pi \mathrm{d}(x^2 + y^2)$ one obtains

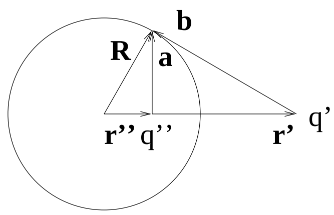
$$\int \mathrm{d}f \sigma(\mathbf{r}) = -\frac{q'|z'|}{2} \int_{z'^2}^{\infty} \frac{\mathrm{d}(x^2 + y^2 + z'^2)}{(x^2 + y^2 + z'^2)^{3/2}} = -q'. \tag{7.37}$$

The force acting on the plate is obtained as

$$\mathbf{K} = \frac{1}{2} \int \mathrm{d}f \mathbf{E}(\mathbf{r}) \sigma(\mathbf{r}) = \frac{q'^2 z' |z'|}{2} \mathbf{e}_z \int \frac{\mathrm{d}(x^2 + y^2 + z'^2)}{(x^2 + y^2 + z'^2)^3} = \frac{q'^2 \mathbf{e}_z}{4z'^2} \operatorname{sign} z'. \tag{7.38}$$

7.c.γ Conducting Sphere

We consider a charge q' located at $\mathbf{r}'$ in the presence of a conducting sphere with radius R and center in the origin. Then there is a vector $\mathbf{r}''$, so that the ratio of the distances of all points $\mathbf{R}$ on the surface of the sphere from $\mathbf{r}'$ and $\mathbf{r}''$ is constant. Be



$$a^2 := (\mathbf{R} - \mathbf{r}'')^2 = R^2 + r''^2 - 2\mathbf{R}\cdot\mathbf{r}'' \tag{7.39}$$
$$b^2 := (\mathbf{R} - \mathbf{r}')^2 = R^2 + r'^2 - 2\mathbf{R}\cdot\mathbf{r}' \tag{7.40}$$

This constant ratio of the distances is fulfilled for $\mathbf{r} \parallel \mathbf{r}''$ and

$$\frac{R^2 + r''^2}{R^2 + r'^2} = \frac{r''}{r'}. \tag{7.41}$$

Then one has

$$R^2 = r'r'' \quad \mathbf{r}'' = \frac{R^2}{r'^2}\mathbf{r}' \tag{7.42}$$
$$\frac{a^2}{b^2} = \frac{r''}{r'} = \frac{R^2}{r'^2} = \frac{r''^2}{R^2}. \tag{7.43}$$

Thus one obtains a constant potential on the sphere with the charge q' at $\mathbf{r}'$ and the charge $q'' = -q'R/r'$ at r''

$$G(\mathbf{r}, \mathbf{r}') = \begin{cases} \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{|\mathbf{r}-\mathbf{r}''|} & \text{for } \operatorname{sign}(r-R) = \operatorname{sign}(r'-R), \\ 0 & \text{otherwise}. \end{cases} \tag{7.44}$$

The potential on the sphere vanishes with this Green's function G. For $r' > R$ it carries the charge q'' and for $r' < R$ the charge $-q'$. Thus if the total charge on the sphere vanishes one has to add a potential Φ, which corresponds to a homogeneously distributed charge $-q''$ and q', resp.

8 Energy, Forces and Stress in Dielectric Media

8.a Electrostatic Energy

By displacing the charge densities $\delta\rho = \delta\rho_f + \delta\rho_P$ the electrostatic energy

$$\delta U = \int d^3r \delta\rho_f \Phi + \int d^3r \delta\rho_P \Phi \tag{8.1}$$

will be added to the system. Simultaneously there are additional potentials Φ_i in the matter guaranteeing that the polarization is in equilibrium, i. e.

$$\delta U = \int d^3r \delta\rho_f \Phi + \int d^3r \delta\rho_P (\Phi + \Phi_i). \tag{8.2}$$

These potentials are so that $\delta U = 0$ holds for a variation of the polarization, so that the polarizations are in equilibrium

$$\Phi + \Phi_i = 0. \tag{8.3}$$

These considerations hold as long as the process is run adiabatically and under the condition that no mechanical energy is added. Thus the matter is in a force-free state (equilibrium $\mathbf{k} = \mathbf{0}$) or it has to be under rigid constraints. Then one obtains with (B.62)

$$\delta U = \int d^3r\, \delta\rho_f\, \Phi = \frac{1}{4\pi}\int d^3r \operatorname{div}\delta\mathbf{D}\,\Phi = -\frac{1}{4\pi}\int d^3r \delta\mathbf{D}\cdot \operatorname{grad}\Phi = \frac{1}{4\pi}\int d^3r \mathbf{E}\cdot\delta\mathbf{D}, \tag{8.4}$$

similarly to the matter-free case (3.25). Then one obtains for the density of the energy at fixed density of matter ρ_m (we assume that apart from the electric field only the density of matter determines the energy-density; in general, however, the state of distortion will be essential)

$$du = \frac{1}{4\pi}\mathbf{E}\cdot d\mathbf{D}. \tag{8.5}$$

If $\mathbf{D} = \epsilon\mathbf{E}$, then one obtains

$$u = u_0(\rho_m) + \frac{1}{4\pi}\int \epsilon(\rho_m)\mathbf{E}\cdot d\mathbf{E} = u_0(\rho_m) + \frac{1}{8\pi}\epsilon(\rho_m)E^2 = u_0(\rho_m) + \frac{D^2}{8\pi\epsilon(\rho_m)}, \tag{8.6}$$

since the dielectric constant depends in general on the density of mass.

8.b Force Density in Isotropic Dielectric Matter

We may determine the force density in a dielectric medium by moving the masses and free charges from $\mathbf{r}$ to $\mathbf{r} + \delta\mathbf{s}(\mathbf{r})$ and calculating the change of energy δU. The energy added to the system is

$$\delta U = \int d^3r\, \mathbf{k}_a(\mathbf{r})\cdot\delta\mathbf{s}(\mathbf{r}), \tag{8.7}$$

where $\mathbf{k}_a$ is an external force density. The internal electric and mechanical force density $\mathbf{k}$ acting against it in equilibrium is

$$\mathbf{k}(\mathbf{r}) = -\mathbf{k}_a(\mathbf{r}), \tag{8.8}$$

so that

$$\delta U = -\int d^3r\, \mathbf{k}(\mathbf{r})\cdot\delta\mathbf{s}(\mathbf{r}) \tag{8.9}$$

holds. We bring now δU into this form

$$\delta U = \int d^3r \left(\frac{\partial u}{\partial\mathbf{D}}\cdot\delta\mathbf{D} + \left.\frac{\partial u}{\partial\rho_m}\right|_{\mathbf{D}}\delta\rho_m\right), \quad u = u(\mathbf{D},\rho_m). \tag{8.10}$$

Since $\partial u/\partial \mathbf{D} = \mathbf{E}/(4\pi)$ we rewrite the first term as in the previous section

$$\delta U = \int \mathrm{d}^3r \left(\Phi(\mathbf{r})\delta\rho_\mathrm{f}(\mathbf{r}) + \left.\frac{\partial u}{\partial\rho_\mathrm{m}}\right|_\mathbf{D} \delta\rho_\mathrm{m} \right). \tag{8.11}$$

From the equation of continuity $\partial\rho/\partial t = -\operatorname{div}\mathbf{j}$ we derive the relation between $\delta\rho$ and $\delta\mathbf{s}$. The equation has to be multiplied by δt and one has to consider that $\mathbf{j}\delta t = \rho\mathbf{v}\delta t = \rho\delta\mathbf{s}$ holds. With $(\partial\rho/\partial t)\delta t = \delta\rho$ we obtain

$$\delta\rho = -\operatorname{div}(\rho\delta\mathbf{s}). \tag{8.12}$$

Then we obtain

$$\begin{aligned} \delta U &= -\int \mathrm{d}^3r \left(\Phi(\mathbf{r}) \operatorname{div}(\rho_\mathrm{f}\delta\mathbf{s}) + \frac{\partial u}{\partial\rho_\mathrm{m}} \operatorname{div}(\rho_\mathrm{m}\delta\mathbf{s}) \right) \\ &= \int \mathrm{d}^3r \left(\operatorname{grad}\Phi(\mathbf{r})\rho_\mathrm{f}(\mathbf{r}) + \left(\operatorname{grad}\frac{\partial u}{\partial\rho_\mathrm{m}} \right)\rho_\mathrm{m}(\mathbf{r}) \right) \cdot \delta\mathbf{s}(\mathbf{r}), \end{aligned} \tag{8.13}$$

where the divergence theorem (B.62) has been used by the derivation of the last line. This yields

$$\mathbf{k}(\mathbf{r}) = \rho_\mathrm{f}(\mathbf{r})\mathbf{E}(\mathbf{r}) - \rho_\mathrm{m}(\mathbf{r}) \operatorname{grad}\left(\frac{\partial u}{\partial\rho_\mathrm{m}}\right). \tag{8.14}$$

The first contribution is the Coulomb force on the free charges. The second contribution has to be rewritten. We substitute (8.6) $u = u_0(\rho_\mathrm{m}) + D^2/(8\pi\epsilon(\rho_\mathrm{m}))$. Then one has

$$\frac{\partial u}{\partial\rho_\mathrm{m}} = \frac{\mathrm{d}u_0}{\mathrm{d}\rho_\mathrm{m}} + \frac{1}{8\pi}D^2\frac{\mathrm{d}(1/\epsilon)}{\mathrm{d}\rho_\mathrm{m}} = \frac{\mathrm{d}u_0}{\mathrm{d}\rho_\mathrm{m}} - \frac{1}{8\pi}E^2\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_\mathrm{m}}. \tag{8.15}$$

The first term can be written

$$-\rho_\mathrm{m} \operatorname{grad}\frac{\mathrm{d}u_0}{\mathrm{d}\rho_\mathrm{m}} = -\operatorname{grad}\left(\rho_\mathrm{m}\frac{\mathrm{d}u_0}{\mathrm{d}\rho_\mathrm{m}} - u_0\right) = -\operatorname{grad}P_0(\rho_\mathrm{m}), \tag{8.16}$$

where we use that $(\mathrm{d}u_0/\mathrm{d}\rho_\mathrm{m}) \operatorname{grad}\rho_m = \operatorname{grad}u_0$. Here P_0 is the hydrostatic pressure of the liquid without electric field

$$\mathbf{k}_{0,\mathrm{hydro}} = -\operatorname{grad}P_0(\rho_\mathrm{m}(\mathbf{r})). \tag{8.17}$$

The hydrostatic force acting on the volume V can be written in terms of a surface integral

$$\mathbf{K}_0 = -\int_V \mathrm{d}^3r \operatorname{grad}P_0(\rho_\mathrm{m}(\mathbf{r})) = -\int_{\partial V} \mathrm{d}\mathbf{f}\, P_0(\rho_\mathrm{m}(\mathbf{r})). \tag{8.18}$$

This is a force which acts on the surface ∂V with the pressure P_0. There remains the electrostrictive contribution

$$\frac{1}{8\pi}\rho_\mathrm{m} \operatorname{grad}\left(E^2\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_\mathrm{m}}\right) = \frac{1}{8\pi}\operatorname{grad}\left(E^2\rho_\mathrm{m}\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_\mathrm{m}}\right) - \frac{1}{8\pi}E^2 \operatorname{grad}\epsilon, \tag{8.19}$$

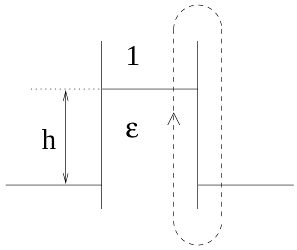
where $(\mathrm{d}\epsilon/\mathrm{d}\rho_\mathrm{m}) \operatorname{grad}\rho_\mathrm{m} = \operatorname{grad}\epsilon$ has been used. Then the total force density is

$$\mathbf{k}(\mathbf{r}) = \rho_\mathrm{f}(\mathbf{r})\mathbf{E}(\mathbf{r}) + \operatorname{grad}\left(-P_0(\rho_\mathrm{m}) + \frac{1}{8\pi}E^2\rho_\mathrm{m}\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_\mathrm{m}}\right) - \frac{1}{8\pi}E^2 \operatorname{grad}\epsilon. \tag{8.20}$$

Applications:
Dielectric fluid between two vertical capacitor plates. What is the difference h in height between between the surface of a fluid between the plates of the capacitor and outside the capacitor? For this purpose we introduce the integral along a closed path which goes up between the plates of the capacitor and outside down

$$\oint \mathbf{k}\cdot\mathrm{d}\mathbf{r} = \oint \operatorname{grad}\left(-P_0 + \frac{1}{8\pi}E^2\rho_\mathrm{m}\frac{\partial\epsilon}{\partial\rho_\mathrm{m}}\right)\cdot\mathrm{d}\mathbf{r} - \frac{1}{8\pi}\oint E^2 \operatorname{grad}\epsilon\cdot\mathrm{d}\mathbf{r} = \frac{1}{8\pi}E^2(\epsilon-1). \tag{8.21}$$

The integral over the gradient along the closed path vanishes, whereas the integral of $E^2 \operatorname{grad} \epsilon$ yields a contribution at the two points where the path of integration intersects the surface. In addition there is the gravitational force. Both have to compensate each other



$$\mathbf{k} + \mathbf{k}_{\rm grav} = \mathbf{0}, \tag{8.22}$$

that is

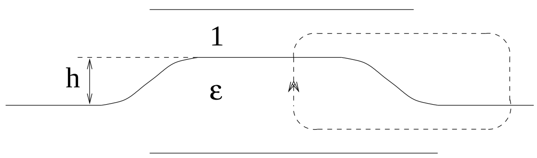
$$\oint \mathrm{d}\mathbf{r} \cdot \mathbf{k}_{\rm grav} = -\rho_{\rm m} g h = -\oint \mathrm{d}\mathbf{r} \cdot \mathbf{k}, \tag{8.23}$$

from which one obtains the height

$$h = \frac{E^2(\epsilon - 1)}{8\pi \rho_{\rm m} g}. \tag{8.24}$$

Dielectric fluid between two horizontal capacitor plates

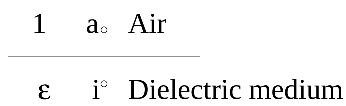
What is the elevation of a dielectric fluid between two horizontal capacitor plates? The problem can be solved in a similar way as between two vertical plates.It is useful, however, to use



$$-\frac{1}{8\pi} E^2 \operatorname{grad} \epsilon = \frac{1}{8\pi} D^2 \operatorname{grad} (\frac{1}{\epsilon}). \tag{8.25}$$

Hydrostatic pressure difference at a boundary

Performing an integration through the boundary from the dielectric medium to air one obtains



$$0 = \int_{\rm i}^{\rm a} \mathbf{k} \cdot \mathrm{d}\mathbf{r} = \int \operatorname{grad} (-P_0 + \frac{1}{8\pi} \rho_{\rm m} E^2 \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_{\rm m}}) \cdot \mathrm{d}\mathbf{r} - \frac{1}{8\pi} \int E_{\rm t}^2 \operatorname{grad} \epsilon \cdot \mathrm{d}\mathbf{r} + \frac{1}{8\pi} \int D_{\rm n}^2 \operatorname{grad} (\frac{1}{\epsilon}) \cdot \mathrm{d}\mathbf{r}. \tag{8.26}$$

This yields the difference in hydrostatic pressure at both sides of the boundary

$$P_{0,\rm i}(\rho_{\rm m}) - P_{0,\rm a} = \frac{1}{8\pi} \left(\rho_{\rm m} \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_{\rm m}} E^2 - (\epsilon - 1) E_{\rm t}^2 + (\frac{1}{\epsilon} - 1) D_{\rm n}^2 \right) \tag{8.27}$$

Pressure in a practically incompressible dielectric medium
From

$$\mathbf{k} + \mathbf{k}_{\rm grav} = -\operatorname{grad} (P_0(\rho_{\rm m})) + \rho_{\rm m} \operatorname{grad} (\frac{1}{8\pi} E^2 \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_{\rm m}}) - \rho_{\rm m} \operatorname{grad} (gz) = \mathbf{0}. \tag{8.28}$$

one obtains for approximately constant $\rho_{\rm m}$

$$P_0 = \rho_{\rm m} (\frac{1}{8\pi} E^2 \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho_{\rm m}} - gz) + \text{const.} \tag{8.29}$$

8.c MAXWELL's Stress Tensor

Now we represent the force density **k** as divergence of a tensor

$$k_\alpha = \nabla_\beta T_{\alpha,\beta}. \tag{8.30}$$

If one has such a representation, then the force acting on a volume V is given by

$$\mathbf{K} = \int_V \mathrm{d}^3 r \mathbf{k}(\mathbf{r}) = \int \mathrm{d}^3 r \mathbf{e}_\alpha \nabla_\beta T_{\alpha,\beta} = \int_{\partial V} \mathrm{d} f_\beta (\mathbf{e}_\alpha T_{\alpha,\beta}). \tag{8.31}$$

The force acting on the volume is such represented by a force acting on the surface. If it were isotropic $T_{\alpha,\beta} = -P\delta_{\alpha,\beta}$, we would call P the pressure acting on the surface. In the general case we consider here one calls T the stress tensor, since the pressure is anisotropic and there can be shear stress.
In order to calculate T we start from

$$k_\alpha = \rho_\mathrm{f} E_\alpha - \rho_\mathrm{m} \nabla_\alpha \left(\frac{\partial u}{\partial \rho_\mathrm{m}}\right). \tag{8.32}$$

We transform

$$\rho_\mathrm{f} E_\alpha = \frac{1}{4\pi} E_\alpha \nabla_\beta D_\beta = \frac{1}{4\pi}\left(\nabla_\beta (E_\alpha D_\beta)\right) - (\nabla_\beta E_\alpha) D_\beta) \tag{8.33}$$

and use $\nabla_\beta E_\alpha = \nabla_\alpha E_\beta$ because of $\operatorname{curl}\mathbf{E} = \mathbf{0}$. This yields

$$k_\alpha = \nabla_\beta (\frac{1}{4\pi} E_\alpha D_\beta) - \rho_\mathrm{m} \nabla_\alpha \left(\frac{\partial u}{\partial \rho_\mathrm{m}}\right) - \frac{1}{4\pi} D_\beta \nabla_\alpha E_\beta. \tag{8.34}$$

Now there is

$$\nabla_\alpha \left(u - \rho_\mathrm{m} \frac{\partial u}{\partial \rho_\mathrm{m}} - \frac{1}{4\pi}\mathbf{D}\cdot\mathbf{E}\right) = -\rho_\mathrm{m} \nabla_\alpha \frac{\partial u}{\partial \rho_\mathrm{m}} - \frac{1}{4\pi} D_\beta \nabla_\alpha E_\beta, \tag{8.35}$$

since $\partial u/\partial D_\beta = E_\beta/(4\pi)$. This yields the expression for the stress tensor

$$T_{\alpha,\beta} = \frac{1}{4\pi} E_\alpha D_\beta + \delta_{\alpha,\beta}\left(u - \rho_\mathrm{m}\frac{\partial u}{\partial \rho_\mathrm{m}} - \frac{1}{4\pi}\mathbf{D}\cdot\mathbf{E}\right). \tag{8.36}$$

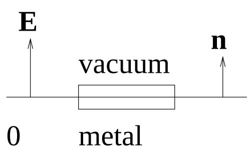
In particular with $u = u_0(\rho_\mathrm{m}) + D^2/(8\pi\epsilon(\rho_\mathrm{m}))$, (8.6) one obtains

$$T_{\alpha,\beta} = \frac{1}{4\pi} E_\alpha D_\beta + \delta_{\alpha,\beta}\left(-P_0(\rho_\mathrm{m}) - \frac{1}{8\pi}\mathbf{D}\cdot\mathbf{E} + \frac{1}{8\pi} E^2 \rho_\mathrm{m} \frac{d\epsilon}{d\rho_\mathrm{m}}\right). \tag{8.37}$$

MAXWELL's stress tensor reads in vacuum

$$T_{\alpha,\beta} = \frac{1}{4\pi} E_\alpha E_\beta - \frac{\delta_{\alpha,\beta}}{8\pi} E^2. \tag{8.38}$$

As an example we consider the electrostatic force on a plane piece of metal of area F. We have to evaluate



$$\mathbf{K} = \int d f_\beta (\mathbf{e}_\alpha T_{\alpha,\beta}) = \left(\frac{1}{4\pi}\mathbf{E}(\mathbf{E}\mathbf{n}) - \frac{1}{8\pi}\mathbf{n}E^2\right) F = \frac{1}{8\pi} E^2 \mathbf{n} F. \tag{8.39}$$

This is in agreement with the result from (7.8).

C Magnetostatics

In this chapter we consider magnetostatics starting from the equations, which were derived at the beginning of section (3.a) for time independent currents.

9 Magnetic Induction and Vector Potential

9.a Ampere's Law

From

$$\operatorname{curl} \mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r}) \tag{9.1}$$

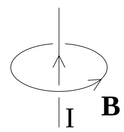
one obtains

$$\int \mathrm{d}\mathbf{f}\cdot\ \operatorname{curl} \mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}\int \mathrm{d}\mathbf{f}\cdot\mathbf{j}(\mathbf{r}), \tag{9.2}$$

which can be written by means of Stokes' theorem (B.56)

$$\oint \mathrm{d}\mathbf{r}\cdot\mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}I. \tag{9.3}$$

The line integral of the magnetic induction $\mathbf{B}$ along a closed line yields $4\pi/c$ times the current I through the line.Here the corkscrew rule applies: If the current moves in the direction of the corkscrew, then the magnetic induction has the direction in which the corkscrew rotates.



9.b Magnetic Flux

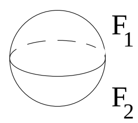
The magnetic flux Ψ^{m} through an oriented area F is defined as the integral

$$\Psi^{\mathrm{m}} = \int_F \mathrm{d}\mathbf{f}\cdot\mathbf{B}(\mathbf{r}). \tag{9.4}$$

The magnetic flux depends only on the boundary ∂F of the area. To show this we consider the difference of the flux through two areas F_1 and F_2 with the same boundary and obtain

$$\Psi^{\mathrm{m}}_1 - \Psi^{\mathrm{m}}_2 = \int_{F_1} \mathrm{d}\mathbf{f}\cdot\mathbf{B}(\mathbf{r}) - \int_{F_2} \mathrm{d}\mathbf{f}\cdot\mathbf{B}(\mathbf{r}) = \int_F \mathrm{d}\mathbf{f}\cdot\mathbf{B}(\mathbf{r}) = \int_V \mathrm{d}^3r \operatorname{div}\mathbf{B}(\mathbf{r}) = 0 \tag{9.5}$$

by means of the divergence theorem (B.59) and $\operatorname{div}\mathbf{B}(\mathbf{r}) = 0$.Suppose F_1 and F_2 are oriented in the same direction (for example upwards). Then the closed surface F is composed of F_1 and F_2, where F_2 is now oriented in the opposite direction.Then F has a definite orientation (for example outwards) and includes the volume V.



9.c Field of a Current Distribution

From curl curl $\mathbf{B}(\mathbf{r}) = (4\pi/c)\,\mathrm{curl}\,\mathbf{j}(\mathbf{r})$ due to (B.26)

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{B}(\mathbf{r}) = \mathrm{grad}\,\mathrm{div}\,\mathbf{B}(\mathbf{r}) - \triangle\mathbf{B}(\mathbf{r}) \tag{9.6}$$

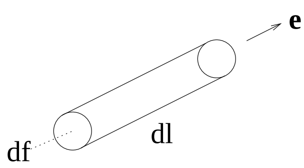
and $\mathrm{div}\,\mathbf{B}(\mathbf{r}) = 0$ one obtains

$$\triangle\mathbf{B}(\mathbf{r}) = -\frac{4\pi}{c}\,\mathrm{curl}\,\mathbf{j}(\mathbf{r}) \tag{9.7}$$

with the solution

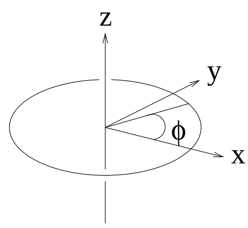
$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\int \mathrm{d}^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{curl}'\mathbf{j}(\mathbf{r}') = -\frac{1}{c}\int \mathrm{d}^3r' \Big(\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big)\times\mathbf{j}(\mathbf{r}') = \frac{1}{c}\int \mathrm{d}^3r' \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3}\times\mathbf{j}(\mathbf{r}'), \tag{9.8}$$

where we have used (B.63) at the second equals sign. The last expression is called the law of Biot and Savart.If the extension of a wire perpendicular to the direction of the current is negligible (filamentary wire) then one can approximate $\mathrm{d}^3r'\mathbf{j}(\mathbf{r}') = \mathrm{d}f'\mathrm{d}l'\,j(\mathbf{r}')\mathbf{e} = I\mathrm{d}\mathbf{r}'$ and obtains



$$\mathbf{B}(\mathbf{r}) = \frac{I}{c}\int \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3}\times\mathrm{d}\mathbf{r}' \tag{9.9}$$

As an example we consider the induction in the middle axis of a current along a circle



$$\mathbf{r} = z\mathbf{e}_z, \quad \mathbf{r}' = (R\cos\phi, R\sin\phi, z') \quad \mathrm{d}\mathbf{r}' = (-R\sin\phi, R\cos\phi, 0)\mathrm{d}\phi \tag{9.10}$$

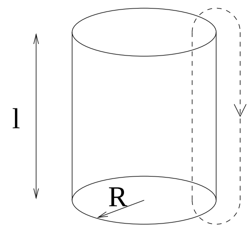
$$(\mathbf{r}'-\mathbf{r})\times\mathrm{d}\mathbf{r}' = (R(z-z')\cos\phi, R(z-z')\sin\phi, R^2)\mathrm{d}\phi \tag{9.11}$$

$$\mathbf{B}(0,0,z) = \frac{2\pi I R^2\mathbf{e}_z}{c(R^2+(z-z')^2)^{3/2}}. \tag{9.12}$$

Starting from this result we calculate the field in the axis of a coil. The number of windings be N and it extends from $z' = -l/2$ to $z' = +l/2$. Then we obtain

$$\mathbf{B}(0,0,z) = \int_{-l/2}^{+l/2} \frac{N\mathrm{d}z'}{l}\frac{2\pi I R^2\mathbf{e}_z}{c(R^2+(z-z')^2)^{3/2}} = \frac{2\pi IN}{cl}\mathbf{e}_z\left(\frac{\frac{l}{2}-z}{\sqrt{R^2+(\frac{l}{2}-z)^2}} + \frac{\frac{l}{2}+z}{\sqrt{R^2+(\frac{l}{2}+z)^2}}\right). \tag{9.13}$$

If the coil is long, $R \ll l$, then one may neglect R^2 and obtains inside the coil



$$\mathbf{B} = \frac{4\pi IN}{cl}\mathbf{e}_z. \tag{9.14}$$

At the ends of the coil the field has decayed to one half of its intensity inside the coil. From Ampere's law one obtains by integration along the path described in the figure

$$\oint \mathrm{d}\mathbf{r}\cdot\mathbf{B} = \frac{4\pi}{c}IN. \tag{9.15}$$

Thus inside the coil one obtains the induction (9.14), whereas the magnetic induction outside is comparatively small.

9.d Vector Potential

We now rewrite the expression for the magnetic induction

$$\mathbf{B}(\mathbf{r}) = -\frac{1}{c}\int \mathrm{d}^3r'\Big(\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big)\times\mathbf{j}(\mathbf{r}') = \frac{1}{c}\int \mathrm{d}^3r'\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\times\mathbf{j}(\mathbf{r}') = \mathrm{curl}\,\mathbf{A}(\mathbf{r}) \tag{9.16}$$

with

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{9.17}$$

One calls **A** the vector potential. Consider the analog relation between charge density ρ and the electric potential ϕ in electrostatics (3.14). We show that **A** is divergence-free

$$\begin{aligned} \operatorname{div}\mathbf{A}(\mathbf{r}) &= \frac{1}{c}\int d^3r' \left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{j}(\mathbf{r}') = -\frac{1}{c}\int d^3r' \left(\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{j}(\mathbf{r}') \\ &= \frac{1}{c}\int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\mathbf{j}(\mathbf{r}') = 0. \end{aligned} \tag{9.18}$$

At the third equals sign we have performed a partial integration (B.62). Finally we have used $\operatorname{div}\mathbf{j}(\mathbf{r}) = 0$.

9.e Force Between Two Circuits

Finally, we consider the force between two circuits. The force exerted by circuit (1) on circuit (2) is

$$\begin{aligned} \mathbf{K}_2 &= \frac{1}{c}\int d^3r\,\mathbf{j}_2(\mathbf{r})\times\mathbf{B}_1(\mathbf{r}) = \frac{1}{c^2}\int d^3r d^3r'\,\mathbf{j}_2(\mathbf{r})\times\left(\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{j}_1(\mathbf{r}')\right) \\ &= \frac{1}{c^2}\int d^3r d^3r' \big(\mathbf{j}_1(\mathbf{r}')\cdot\mathbf{j}_2(\mathbf{r})\big)\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{c^2}\int d^3r d^3r' \left(\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{j}_2(\mathbf{r})\right)\mathbf{j}_1(\mathbf{r}') \end{aligned} \tag{9.19}$$

where (B.14) has been applied. Since due to (B.62)

$$\int d^3r \left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{j}_2(\mathbf{r}) = -\int d^3r \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla\cdot\mathbf{j}_2(\mathbf{r}) \tag{9.20}$$

and $\operatorname{div}\mathbf{j}_2(\mathbf{r}) = 0$, one obtains finally for the force

$$\mathbf{K}_2 = \frac{1}{c^2}\int d^3r d^3r' \big(\mathbf{j}_1(\mathbf{r}')\cdot\mathbf{j}_2(\mathbf{r})\big)\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{9.21}$$

The force acting on circuit (1) is obtained by exchanging 1 and 2. Simultaneously, one can exchange **r** and **r**′. One sees then that

$$\mathbf{K}_1 = -\mathbf{K}_2 \tag{9.22}$$

holds.

Exercise Calculate the force between two wires of length l carrying currents I_1 and I_2 which run parallel in a distance r ($r \ll l$). Kohlrausch and Weber measured this force in order to determine the velocity of light.

10 Loops of Current as Magnetic Dipoles

10.a Localized Current Distribution and Magnetic Dipole

We consider a distribution of currents which vanishes outside a sphere of radius R ($\mathbf{j}(\mathbf{r}') = \mathbf{0}$ for $r' > R$) and determine the magnetic induction $\mathbf{B}(\mathbf{r})$ for $r > R$. We may expand the vector potential $\mathbf{A}(\mathbf{r})$ (9.17) similar to the electric potential $\Phi(\mathbf{r})$ in section (4)

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{cr}\int d^3r' \mathbf{j}(\mathbf{r}') + \frac{x_\alpha}{cr^3}\int d^3r' x'_\alpha \mathbf{j}(\mathbf{r}') + ... \tag{10.1}$$

Since no current flows through the surface of the sphere one obtains

$$0 = \int d\mathbf{f}\cdot g(\mathbf{r})\mathbf{j}(\mathbf{r}) = \int d^3r \operatorname{div}(g(\mathbf{r})\mathbf{j}(\mathbf{r})) = \int d^3r \operatorname{grad} g(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}) + \int d^3r g(\mathbf{r}) \operatorname{div}\mathbf{j}(\mathbf{r}), \tag{10.2}$$

where the integrals are extended over the surface and the volume of the sphere, respectively. From the equation of continuity (1.12,3.1) it follows that

$$\int d^3r \operatorname{grad} g(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}) = 0. \tag{10.3}$$

This is used to simplify the integral in the expansion (10.1). With $g(\mathbf{r}) = x_\alpha$ one obtains

$$\int d^3r\, j_\alpha(\mathbf{r}) = 0. \tag{10.4}$$

Thus the first term in the expansion vanishes. There is no contribution to the vector potential decaying like $1/r$ in magnetostatics, i.e. there is no magnetic monopole. With $g(\mathbf{r}) = x_\alpha x_\beta$ one obtains

$$\int d^3r\Big(x_\alpha j_\beta(\mathbf{r}) + x_\beta j_\alpha(\mathbf{r})\Big) = 0. \tag{10.5}$$

Thus we can rewrite

$$\int d^3r x_\alpha j_\beta = \frac{1}{2}\int d^3r(x_\alpha j_\beta - x_\beta j_\alpha) + \frac{1}{2}\int d^3r(x_\alpha j_\beta + x_\beta j_\alpha). \tag{10.6}$$

The second integral vanishes, as we have seen. The first one changes its sign upon exchanging the indices α and β. One introduces

$$\int d^3r x_\alpha j_\beta = \frac{1}{2}\int d^3r(x_\alpha j_\beta - x_\beta j_\alpha) = c\epsilon_{\alpha,\beta,\gamma} m_\gamma \tag{10.7}$$

and calls the resulting vector

$$\mathbf{m} = \frac{1}{2c}\int d^3r'(\mathbf{r}'\times\mathbf{j}(\mathbf{r}')) \tag{10.8}$$

magnetic dipole moment.. Then one obtains

$$A_\beta(\mathbf{r}) = \frac{x_\alpha}{cr^3} c\epsilon_{\alpha,\beta,\gamma} m_\gamma + ... \tag{10.9}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m}\times\mathbf{r}}{r^3} + ... \tag{10.10}$$

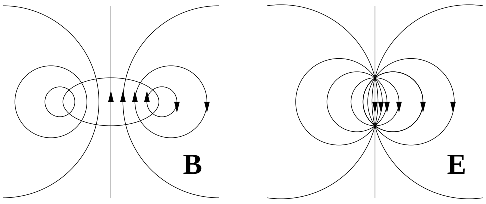
with $\mathbf{B}(\mathbf{r}) = \operatorname{curl}\mathbf{A}(\mathbf{r})$ one obtains

$$\mathbf{B}(\mathbf{r}) = \frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r}) - \mathbf{m}r^2}{r^5} + ... \tag{10.11}$$

This is the field of a magnetic dipole. It has the same form as the electric field of an electric dipole (4.12)

$$\mathbf{E}(\mathbf{r}) = -\operatorname{grad}\Big(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\Big) = \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r}) - \mathbf{p}r^2}{r^5}, \tag{10.12}$$

but there is a difference at the location of the dipole. This can be seen in the accompanying figure. Calculate the $\delta^3(\mathbf{r})$-contribution to both dipolar moments. Compare (B.71).



10.b Magnetic Dipolar Moment of a Current Loop

The magnetic dipolar moment of a current on a closed curve yields

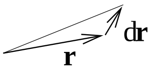
$$\mathbf{m} = \frac{I}{2c}\int \mathbf{r}\times \mathrm{d}\mathbf{r} = \frac{I}{c}\mathbf{f}, \tag{10.13}$$

e.g.

$$m_z = \frac{I}{2c}\int (x\mathrm{d}y - y\mathrm{d}x) = \frac{I}{c}f_z. \tag{10.14}$$

Here f_α is the projection of the area inside the loop onto the plane spanned by the two other axes

$$\mathrm{d}\mathbf{f} = \frac{1}{2}\mathbf{r}\times \mathrm{d}\mathbf{r}. \tag{10.15}$$



If $\mathbf{j} = \sum_i q_i \mathbf{v}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$, then using (10.8) the magnetic moment reads

$$\mathbf{m} = \frac{1}{2c}\sum_i q_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i \frac{q_i}{2m_i c}\mathbf{l}_i, \tag{10.16}$$

where m_i is the mass and $\mathbf{l}_i$ the angular momentum. If only one kind of charges is dealt with, then one has

$$\mathbf{m} = \frac{q}{2mc}\mathbf{l}. \tag{10.17}$$

This applies for orbital currents. For spins, however, one has

$$\mathbf{m} = \frac{q}{2mc}g\mathbf{s}, \tag{10.18}$$

where $\mathbf{s}$ is the angular momentum of the spin. The gyromagnetic factor for electrons is $g = 2.0023$ and the components of the spin $\mathbf{s}$ are $\pm\hbar/2$. Since in quantum mechanics the orbital angular momentum assumes integer multiples of $\hbar$, one introduces as unit for the magnetic moment of the electron Bohr's magneton, $\mu_B = \frac{e_0\hbar}{2m_0c} = 0.927\cdot 10^{-20}\ \mathrm{dyn}^{1/2}\ \mathrm{cm}^2$.

10.c Force and Torque on a Dipole in an External Magnetic Field

10.c.α Force

An external magnetic induction $\mathbf{B}_\mathrm{a}$ exerts on a loop of a current the Lorentz force

$$\mathbf{K} = \frac{1}{c}\int \mathrm{d}^3r\mathbf{j}(\mathbf{r})\times\mathbf{B}_\mathrm{a}(\mathbf{r}) = -\frac{1}{c}\mathbf{B}_\mathrm{a}(0)\times\int \mathrm{d}^3r\mathbf{j}(\mathbf{r}) - \frac{1}{c}\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}\times\int \mathrm{d}^3r x_\alpha j_\beta(\mathbf{r})\mathbf{e}_\beta - ... = -\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}\times \mathbf{e}_\beta\epsilon_{\alpha,\beta,\gamma}m_\gamma. \tag{10.19}$$

We rewrite $m_\gamma\epsilon_{\alpha,\beta,\gamma}\mathbf{e}_\beta = m_\gamma\mathbf{e}_\gamma\times\mathbf{e}_\alpha = \mathbf{m}\times\mathbf{e}_\alpha$ and find

$$\mathbf{K} = -\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha}\times(\mathbf{m}\times\mathbf{e}_\alpha) = (\mathbf{m}\cdot\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha})\mathbf{e}_\alpha - (\mathbf{e}_\alpha\cdot\frac{\partial \mathbf{B}_\mathrm{a}}{\partial x_\alpha})\mathbf{m}. \tag{10.20}$$

The last term vanishes because of $\operatorname{div}\mathbf{B} = 0$. The first term on the right hand side can be written $(\mathbf{m}\cdot\frac{\partial\mathbf{B}_\mathrm{a}}{\partial x_\alpha})\mathbf{e}_\alpha = m_\gamma\frac{\partial B_{\mathrm{a},\gamma}}{\partial x_\alpha}\mathbf{e}_\alpha = m_\gamma\frac{\partial B_{\mathrm{a},\alpha}}{\partial x_\gamma}\mathbf{e}_\alpha = (\mathbf{m}\nabla)\mathbf{B}_\mathrm{a}$, where we have used $\operatorname{curl}\mathbf{B}_\mathrm{a} = \mathbf{0}$ in the region of the dipole. Thus we obtain the force

$$\mathbf{K} = (\mathbf{m}\operatorname{grad})\mathbf{B}_\mathrm{a} \tag{10.21}$$

acting on the magnetic dipole expressed by the vector gradient (B.18). This is in analogy to (4.35), where we obtained the force $(\mathbf{p}\operatorname{grad})\mathbf{E}_\mathrm{a}$ acting on an electric dipole.

10.c.β Torque

The torque on a magnetic dipole is given by

$$\mathbf{M}_\mathrm{mech} = \frac{1}{c}\int \mathrm{d}^3r\mathbf{r}\times(\mathbf{j}\times\mathbf{B}_\mathrm{a}) = -\frac{1}{c}\mathbf{B}_\mathrm{a}\int \mathrm{d}^3r(\mathbf{r}\cdot\mathbf{j}) + \frac{1}{c}\int \mathrm{d}^3r(\mathbf{B}_\mathrm{a}\cdot\mathbf{r})\mathbf{j}. \tag{10.22}$$

The first integral vanishes, which is easily seen from (10.3) and $g = r^2/2$. The second integral yields

$$\mathbf{M}_\mathrm{mech} = \frac{1}{c}\mathbf{e}_\beta B_{\mathrm{a},\alpha}\int \mathrm{d}^3r x_\alpha j_\beta = B_{\mathrm{a},\alpha}\mathbf{e}_\beta\epsilon_{\alpha,\beta,\gamma}m_\gamma = \mathbf{m}\times\mathbf{B}_\mathrm{a} \tag{10.23}$$

Analogously the torque on an electric dipole was $\mathbf{p}\times\mathbf{E}_\mathrm{a}$, (4.36).
From the law of force one concludes the energy of a magnetic dipole in an external field as

$$U = -\mathbf{m}\cdot\mathbf{B}_\mathrm{a}. \tag{10.24}$$

This is correct for permanent magnetic moments. However, the precise derivation of this expression becomes clear only when we treat the law of induction (section 13).

11 Magnetism in Matter. Field of a Coil

11.a Magnetism in Matter

In a similar way as we separated the polarization charges from freely accessible charges, we divide the current density into a freely moving current density $\mathbf{j}_\mathrm{f}$ and the density of the magnetization current $\mathbf{j}_\mathrm{M}$, which may come from orbital currents of electrons

$$\mathbf{j}(\mathbf{r}) = \mathbf{j}_\mathrm{f}(\mathbf{r}) + \mathbf{j}_\mathrm{M}(\mathbf{r}). \tag{11.1}$$

We introduce the magnetization as the density of magnetic dipoles

$$\mathbf{M} = \frac{\sum \mathbf{m}_i}{\Delta V} \tag{11.2}$$

and conduct the continuum limit

$$\sum_i \mathbf{m}_i f(\mathbf{r}_i) \to \int \mathrm{d}^3r' \mathbf{M}(\mathbf{r}') f(\mathbf{r}'). \tag{11.3}$$

Then using (10.10) we obtain for the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathrm{d}^3r' \frac{\mathbf{j}_\mathrm{f}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int \mathrm{d}^3r' \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{11.4}$$

The second integral can be rewritten

$$\int \mathrm{d}^3r' \mathbf{M}(\mathbf{r}')\times\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \int \mathrm{d}^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\times\mathbf{M}(\mathbf{r}'), \tag{11.5}$$

so that one obtains

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathrm{d}^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\mathbf{j}_\mathrm{f}(\mathbf{r}') + c\,\mathrm{curl}'\mathbf{M}(\mathbf{r}')). \tag{11.6}$$

Thus one interprets

$$\mathbf{j}_\mathrm{M}(\mathbf{r}') = c\,\mathrm{curl}'\mathbf{M}(\mathbf{r}') \tag{11.7}$$

as the density of the magnetization current. Then one obtains for the magnetic induction

$$\mathrm{curl}\,\mathbf{B}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}_\mathrm{f}(\mathbf{r}) + 4\pi\,\mathrm{curl}\,\mathbf{M}(\mathbf{r}). \tag{11.8}$$

Now one introduces the magnetic field strength

$$\mathbf{H}(\mathbf{r}) := \mathbf{B}(\mathbf{r}) - 4\pi\mathbf{M}(\mathbf{r}) \tag{11.9}$$

for which MAXWELL's equation

$$\mathrm{curl}\,\mathbf{H}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}_\mathrm{f}(\mathbf{r}) \tag{11.10}$$

holds. MAXWELL's equation $\mathrm{div}\,\mathbf{B}(\mathbf{r}) = 0$ remains unchanged.
One obtains for paramagnetic and diamagnetic materials in not too strong fields

$$\mathbf{M} = \chi_\mathrm{m}\mathbf{H}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \mu = 1 + 4\pi\chi_\mathrm{m}, \tag{11.11}$$

where χ_m is the magnetic susceptibility and μ the permeability. In superconductors (of first kind) one finds complete diamagnetism $\mathbf{B} = \mathbf{0}$. There the magnetic induction is completely expelled from the interior by surface currents.
In analogy to the arguments for the dielectric displacement and the electric field one obtains that the normal component B_n is continuous, and in the absence of conductive currents also the tangential components $\mathbf{H}_\mathrm{t}$ are continuous across the boundary.
In the GAUSSian system of units the fields **M** and **H** are measured just as **B** in $\mathrm{dyn}^{1/2}\,\mathrm{cm}^{-1}$, whereas in SI-units **B** is measured in $\mathrm{Vs/m}^2$, **H** and **M** in A/m. The conversion factors for **H** and **M** differ by a factor 4π. For more information see appendix A.

11.b Field of a coil

The field of a coil along its axis was determined in (9.13). We will now determine the field of a cylindrical coil in general. In order to do so we firstly consider an electric analogy. The field between two charges q and $-q$ at $\mathbf{r}_2$ and $\mathbf{r}_1$ is equivalent to a line of electric dipoles $\mathrm{d}\mathbf{p} = q\mathrm{d}\mathbf{r}'$ from $\mathbf{r}' = \mathbf{r}_1$ to $\mathbf{r}' = \mathbf{r}_2$. Indeed we obtain for the potential

$$\Phi(\mathbf{r}) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{p}}{|\mathbf{r}-\mathbf{r}'|^3} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{q(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\frac{q}{2}\int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{\mathrm{d}(\mathbf{r}-\mathbf{r}')^2}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{q}{|\mathbf{r}-\mathbf{r}_2|} - \frac{q}{|\mathbf{r}-\mathbf{r}_1|} \tag{11.12}$$

and thus for the field

$$\mathbf{E}(\mathbf{r}) = q\left(\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3} - \frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3}\right). \tag{11.13}$$

The magnetic analogy is to think of a long thin coil as consisting of magnetic dipoles

$$\mathrm{d}\mathbf{m} = \frac{\mathrm{d}I}{\mathrm{d}l}\frac{f}{c}\mathrm{d}\mathbf{r} = \frac{NIf}{lc}\mathrm{d}\mathbf{r}. \tag{11.14}$$

If we consider that the field of the electric and the magnetic dipole have the same form (10.11, 10.12) except at the point of the dipole, then it follows that we may replace q by $q_\mathrm{m} = NIf/(lc)$ in order to obtain the magnetic induction

$$\mathbf{B}(\mathbf{r}) = q_\mathrm{m}\left(\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3} - \frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3}\right). \tag{11.15}$$

Thus the field has a form which can be described by two magnetic monopoles with strengths q_m und $-q_\mathrm{m}$. However, at the positions of the dipoles the field differs in the magnetic case. There, i.e. inside the coil an additional field $B = 4\pi NI/(lc)$ flows back so that the field is divergency free and fulfills Ampere's law.
In order to obtain the result in a more precise way one uses the following consideration: We represent the current density in analogy to (11.7) as curl of a fictitious magnetization $\mathbf{j}_\mathrm{f} = c\,\mathrm{curl}\,\mathbf{M}_\mathrm{f}(\mathbf{r})$ inside the coil, outside $\mathbf{M}_\mathrm{f} = \mathbf{0}$. For a cylindrical (its cross-section needs not be circular) coil parallel to the z-axis one puts simply $\mathbf{M}_\mathrm{f} = NI\mathbf{e}_z/(cl)$. Then one obtains from

$$\mathrm{curl}\,\mathbf{B} = \frac{4\pi}{c}\mathbf{j}_\mathrm{f}(\mathbf{r}) = 4\pi\,\mathrm{curl}\,\mathbf{M}_\mathrm{f} \tag{11.16}$$

the induction $\mathbf{B}$ in the form

$$\mathbf{B}(\mathbf{r}) = 4\pi\mathbf{M}_\mathrm{f}(\mathbf{r}) - \mathrm{grad}\,\Psi(\mathbf{r}). \tag{11.17}$$

The function Ψ is determined from

$$\mathrm{div}\,\mathbf{B} = 4\pi\,\mathrm{div}\,\mathbf{M}_\mathrm{f} - \triangle\Psi = 0 \tag{11.18}$$

as

$$\Psi(\mathbf{r}) = -\int \mathrm{d}^3r'\,\frac{\mathrm{div}'\mathbf{M}_\mathrm{f}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{11.19}$$

The divergency yields in the present case of a cylindrical coil a contribution $\delta(z-z_1)NI/(cl)$ at the covering and a contribution $-\delta(z-z_2)NI/(cl)$ at the basal surface of the coil, since the component of $\mathbf{B}$ normal to the surface jumps by $NI/(cl)$, which yields

$$\Psi(\mathbf{r}) = \frac{NI}{cl}\left(\int_{F_2}\frac{\mathrm{d}^2r'}{|\mathbf{r}-\mathbf{r}'|} - \int_{F_1}\frac{\mathrm{d}^2r'}{|\mathbf{r}-\mathbf{r}'|}\right), \tag{11.20}$$

where F_2 is the covering and F_1 the basal surface. Thus one obtains an induction, as if there were magnetic charge densities $\pm NI/(cl)$ per area at the covering and the basal surface. This contribution yields a discontinuity of the induction at these parts of the surface which is compensated by the additional contribution $4\pi\mathbf{M}_\mathrm{f}$ inside the coil. The total strength of pole $\pm q_\mathrm{m}$ is the area of the basal (ground) surface times the charge density per area. One calls $\Psi(\mathbf{r})$ the magnetic potential. In view of the additional contribution $4\pi\mathbf{M}_\mathrm{f}(\mathbf{r})$ in (11.17) it is in contrast to the potentials $\Phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ only of limited use. We will not use it in the following.
Exercise Calculate magnetic field and magnetic induction for the coil filled by a core of permeability μ.
Exercise Show that the z-component of the magnetic induction is proportional to the difference of the solid angles under which the (transparently thought) coil appears from outside and from inside.

D Law of Induction

12 Faraday's Law of Induction

The force acting on charges is $q(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c)$. It does not matter for the charges, whether the force is exerted by the electric field or by the magnetic induction. Thus they experience in a time-dependent magnetic field an effective electric field

$$\mathbf{E}^{(\mathrm{ind})}=\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B} \tag{12.1}$$

with $\operatorname{curl}\mathbf{E}=-\dot{\mathbf{B}}/c$. Therefore the voltage along a loop of a conductor is given by

$$V^{(\mathrm{ind})}=\oint\mathbf{E}\cdot\mathrm{d}\mathbf{r}+\oint(\frac{\mathbf{v}}{c}\times\mathbf{B})\cdot\mathrm{d}\mathbf{r}. \tag{12.2}$$

The first integral gives a contribution due to the variation of the magnetic induction. For a fixed loop and varying **B** one obtains (since $\mathbf{v}\parallel\mathrm{d}\mathbf{r}$)

$$V^{(\mathrm{ind})}=\oint\mathbf{E}\cdot\mathrm{d}\mathbf{r}=\int\operatorname{curl}\mathbf{E}\cdot\mathrm{d}\mathbf{f}=-\frac{1}{c}\int\frac{\partial\mathbf{B}}{\partial t}\cdot\mathrm{d}\mathbf{f}=-\frac{1}{c}\left.\frac{\mathrm{d}\Psi^{\mathrm{m}}}{\mathrm{d}t}\right|_{\mathrm{loop\ fixed}}. \tag{12.3}$$

The second integral in (12.2) gives a contribution due to the motion of the loop. In order to investigate a loop which moves (and is distorted) we use a parameter representation of the loop $\mathbf{r}=\mathbf{r}(t,p)$ with the body-fixed parameter p. For fixed t we have $\mathrm{d}\mathbf{r}=(\partial\mathbf{r}/\partial p)\mathrm{d}p$ and

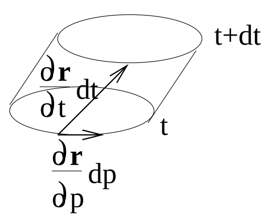
$$\mathbf{v}=\frac{\partial\mathbf{r}}{\partial t}+\lambda(p,t)\frac{\partial\mathbf{r}}{\partial p} \tag{12.4}$$

with a $\lambda=\mathrm{d}p/\mathrm{d}t$ which depends on the motion of the charges in the conductor. This yields

$$\mathrm{d}t\oint\Big(\frac{\mathbf{v}}{c}\times\mathbf{B}\Big)\cdot\mathrm{d}\mathbf{r}=-\frac{1}{c}\int\Big(\frac{\partial\mathbf{r}}{\partial t}\times\frac{\partial\mathbf{r}}{\partial p}\Big)\cdot\mathbf{B}\,\mathrm{d}p\,\mathrm{d}t=-\frac{1}{c}\int\mathrm{d}\mathbf{f}\cdot\mathbf{B}, \tag{12.5}$$

since $\frac{\partial\mathbf{r}}{\partial t}\times\frac{\partial\mathbf{r}}{\partial p}\,\mathrm{d}p\,\mathrm{d}t$ is the element of the area which in time $\mathrm{d}t$ is swept over by the conductor element $\mathrm{d}p$. Therefore we obtain

$$\oint(\frac{\mathbf{v}}{c}\times\mathbf{B})\cdot\mathrm{d}\mathbf{r}=-\frac{1}{c}\left.\frac{\mathrm{d}\Psi^{m}}{\mathrm{d}t}\right|_{\mathbf{B}\,\mathrm{fixed}}. \tag{12.6}$$

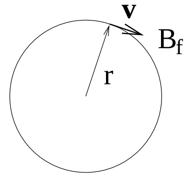


The total induced voltage is composed by the change of the magnetic flux due to the change of the magnetic induction (12.3) and by the motion of the loop (12.6)

$$V^{(\mathrm{ind})}=-\frac{1}{c}\frac{\mathrm{d}\Psi^{\mathrm{m}}}{\mathrm{d}t}, \tag{12.7}$$

and is thus given by the total change of the magnetic flux through the loop. Thus it does not matter for a generator whether the generating magnetic field rotates or whether the coil is rotating.

The betatron (non-relativistic) The electrons move along circular orbits and are kept on these by the Lorentz force exerted by the guide field B_{f}. Thus the centrifugal force and the Lorentz force have to compensate each other



$$\frac{mv^2}{r}=e_0\frac{v}{c}B_{\mathrm{f}}\quad\rightarrow\quad mv=\frac{e_0}{c}B_{\mathrm{f}}r. \tag{12.8}$$

The electrons are accelerated by the induction

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = -e_0 E = \frac{e_0}{2\pi r}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{c}\int B\,\mathrm{d}f = \frac{e_0}{2\pi r c} r^2 \pi \frac{\mathrm{d}\overline{B}}{\mathrm{d}t}. \tag{12.9}$$

Here $\overline{B}$ is the averaged magnetic induction inside the circle. Thus one has

$$mv = \frac{e_0}{2}\overline{B}\frac{r}{c} = \frac{e_0}{c}B_{\mathrm{f}} r, \tag{12.10}$$

from which the betatron condition $B_{\mathrm{f}} = \overline{B}/2$ follows.

13 Inductances and Electric Circuits

13.a Inductances

The magnetic flux through a coil and a circuit $\#j$, resp. is given by

$$\Psi_j^{\mathrm{m}} = \int \mathrm{d}\mathbf{f}_j \cdot \mathbf{B}(\mathbf{r}_j) = \int \mathrm{d}\mathbf{f}_j \cdot \operatorname{curl} \mathbf{A}(\mathbf{r}_j) = \oint \mathrm{d}\mathbf{r}_j \cdot \mathbf{A}(\mathbf{r}_j). \tag{13.1}$$

Several circuits generate the vector-potential

$$\mathbf{A}(\mathbf{r}) = \sum_k \frac{I_k}{c} \oint \frac{\mathrm{d}\mathbf{r}_k}{|\mathbf{r} - \mathbf{r}_k|}. \tag{13.2}$$

Therefore the magnetic flux can be expressed by

$$\frac{1}{c}\Psi_j^{\mathrm{m}} = \sum_k L_{j,k} I_k \tag{13.3}$$

with

$$L_{j,k} = \frac{1}{c^2} \int \frac{\mathrm{d}\mathbf{r}_j \cdot \mathrm{d}\mathbf{r}_k}{|\mathbf{r}_j - \mathbf{r}_k|}. \tag{13.4}$$

Therefore one has $L_{j,k} = L_{k,j}$. For $j \neq k$ they are called mutual inductances, for $j = k$ self-inductances. In calculating the self-inductances according to (13.4) logarithmic divergencies appear, when $\mathbf{r}_j$ approaches $\mathbf{r}_k$, if the current distribution across the cross-section is not taken into account. The contributions $|\mathbf{r}_j - \mathbf{r} - k| < r_0/(2\mathrm{e}^{1/4})$ have to be excluded from the integral, where r_0 is the radius of the circular cross-section of the wire (compare Becker-Sauter).
The dimension of the inductances is given by s^2/cm. The conversion into the SI-system is given by $1\mathrm{s}^2/\mathrm{cm} \,\hat{=}\, 9 \cdot 10^{11}\mathrm{Vs/A} = 9 \cdot 10^{11}$ H (Henry).
If the regions in which the magnetic flux is of appreciable strength is filled with a material of permeability μ, then from $\operatorname{curl} \mathbf{H} = 4\pi\mathbf{j}_\mathrm{f}/c$ one obtains $\operatorname{curl}(\mathbf{B}/\mu) = 4\pi\mathbf{j}_\mathrm{f}/c$, so that

$$L_{j,k}^{\mathrm{Mat}} = \mu L_{j,k}^{\mathrm{Vak}}. \tag{13.5}$$

holds. Thus one obtains large inductances by cores of high permeability $\mu \approx 10^3 ... 10^4$ in the yoke.
Inductance of a long coil If a closed magnetic yoke of length l and cross-section f is surrounded by N windings of wire, through which a current I flows, then from Ampere's law $Hl = 4\pi IN/c$ one obtains the magnetic induction $B = 4\pi IN\mu/(cl)$. The magnetic flux can then be written $Bf = cL_0NI$ with $L_0 = 4\pi\mu f/c^2 l$. For N turns the magnetic flux is to be multiplied by N, which yields the self-induction $L = L_0 N^2$. For mutual inductances between two circuits with N_1 and N_2 turns one obtains $L_{1,2} = L_0 N_1 N_2$. Thus we obtain in general

$$L_{i,j} = L_0 N_i N_j, \quad L_0 = \frac{4\pi\mu f}{c^2 l}. \tag{13.6}$$

13.b Elements of Circuits

We consider now circuits, which contain the following elements: voltage sources, ohmic resistors, inductances, and capacitors. Whereas we have already introduced inductances and capacitors, we have to say a few words on the two other elements.
Voltage sources A voltage source or electromotive force with voltage $V^{(\mathrm{e})}(t)$ feeds the power $V^{(\mathrm{e})}I$ into the system. An example is a battery which transforms chemical energy into electromagnetic one. The voltages $V^{(\mathrm{ind})}$ of the inductances are also called electromotive forces.
Ohmic resistors In many materials the current density and the electric field are proportional if the field is not too strong. The coefficient of proportionality σ is called conductivity

$$\mathbf{j} = \sigma\mathbf{E}. \tag{13.7}$$

For a wire of length l and cross-section f one obtains

$$I = jf = \sigma f E = \sigma \frac{f}{l} V^{(R)}. \tag{13.8}$$

Here $V^{(R)}$ is the OHMic voltage drop along the conductor. Thus one has

$$V^{(R)} = RI, \quad R = \frac{l}{\sigma f} \tag{13.9}$$

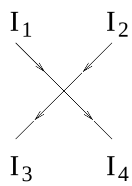
with the OHMic resistance R. In GAUSSian units the conductivity σ is measured in 1/s and the resistance R in s/cm. The conversion into the SI-system is obtained by $c^{-1} \widehat{=} 30\Omega$. The electromagnetic energy is dissipated in an OHMic resistor into heat at the rate $V^{(R)} I$.

13.c KIRCHHOFF's Rules

KIRCHHOFF's first Law (Current Law)

KIRCHHOFF's first law states that at each electrical contact, where several wires are joined, the sum of the incoming currents equals the sum of the outgoing currents

$$\sum I_{\text{incoming}} = \sum I_{\text{outgoing}}. \tag{13.10}$$

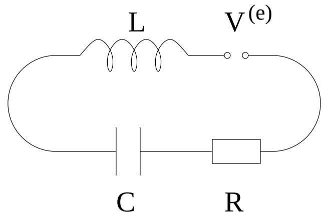


This rule is the macroscopic form of $\operatorname{div} \mathbf{j} = 0$. In the figure aside it implies $I_1 + I_2 = I_3 + I_4$.

KIRCHHOFF's second Law (Voltage Law)

The second law says that along a closed path the sum of electromotive forces equals the sum of the other voltage drops

$$\sum \left(V^{(e)} + V^{(ind)} \right) = \sum \left(V^{(R)} + V^{(C)} \right), \tag{13.11}$$



where

$$V^{(ind)} = -\mathrm{d}(LI)/\mathrm{d}t, \quad V^{(C)} = q/C, \quad \mathrm{d}V^{(C)}/\mathrm{d}t = I/C. \tag{13.12}$$

This rule is FARADAY's induction law in macroscopic form.

13.d Energy of Inductances

In order to determine the energies of inductances we consider circuits with electromotive forces, OHMic resistors and inductive couplings

$$V_j^{(e)} + V_j^{(ind)} = R_j I_j. \tag{13.13}$$

The variation of the electromagnetic energy as a function of time is then given by

$$\dot{U}_{em} = \sum_j I_j V_j^{(e)} - \sum_j R_j I_j^2 + L_{mech} = -\sum_j I_j V_j^{(ind)} + L_{mech} \tag{13.14}$$

with

$$V_j^{(ind)} = -\frac{1}{c} \dot{\Psi}_j^m = -\frac{\mathrm{d}}{\mathrm{d}t} (\sum_k L_{j,k} I_k). \tag{13.15}$$

Here L_{mech} is the mechanical power fed into the system.
Now we consider various cases:

13.d.α Constant Inductances

We keep the circuits fixed, then $L_{j,k} =$ const, $L_{\rm mech} = 0$ holds. From this it follows that

$$\dot{U}_{\rm em} = \sum_{j,k} I_j L_{j,k} \dot{I}_k, \tag{13.16}$$

from which we obtain the energies of the inductances

$$U_{\rm em} = \frac{1}{2} \sum_{j,k} I_j L_{j,k} I_k. \tag{13.17}$$

13.d.β Moving Loops of Currents

Now we move the circuits against each other. This yields

$$\begin{aligned} L_{\rm mech} &= \dot{U}_{\rm em} + \sum_j I_j V_j^{\rm (ind)} = \sum_{j,k} (I_j L_{j,k} \dot{I}_k + \frac{1}{2} I_j \dot{L}_{j,k} I_k) - \sum_{j,k} (I_j \dot{L}_{j,k} I_k + I_j L_{j,k} \dot{I}_k) \\ &= -\frac{1}{2} \sum_{j,k} I_j \dot{L}_{j,k} I_k = -\left.\frac{\partial U_{\rm em}}{\partial t}\right|_I . \end{aligned} \tag{13.18}$$

Thus the mechanical work to be done is not given by the change of the electromagnetic energy $U_{\rm em}$ at constant currents I, but by its negative.

13.d.γ Constant Magnetic Fluxes

In case there are no electromotive forces $V_j^{\rm (e)} = 0$ and no resistors $R_j = 0$ in the loops, then according to (13.13) we have $V^{\rm (ind)} = 0$, from which we conclude that the magnetic fluxes $\Psi_j^{\rm m}$ remain unchanged. Thus the induction tries to keep the magnetic fluxes unaltered (example superconducting loop-currents). If we express the energy $U_{\rm em}$ in terms of the fluxes

$$U_{\rm em} = \frac{1}{2c^2} \sum_{j,k} \Psi_j^{\rm m} (L^{-1})_{j,k} \Psi_k^{\rm m}, \tag{13.19}$$

and use the matrix identity $\dot{L}^{-1} = -L^{-1} \dot{L} L^{-1}$ then we obtain (the identity can be obtained by differentiating $LL^{-1} = 1$ and solving for $\dot{L}^{-1}$)

$$\left.\frac{\partial U_{\rm em}}{\partial t}\right|_{\Psi^{\rm m}} = -\frac{1}{2} \sum_{j,k} I_j \dot{L}_{j,k} I_k = L_{\rm mech}. \tag{13.20}$$

The mechanical power is thus the rate by which the electromagnetic energy changes at constant magnetic fluxes.

13.d.δ Force between two Electric Circuits

After these considerations we return to the force between two electric circuits. In section (9.e) we calculated the force from circuit 1 on circuit 2 as (9.21)

$$\mathbf{K}_2 = \frac{1}{c^2} \int {\rm d}^3 r {\rm d}^3 r' (\mathbf{j}_1(\mathbf{r}') \cdot \mathbf{j}_2(\mathbf{r})) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{13.21}$$

Now if we consider two filamentary wires

$$\mathbf{r} = \mathbf{r}_2 + \mathbf{a} \qquad \mathbf{r}' = \mathbf{r}_1 \tag{13.22}$$

$${\rm d}^3 r' \mathbf{j}_1(\mathbf{r}') \to {\rm d}\mathbf{r}_1 I_1, \qquad {\rm d}^3 r \mathbf{j}_2(\mathbf{r}) \to {\rm d}\mathbf{r}_2 I_2, \tag{13.23}$$

we obtain

$$\mathbf{K}_2 = \frac{I_1 I_2}{c^2} \int ({\rm d}\mathbf{r}_1 \cdot {\rm d}\mathbf{r}_2) \nabla_2 \frac{1}{|\mathbf{r}_2 + \mathbf{a} - \mathbf{r}_1|} = I_1 I_2 \nabla_a L_{1,2}(\mathbf{a}). \tag{13.24}$$

Thus

$$L_{\rm mech} = -\mathbf{K}_2 \cdot \dot{\mathbf{a}} = -I_1 I_2 \dot{L}_{1,2} \tag{13.25}$$

is in agreement with (13.18).

13.d.ϵ Energy of a Magnetic Dipole in an External Magnetic Induction

On the other hand we may now write the interaction energy between a magnetic dipole generated by a density of current $\mathbf{j}$ in an external magnetic field $\mathbf{B}_\mathrm{a}$ generated by a density of current $\mathbf{j}_\mathrm{a}$

$$\begin{aligned} U &= \frac{1}{c^2}\int \mathrm{d}^3r\mathrm{d}^3r'(\mathbf{j}(\mathbf{r})\cdot\mathbf{j}_\mathrm{a}(\mathbf{r}'))\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{c^2}\int \mathrm{d}^3r\mathbf{j}(\mathbf{r})\cdot\int \mathrm{d}^3r'\frac{\mathbf{j}_\mathrm{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{c}\int \mathrm{d}^3r\mathbf{j}(\mathbf{r})\cdot\mathbf{A}_\mathrm{a}(\mathbf{r}) \\ &= \frac{1}{c}\int \mathrm{d}^3r\mathbf{j}(\mathbf{r})\cdot(\mathbf{A}_\mathrm{a}(0)+x_\alpha\nabla_\alpha\mathbf{A}_\mathrm{a}|_{r=0}+...) = \frac{1}{c}\int \mathrm{d}^3rx_\alpha j_\beta\nabla_\alpha A_{\mathrm{a},\beta} \\ &= \epsilon_{\alpha,\beta,\gamma}m_\gamma\nabla_\alpha A_{\mathrm{a},\beta} = \mathbf{m}\cdot\mathbf{B}_\mathrm{a}. \end{aligned} \tag{13.26}$$

This is the correct expression for the interaction energy of a magnetic dipole $\mathbf{m}$ in an external magnetic induction $\mathbf{B}_\mathrm{a}$.

13.d.ζ Permanent Magnetic Moments

Permanent magnetic moments may be considered as loop currents with large self inductance $L_{j,j}$ and constant flux Ψ^m_j. For further calculation we first solve (13.3) for I_j

$$I_j = \frac{\Psi^\mathrm{m}_j}{cL_{j,j}} - \sum_{k\neq j}\frac{L_{j,k}I_k}{L_{j,j}}. \tag{13.27}$$

Upon moving the magnetic moments the mutual inductances change, and one obtains

$$\dot{I}_j = -\frac{1}{L_{j,j}}\Big(\sum_{k\neq j}\dot{L}_{j,k}I_k + \sum_{k\neq j}L_{j,k}\dot{I}_k\Big). \tag{13.28}$$

If the self-inductances $L_{j,j}$ are very large in comparison to the mutual inductances, the currents vary only a little bit, and the second sum is negligible. Then one obtains from the self-inductance contribution of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}L_{j,j}I_j^2\right) = L_{j,j}I_j\dot{I}_j = -I_j\sum_{k\neq j}\dot{L}_{j,k}I_k. \tag{13.29}$$

Thus one obtains from a change of $L_{j,k}$ a contribution $\dot{L}_{j,k}I_jI_k$ directly from the interaction between the currents I_j and I_k, which yields a contribution of the form (13.26) to U_em and two contributions with the opposite sign from $\frac{1}{2}L_{j,j}I_j^2$ and $\frac{1}{2}L_{k,k}I_k^2$. This explains the difference between (10.24) and (13.26).

E MAXWELL's Equations

14 Complete Set of MAXWELL's Equations

14.a Consistency of MAXWELL's Equations

In section (1) we have introduced the four MAXWELL's equations (1.13-1.16)

$$\operatorname{curl}\mathbf{B}(\mathbf{r},t) - \frac{\partial\mathbf{E}(\mathbf{r},t)}{c\partial t} = \frac{4\pi}{c}\mathbf{j}(\mathbf{r},t) \tag{14.1}$$

$$\operatorname{div}\mathbf{E}(\mathbf{r},t) = 4\pi\rho(\mathbf{r},t) \tag{14.2}$$

$$\operatorname{curl}\mathbf{E}(\mathbf{r},t) + \frac{\partial\mathbf{B}(\mathbf{r},t)}{c\partial t} = \mathbf{0} \tag{14.3}$$

$$\operatorname{div}\mathbf{B}(\mathbf{r},t) = 0. \tag{14.4}$$

These are eight component equations for six components B_α and E_α. Thus the equations cannot be independent from each other. Indeed calculating the divergence of the first equation and comparing it with the second equation we find

$$-\frac{1}{c}\operatorname{div}\dot{\mathbf{E}} = \frac{4\pi}{c}\operatorname{div}\mathbf{j} = -\frac{4\pi}{c}\dot{\rho}, \tag{14.5}$$

from which we see that the equation of continuity (1.12) is contained in both equations, and these equations can only be fulfilled if charge is conserved. But it also follows that

$$\frac{\partial}{\partial t}(\operatorname{div}\mathbf{E} - 4\pi\rho) = 0. \tag{14.6}$$

Thus if at a certain time equation (14.2) and at all times the equation of continuity is fulfilled, then equation (14.1) guarantees that (14.2) is fulfilled at all times.
Similarly, it follows from the divergence of (14.3) that

$$\frac{\partial}{\partial t}(\operatorname{div}\mathbf{B}) = 0. \tag{14.7}$$

Thus if (14.4) is fulfilled at a certain time, then due to equation (14.3) it is fulfilled at all times.
Equations (14.1) and (14.3) allow the calculation of **B** and **E** if **j** is given at all times and **B** and **E** are given at a time t_0 and (14.2) and (14.4) are fulfilled at that time. Then ρ is determined by the equation of continuity.
The only contribution we have not yet considered is the contribution proportional to $\dot{\mathbf{E}}$ in (14.1). It was found by MAXWELL. He called $\dot{\mathbf{E}}/(4\pi)$ displacement current, since (14.1) may be rewritten

$$\operatorname{curl}\mathbf{B} = \frac{4\pi}{c}(\mathbf{j} + \frac{1}{4\pi}\dot{\mathbf{E}}). \tag{14.8}$$

With the introduction of this term the system of equations (14.1-14.4) became consistent. Simultaneously this system allowed the description of electromagnetic waves.

14.b MAXWELL's Equations for Freely Moving Charges and Currents

The density of the charges and currents are separated into (compare sections 6.a and 11)

$$\rho = \rho_{\rm f} + \rho_{\rm P} \tag{14.9}$$

$$\mathbf{j} = \mathbf{j}_{\rm f} + \mathbf{j}_{\rm P} + \mathbf{j}_{\rm M}. \tag{14.10}$$

Here $\rho_{\rm f}$ and $\mathbf{j}_{\rm f}$ are the freely moving contributions, whereas $\rho_{\rm P}$ and the newly introduced $\mathbf{j}_{\rm P}$ are the polarization contributions. We expressed the electric dipole moment in the volume ΔV by the dipole moments $\mathbf{p}_i$, and those by the pairs of charges $\pm q_i$ at distance $\mathbf{a}_i$

$$\mathbf{P}\Delta V = \sum \mathbf{p}_i = \sum q_i \mathbf{a}_i \tag{14.11}$$

$$\mathbf{j}_{\rm P}\Delta V = \sum \dot{\mathbf{p}}_i = \sum q_i \dot{\mathbf{a}}_i \tag{14.12}$$

with $\mathbf{j}_{\rm P} = \dot{\mathbf{P}}$ (in matter at rest). In addition, there is a current density from the magnetization as introduced in section 11

$$\mathbf{j}_{\rm M} = c\,\mathrm{curl}\,\mathbf{M}. \tag{14.13}$$

For these charge and current densities one obtains

$$\frac{\partial\rho_{\rm f}}{\partial t} + \mathrm{div}\,\mathbf{j}_{\rm f} = 0 \tag{14.14}$$

$$\frac{\partial\rho_{\rm P}}{\partial t} + \mathrm{div}\,\mathbf{j}_{\rm P} = 0 \tag{14.15}$$

$$\mathrm{div}\,\mathbf{j}_{\rm M} = 0. \tag{14.16}$$

By inserting these charge and current densities into (14.1) one obtains

$$\mathrm{curl}\,\mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}(\mathbf{j}_{\rm f} + \dot{\mathbf{P}} + c\,\mathrm{curl}\,\mathbf{M}), \tag{14.17}$$

from which it follows that

$$\mathrm{curl}\,(\mathbf{B} - 4\pi\mathbf{M}) - \frac{\partial}{c\partial t}(\mathbf{E} + 4\pi\mathbf{P}) = \frac{4\pi}{c}\mathbf{j}_{\rm f}. \tag{14.18}$$

If we now introduce the magnetic field $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$ and the dielectric displacement $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ in (11.9) and (6.6), eq. (11.10) becomes

$$\mathrm{curl}\,\mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} = \frac{4\pi}{c}\mathbf{j}_{\rm f}. \tag{14.19}$$

Similarly, one obtains from (14.2) as in (6.7)

$$\mathrm{div}\,\mathbf{D} = 4\pi\rho_{\rm f}. \tag{14.20}$$

MAXWELL's equations (14.3) and (14.4) remain unchanged. Equations (14.19, 14.20) are called MAXWELL's equations in matter.

15 Energy and Momentum Balance

15.a Energy

We consider a volume of a system with freely moving charges and matter at rest. The force density on the freely moving charges is given by $\mathbf{k} = \rho_f(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c)$. If the charges are moved with velocity $\mathbf{v}$, the power $-\int d^3r\mathbf{k} \cdot \mathbf{v} = -\int d^3r\mathbf{j}_f \cdot \mathbf{E}$ has to be fed into the system against the force density. We rewrite this expression by using (14.19), (B.30) and (14.3)

$$\begin{aligned} -\mathbf{j}_f \cdot \mathbf{E} &= -\frac{c}{4\pi}\mathbf{E} \cdot \operatorname{curl} \mathbf{H} + \frac{1}{4\pi}\mathbf{E} \cdot \dot{\mathbf{D}} = \frac{c}{4\pi} \operatorname{div}(\mathbf{E} \times \mathbf{H}) - \frac{c}{4\pi}\mathbf{H} \cdot \operatorname{curl} \mathbf{E} + \frac{1}{4\pi}\mathbf{E} \cdot \dot{\mathbf{D}} \\ &= \frac{c}{4\pi} \operatorname{div}(\mathbf{E} \times \mathbf{H}) + \frac{1}{4\pi}(\mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}}). \end{aligned} \tag{15.1}$$

These contributions are interpreted in the following way: In matter at rest the second contribution is the temporal change of the energy density $u(\rho_m, \mathbf{D}, \mathbf{B})$ with

$$du = \frac{\partial u}{\partial \rho_m} d\rho_m + \frac{1}{4\pi}\mathbf{E} \cdot d\mathbf{D} + \frac{1}{4\pi}\mathbf{H} \cdot d\mathbf{B}. \tag{15.2}$$

For simplicity we assume that the energy of the matter depends on its density ρ_m, but not on the complete state of strain. We have seen earlier that $\partial u/\partial \mathbf{D} = \mathbf{E}/(4\pi)$ holds. Similarly, one can show from the law of induction that $\partial u/\partial \mathbf{B} = \mathbf{H}/(4\pi)$ holds for rigid matter. We give a short account of the derivation

$$\begin{aligned} \delta U_{em} &= -\sum_j V_j^{(ind)} \delta t I_j = \frac{1}{c}\sum_j I_j \delta\Psi_j^m = \frac{1}{c}\sum_j I_j \int d\mathbf{f}_j \cdot \delta\mathbf{B}(\mathbf{r}) = \frac{1}{c}\sum_j I_j \int d\mathbf{r} \cdot \delta\mathbf{A}(\mathbf{r}) \\ &= \frac{1}{c}\int d^3r \mathbf{j}_f(\mathbf{r}) \cdot \delta\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r \operatorname{curl} \mathbf{H}(\mathbf{r}) \cdot \delta\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r \mathbf{H}(\mathbf{r}) \cdot \operatorname{curl} \delta\mathbf{A}(\mathbf{r}) \\ &= \frac{1}{4\pi}\int d^3r \mathbf{H}(\mathbf{r}) \cdot \delta\mathbf{B}(\mathbf{r}). \end{aligned} \tag{15.3}$$

Since the matter is pinned, $\partial u/\partial \rho_m \dot{\rho}_m$ does not contribute. Therefore we write the energy of volume V as

$$U(V) = \int_V d^3r\, u(\rho_m(\mathbf{r}), \mathbf{D}(\mathbf{r}), \mathbf{B}(\mathbf{r})) \tag{15.4}$$

and introduce the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \tag{15.5}$$

Then one has

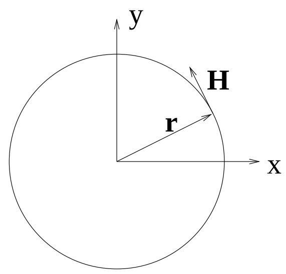
$$-\int_V d^3r \mathbf{j}_f \cdot \mathbf{E} = \dot{U}(V) + \int_V d^3r \operatorname{div} \mathbf{S} = \dot{U}(V) + \int_{\partial V} d\mathbf{f} \cdot \mathbf{S}(\mathbf{r}). \tag{15.6}$$

The energy added to volume V is partially stored in the volume. This stored part is given by ($\dot{U}$). Another part is transported through the surface of the system. This transport of energy is given by the energy current through the surface expressed by the surface integral over $\mathbf{S}$. Similar to the transport of the charge $\int d\mathbf{f} \cdot \mathbf{j}_f$ through a surface per unit time, one has (in matter at rest) the energy transport $\int d\mathbf{f} \cdot \mathbf{S}$ through a surface. Thus the Poynting vector is the density of the electromagnetic energy current.
We note that for $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$ one obtains the energy density

$$u = u^0(\rho_m) + \frac{1}{8\pi}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}). \tag{15.7}$$

Example: Current-carrying straight wire

We consider a straight wire which carries the current I in the direction of the z-axis. Due to AMPERE's law the integral along a concentric circle with radius r around the conductor yields



$$\oint \mathbf{H}\cdot \mathrm{d}\mathbf{r} = \frac{4\pi}{c} I, \quad \mathbf{H} = \frac{2I}{cr}\mathbf{e}_\phi. \tag{15.8}$$

There is a voltage drop along the wire due to the OHMic resistance $V^{(R)}$, which is related to the electric field parallel to the wire, $\mathbf{E} = E_0\mathbf{e}_z$. This yields the POYNTING vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} = -\frac{IE_0\mathbf{e}_r}{2\pi r} \tag{15.9}$$

with the energy flux

$$\int \mathbf{S}\cdot \mathrm{d}\mathbf{f} = -IE_0 l = -IV^{(R)} \tag{15.10}$$

through the lateral surface of the cylinder of the wire of length l in outward direction. In other words, the OHMic power $IV^{(R)}$ flows into the wire. There it is transformed into heat.

15.b Momentum Balance

We consider the momentum balance only for the vacuum with charge densities ρ and current densities $\mathbf{j}$. If we keep the system at rest, a force density $-\mathbf{k}$ has to act against the LORENTZ force density $\mathbf{k} = \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}/c$, so that the momentum $-\int_V \mathrm{d}^3r\, \mathbf{k}$ is added to the volume V per unit time. We transform by means of (14.1) and (14.3)

$$-\mathbf{k} = -\rho\mathbf{E} - \frac{1}{c}\mathbf{j}\times\mathbf{B} = -\frac{1}{4\pi}\mathbf{E}\,\mathrm{div}\,\mathbf{E} + \frac{1}{4\pi}\mathbf{B}\times\,\mathrm{curl}\,\mathbf{B} + \frac{1}{4\pi c}\dot{\mathbf{E}}\times\mathbf{B}. \tag{15.11}$$

With (14.3) and (14.4)

$$\dot{\mathbf{E}}\times\mathbf{B} = (\mathbf{E}\times\mathbf{B})^{\dot{}} - \mathbf{E}\times\dot{\mathbf{B}} = (\mathbf{E}\times\mathbf{B})^{\dot{}} + c\mathbf{E}\times\,\mathrm{curl}\,\mathbf{E} \tag{15.12}$$

$$\mathbf{B}\,\mathrm{div}\,\mathbf{B} = 0 \tag{15.13}$$

one obtains

$$-\mathbf{k} = \frac{1}{4\pi c}(\mathbf{E}\times\mathbf{B})^{\dot{}} + \frac{1}{4\pi}(\mathbf{E}\times\,\mathrm{curl}\,\mathbf{E} - \mathbf{E}\,\mathrm{div}\,\mathbf{E} + \mathbf{B}\times\,\mathrm{curl}\,\mathbf{B} - \mathbf{B}\,\mathrm{div}\,\mathbf{B}). \tag{15.14}$$

One has

$$\mathbf{E}_c\times(\nabla\times\mathbf{E}) - \mathbf{E}_c(\nabla\mathbf{E}) = \nabla(\mathbf{E}\cdot\mathbf{E}_c) - \mathbf{E}(\nabla\mathbf{E}_c) - \mathbf{E}_c(\nabla\cdot\mathbf{E}) = \frac{1}{2}\nabla E^2 - (\nabla\mathbf{E})\mathbf{E}. \tag{15.15}$$

We have indicated quantities on which the ∇-operator does not act with an index $_c$. The ∇-operator acts on both factors $\mathbf{E}$ in the last term of the expression above. Then we may write

$$-\mathbf{k} = \frac{\partial}{\partial t}\mathbf{g}_s - \nabla_\beta T_{\alpha,\beta}\mathbf{e}_\alpha, \tag{15.16}$$

with

$$\mathbf{g}_s = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}, \tag{15.17}$$

$$T_{\alpha,\beta} = \frac{1}{4\pi}(E_\alpha E_\beta + B_\alpha B_\beta) - \frac{\delta_{\alpha,\beta}}{8\pi}(E^2 + B^2). \tag{15.18}$$

Here $\mathbf{g}_s$ is called the density of the electromagnetic momentum and $T_{\alpha,\beta}$ are the components of the electromagnetic stress tensor, whose electrostatic part (8.38) we already know. With these quantities we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}^3r\,\mathbf{g}_s(\mathbf{r}) = -\int_V \mathrm{d}^3r\,\mathbf{k} + \int_{\partial V}\mathbf{e}_\alpha T_{\alpha,\beta}\mathrm{d}f_\beta. \tag{15.19}$$

This is the momentum balance for the volume V. The left handside gives the rate of change of momentum in the volume V, the right handside the rate of momentum added to the volume. It consists of two contributions: the first one is the momentum which is added by the action of the reactive force against the LORENTZ force density $\mathbf{k}$. The second contribution acts by means of stress on the surface. It may also be considered as a flux of momentum through the surface. Thus the stress tensor is apart from its sign the density of momentum flux. It carries two indices. One (α) relates to the components of momentum, the other one (β) to the direction of the flux.

We have only considered the electromagnetic momentum in vacuum, whereas we have considered the electromagnetic energy also in matter. Why is it more difficult to determine momentum in matter? In both cases we consider the system at rest. If one pins the matter, the acting forces do not contribute to the balance of energy, since the power is given by force times velocity. Since velocity vanishes, the forces acting on the matter do not contribute to the balance of the energy. This is different for the balance of momentum. There all forces contribute. One could imagine starting out from a force-free state. Then, however, we have the problem that by moving the free charges, forces will appear which we would have to know. Therefore we can consider here the energy balance in matter, whereas the momentum balance in matter would be more difficult.

In literature there are inconsistent statements: In 1908 MINKOWSKI gave $\mathbf{D}\times\mathbf{B}/(4\pi c)$ for the electromagnetic momentum density in matter. This can also be found in the book by SOMMERFELD (however with words of caution). On the other hand in 1910 ABRAHAM gave $\mathbf{E}\times\mathbf{H}/(4\pi c)$. This is also found in the textbook by LANDAU and LIFSHITZ.

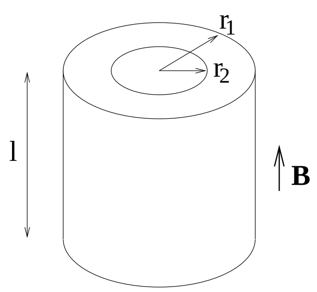
There are two points to be considered, which are often overlooked:

i) The interaction between the electromagnetic field and matter has to be taken into account. Matter cannot be considered rigid.

ii) One has to define precisely what is meant by the electromagnetic momentum, since otherwise any difference can be attributed to the mechanical momentum and the statement is empty. Without derivation it should just be mentioned that a model system can be given which yields the following: The momentum density in the local rest system is $\mathbf{E}\times\mathbf{H}/(4\pi c) = \mathbf{S}/c^2$. However, in homogeneous matter there is a further conserved quantity which in the local rest system is given by $\mathbf{D}\times\mathbf{B}/(4\pi c)$. If one goes through SOMMERFELD's argument, one realizes that it can be carried through only for a space-independent dielectric constant ϵ.

Example: Cylindric capacitor in a magnetic field

We consider a cylindric capacitor of length l with outer radius r_1 and inner radius r_2 with charge q outside and $-q$ inside. We assume that between both cylinders is vacuum. Parallel to the axis be a magnetic field B_0. Then one has in cylinder coordinates



$$\mathbf{E} = -\frac{2q}{lr}\mathbf{e}_r, \quad \mathbf{B} = B_0\mathbf{e}_z, \quad \mathbf{g}_s = \frac{1}{4\pi c}\frac{2qB_0}{lr}\mathbf{e}_\phi. \tag{15.20}$$

From this we calculate the angular momentum $\mathbf{L}$ in z-direction

$$L_z = \int dz d^2r(\mathbf{r}\times\mathbf{g}_s)_z = \int dz d^2r r\frac{2qB_0}{4\pi c l r} = \frac{qB_0}{2c}(r_1^2 - r_2^2). \tag{15.21}$$

If the capacitor is decharged, the decharging current flows through the magnetic field. Then the LORENTZ force acts which gives the system a mechanical torque $\mathbf{M}_{\text{mech}}$

$$\mathbf{M}_{\text{mech}} = \int d^3r\, \mathbf{r}\times(\frac{1}{c}\mathbf{j}\times\mathbf{B}) = \frac{I}{c}\int \mathbf{r}\times(d\mathbf{r}\times\mathbf{B}) = \frac{I}{c}\int((\mathbf{r}\cdot\mathbf{B})d\mathbf{r} - (\mathbf{r}\cdot d\mathbf{r})\mathbf{B}), \tag{15.22}$$

from which one obtains

$$M_{\text{mech},z} = -\frac{IB_0}{c}\int_{r_1}^{r_2} r dr = \frac{IB_0}{2c}(r_1^2 - r_2^2) \tag{15.23}$$

and thus the mechanical angular momentum

$$L_z = \frac{qB_0}{2c}(r_1^2 - r_2^2). \tag{15.24}$$

Thus the electromagnetic angular momentum (15.21) is transformed into a mechanical angular momentum during decharging. Instead of decharging the capacitor one may switch off the magnetic field. Then the electric field

$$\oint \mathbf{E}^{(\mathrm{ind})} \cdot \mathrm{d}\mathbf{r} = -\frac{1}{c} \int \dot{\mathbf{B}} \cdot \mathrm{d}\mathbf{f} = -\frac{1}{c}\pi r^2 \dot{B}_0, \quad \mathbf{E}^{(\mathrm{ind})} = -\frac{1}{2c} r \dot{B}_0 \mathbf{e}_\phi \tag{15.25}$$

is induced, which exerts the torque

$$\mathbf{M}_{\mathrm{mech}} = q\mathbf{r}_1 \times \mathbf{E}^{(\mathrm{ind})}(\mathbf{r}_1) - q\mathbf{r}_2 \times \mathbf{E}^{(\mathrm{ind})}(\mathbf{r}_2) \tag{15.26}$$

$$M_{\mathrm{mech},z} = qr_1(-\frac{1}{2c} r_1 \dot{B}_0) - qr_2(-\frac{1}{2c} r_2 \dot{B}_0) \tag{15.27}$$

so that the capacitor receives the mechanical component of the angular momentum

$$L_z = \frac{qB_0}{2c}(r_1^2 - r_2^2). \tag{15.28}$$

In both cases the electromagnetic angular momentum is transformed into a mechanical one.

F Electromagnetic Waves

16 Electromagnetic Waves in Vacuum and in Homogeneous Isotropic Insulators

16.a Wave Equation

We consider electromagnetic waves in a homogeneous isotropic insulator including the vacuum. More precisely we require that the dielectric constant ϵ and the permeability μ are independent of space and time. Further we require that there are no freely moving currents and charges $\rho_{\rm f} = 0$, $\mathbf{j}_{\rm f} = \mathbf{0}$. Thus the matter is an insulator. Then MAXWELL's equations read, expressed in terms of $\mathbf{E}$ and $\mathbf{H}$ by means of $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$

$$\operatorname{div}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{H} = 0, \tag{16.1}$$

$$\operatorname{curl}\mathbf{H} = \frac{\epsilon}{c}\dot{\mathbf{E}}, \qquad \operatorname{curl}\mathbf{E} = -\frac{\mu}{c}\dot{\mathbf{H}}. \tag{16.2}$$

From these equations one obtains

$$\operatorname{curl}\,\operatorname{curl}\mathbf{H} = \frac{\epsilon}{c}\operatorname{curl}\dot{\mathbf{E}} = -\frac{\epsilon\mu}{c^2}\ddot{\mathbf{H}} \tag{16.3}$$

With

$$\operatorname{curl}\,\operatorname{curl}\mathbf{H} = \nabla\times(\nabla\times\mathbf{H}) = -\triangle\mathbf{H} + \nabla(\nabla\cdot\mathbf{H}) \tag{16.4}$$

one obtains for $\mathbf{H}$ using (16.1) and similarly for $\mathbf{E}$

$$\triangle\mathbf{H} = \frac{1}{c'^2}\ddot{\mathbf{H}}, \tag{16.5}$$

$$\triangle\mathbf{E} = \frac{1}{c'^2}\ddot{\mathbf{E}}, \tag{16.6}$$

$$c' = \frac{c}{\sqrt{\epsilon\mu}}. \tag{16.7}$$

The equations (16.5) and (16.6) are called wave equations.

16.b Plane Waves

Now we look for particular solutions of the wave equations and begin with solutions which depend only on z and t, $\mathbf{E} = \mathbf{E}(z,t)$, $\mathbf{H} = \mathbf{H}(z,t)$. One obtains for the z-components

$$\operatorname{div}\mathbf{E} = 0 \quad\rightarrow\quad \frac{\partial E_z}{\partial z} = 0 \tag{16.8}$$

$$(\operatorname{curl}\mathbf{H})_z = 0 = \frac{\epsilon}{c}\dot{E}_z \quad\rightarrow\quad \frac{\partial E_z}{\partial t} = 0. \tag{16.9}$$

Thus only a static homogeneous field is possible with this ansatz in z-direction, i.e. a constant field E_z. The same is true for H_z. We already see that electromagnetic waves are transversal waves.
For the x- and y-components one obtains

$$(\nabla\times\mathbf{H})_x = \frac{\epsilon}{c}\dot{E}_x \;\rightarrow\; -\nabla_z H_y = \frac{\epsilon}{c}\dot{E}_x \;\rightarrow\; -\nabla_z(\sqrt{\mu}H_y) = \frac{1}{c'}(\sqrt{\epsilon}E_x)^{\dot{}} \tag{16.10}$$

$$(\nabla\times\mathbf{E})_y = -\frac{\mu}{c}\dot{H}_y \;\rightarrow\; \nabla_z E_x = -\frac{\mu}{c}\dot{H}_y \;\rightarrow\; \nabla_z(\sqrt{\epsilon}E_x) = -\frac{1}{c'}(\sqrt{\mu}H_y)^{\dot{}}. \tag{16.11}$$

E_x is connected with H_y, and in the same way E_y with $-H_x$. We may combine the equations (16.10) and (16.11)

$$\frac{\partial}{\partial t}(\sqrt{\epsilon}E_x \pm \sqrt{\mu}H_y) = \mp c'\frac{\partial}{\partial z}(\sqrt{\epsilon}E_x \pm \sqrt{\mu}H_y). \tag{16.12}$$

The solution of this equation and the corresponding one for E_y with $-H_x$ is

$$\sqrt{\epsilon}E_x \pm \sqrt{\mu}H_y = 2f_\pm(z \mp c't), \tag{16.13}$$
$$\sqrt{\epsilon}E_y \mp \sqrt{\mu}H_x = 2g_\pm(z \mp c't), \tag{16.14}$$

with arbitrary (differentiable) functions $f_\pm$ and $g_\pm$, from which one obtains

$$\sqrt{\epsilon}E_x = f_+(z - c't) + f_-(z + c't) \tag{16.15}$$
$$\sqrt{\mu}H_y = f_+(z - c't) - f_-(z + c't) \tag{16.16}$$
$$\sqrt{\epsilon}E_y = g_+(z - c't) + g_-(z + c't) \tag{16.17}$$
$$\sqrt{\mu}H_x = -g_+(z - c't) + g_-(z + c't). \tag{16.18}$$

This is the superposition of waves of arbitrary shapes, which propagate upward (f_+, g_+) and downward (f_-, g_-), resp, with velocity c'. Thus $c' = c/\sqrt{\epsilon\mu}$ is the velocity of propagation of the electromagnetic wave (light) in the corresponding medium. In particular we find that c is the light velocity in vacuum.
We calculate the density of energy

$$u = \frac{1}{8\pi}(\epsilon E^2 + \mu H^2) = \frac{1}{4\pi}(f_+^2 + g_+^2 + f_-^2 + g_-^2) \tag{16.19}$$

and the density of the energy current by means of the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} = \frac{c'\mathbf{e}_z}{4\pi}(f_+^2 + g_+^2 - f_-^2 - g_-^2), \tag{16.20}$$

where a homogeneous field in z-direction is not considered. Comparing the expressions for u and $\mathbf{S}$ separately for the waves moving up and down, one observes that the energy of the wave is transported with velocity $\pm c'\mathbf{e}_z$, since $\mathbf{S} = \pm c'\mathbf{e}_z u$. We remark that the wave which obeys $E_y = 0$ and $H_x = 0$, that is $g_\pm = 0$, is called linearly polarized in x-direction. For the notation of the direction of polarization one always considers that of the vector **E**.

16.c Superposition of Plane Periodic Waves

In general one may describe the electric field in terms of a Fourier integral

$$\mathbf{E}(\mathbf{r},t) = \int d^3k d\omega \mathbf{E}_0(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{16.21}$$

analogously for **H**. Then the fields are expressed as a superposition of plane periodic waves.

16.c.α Insertion on Fourier Series and Integrals

The Fourier series of a function with period L, $f(x+L) = f(x)$ reads

$$f(x) = \hat{c}\sum_{n=-\infty}^{\infty} f_n e^{2\pi i n x/L}. \tag{16.22}$$

f_n are the Fourier coefficients of f. This representation is possible for square integrable functions with a finite number of points of discontinuity. $\hat{c}$ is an appropriate constant. The back-transformation, that is the calculation of the Fourier coefficients is obtained from

$$\int_{-L/2}^{L/2} dx e^{-2\pi i n x/L} f(x) = \hat{c}L f_n, \tag{16.23}$$

as can be seen easily by inserting in (16.22) and exchanging summation and integration. The Fourier transform for a (normally not-periodic) function defined from $-\infty$ to $+\infty$ can be obtained by performing the limit $L \to \infty$ and introducing

$$k := \frac{2\pi n}{L}, \quad f_n = f_0(k), \quad \hat{c} = \Delta k = \frac{2\pi}{L}. \tag{16.24}$$

Then (16.22) transforms into

$$f(x) = \sum \Delta k f_0(k) e^{ikx} \to \int_{-\infty}^{\infty} dk f_0(k) e^{ikx} \tag{16.25}$$

and the back-transformation (16.23) into

$$\int_{-\infty}^{\infty} dx f(x) e^{-ikx} = 2\pi f_0(k). \tag{16.26}$$

This allows us, e.g., to give the back-transformation from (16.21) to

$$\mathbf{E}_0(\mathbf{k}, \omega) = \frac{1}{(2\pi)^4} \int d^3r dt e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathbf{E}(\mathbf{r}, t). \tag{16.27}$$

16.c.β Back to Maxwell's Equations

The representation by the Fourier transform has the advantage that the equations become simpler. Applying the operations ∇ and $\partial/\partial t$ on the exponential function

$$\nabla e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = i\mathbf{k} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad \frac{\partial}{\partial t} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = -i\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{16.28}$$

in Maxwell's equations yields for the Fourier components

$$\nabla \cdot \mathbf{E} = 0 \quad \to \quad i\mathbf{k} \cdot \mathbf{E}_0(\mathbf{k}, \omega) = 0 \tag{16.29}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \to \quad i\mathbf{k} \cdot \mathbf{H}_0(\mathbf{k}, \omega) = 0 \tag{16.30}$$

$$\nabla \times \mathbf{H} = \frac{\epsilon}{c}\dot{\mathbf{E}} \quad \to \quad i\mathbf{k} \times \mathbf{H}_0(\mathbf{k}, \omega) = -i\frac{\epsilon}{c}\omega \mathbf{E}_0(\mathbf{k}, \omega) \tag{16.31}$$

$$\nabla \times \mathbf{E} = -\frac{\mu}{c}\dot{\mathbf{H}} \quad \to \quad i\mathbf{k} \times \mathbf{E}_0(\mathbf{k}, \omega) = i\frac{\mu}{c}\omega \mathbf{H}_0(\mathbf{k}, \omega). \tag{16.32}$$

The advantage of this representation is that only Fourier components with the same $\mathbf{k}$ and ω are connected to each other. For $\mathbf{k} = \mathbf{0}$ one obtains $\omega = 0$, where $\mathbf{E}_0$, $\mathbf{H}_0$ are arbitrary. These are the static homogeneous fields. For $\mathbf{k} \neq \mathbf{0}$ one obtains from (16.29) and (16.30)

$$\mathbf{E}_0(\mathbf{k}, \omega) \perp \mathbf{k}, \quad \mathbf{H}_0(\mathbf{k}, \omega) \perp \mathbf{k}. \tag{16.33}$$

From the two other equations (16.31) and (16.32) one obtains

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0(\mathbf{k}, \omega)) = \frac{\mu}{c}\omega \mathbf{k} \times \mathbf{H}_0(\mathbf{k}, \omega) = -\frac{\epsilon\mu}{c^2}\omega^2 \mathbf{E}_0(\mathbf{k}, \omega). \tag{16.34}$$

From this one obtains

$$\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_0(\mathbf{k}, \omega)) - k^2 \mathbf{E}_0(\mathbf{k}, \omega) = -\frac{1}{c'^2}\omega^2 \mathbf{E}_0(\mathbf{k}, \omega), \tag{16.35}$$

analogously for $\mathbf{H}_0$. The first term on the left hand-side of (16.35) vanishes because of (16.29). Thus there are non-vanishing solutions, if the condition $\omega = \pm c'k$ is fulfilled. This is the dispersion relation for electromagnetic waves that is the relation between frequency and wave-vector for electromagnetic waves. Taking these conditions into account we may write

$$\mathbf{E}_0(\mathbf{k}, \omega) = \frac{1}{2}\delta(\omega - c'k)\mathbf{E}_1(\mathbf{k}) + \frac{1}{2}\delta(\omega + c'k)\mathbf{E}_2(\mathbf{k}). \tag{16.36}$$

and thus

$$\mathbf{E}(\mathbf{r},t) = \int \mathrm{d}^3k \left(\frac{1}{2}\mathbf{E}_1(\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-c'kt)} + \frac{1}{2}\mathbf{E}_2(\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+c'kt)}\right). \tag{16.37}$$

Since the electric field has to be real, it must coincide with its conjugate complex.

$$\begin{aligned}\mathbf{E}^*(\mathbf{r},t) &= \int \mathrm{d}^3k \left(\frac{1}{2}\mathbf{E}_1^*(\mathbf{k})\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-c'kt)} + \frac{1}{2}\mathbf{E}_2^*(\mathbf{k})\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+c'kt)}\right)\\ &= \int \mathrm{d}^3k \left(\frac{1}{2}\mathbf{E}_1^*(-\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+c'kt)} + \frac{1}{2}\mathbf{E}_2^*(-\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-c'kt)}\right). \end{aligned} \tag{16.38}$$

From comparison of the coefficients one obtains

$$\mathbf{E}_2^*(\mathbf{k}) = \mathbf{E}_1(-\mathbf{k}). \tag{16.39}$$

Thus we obtain

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \int \mathrm{d}^3k \left(\frac{1}{2}\mathbf{E}_1(\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-c'kt)} + \frac{1}{2}\mathbf{E}_1^*(-\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+c'kt)}\right)\\ &= \int \mathrm{d}^3k \left(\frac{1}{2}\mathbf{E}_1(\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-c'kt)} + \frac{1}{2}\mathbf{E}_1^*(\mathbf{k})\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-c'kt)}\right)\\ &= \Re\Big(\int \mathrm{d}^3k\mathbf{E}_1(\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-c'kt)}\Big). \end{aligned} \tag{16.40}$$

Eq. (16.32) yields for $\mathbf{H}_0$

$$\mathbf{H}_0(\mathbf{k},\omega) = \frac{c}{\mu\omega}\mathbf{k}\times\mathbf{E}_0(\mathbf{k},\omega) = \sqrt{\frac{\epsilon}{\mu}}\left(\delta(\omega-c'k)\frac{\mathbf{k}}{2k}\times\mathbf{E}_1(\mathbf{k}) - \delta(\omega+c'k)\frac{\mathbf{k}}{2k}\times\mathbf{E}_2(\mathbf{k})\right) \tag{16.41}$$

and thus for $\mathbf{H}$

$$\mathbf{H}(\mathbf{r},t) = \Re\Big(\int \mathrm{d}^3k\sqrt{\frac{\epsilon}{\mu}}\frac{\mathbf{k}}{k}\times\mathbf{E}_1(\mathbf{k})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-c'kt)}\Big). \tag{16.42}$$

If only one Fourier component $\mathbf{E}_1(\mathbf{k}) = \delta^3(\mathbf{k}-\mathbf{k}_0)\mathbf{E}_{1,0}$ (idealization) contributes then one has a monochromatic wave

$$\mathbf{E}(\mathbf{r},t) = \Re(\mathbf{E}_{1,0}\mathrm{e}^{\mathrm{i}(\mathbf{k}_0\cdot\mathbf{r}-c'k_0t)}) \tag{16.43}$$

$$\mathbf{H}(\mathbf{r},t) = \sqrt{\frac{\epsilon}{\mu}}\Re\Big(\frac{\mathbf{k}_0}{k_0}\times\mathbf{E}_{1,0}\mathrm{e}^{\mathrm{i}(\mathbf{k}_0\cdot\mathbf{r}-c'k_0t)}\Big). \tag{16.44}$$

The wave (light) is called linearly polarized, if $\mathbf{E}_{1,0} = \mathbf{e}_1 E_{1,0}$ with a real unit-vector $\mathbf{e}_1$, it is called circularly polarized if $\mathbf{E}_{1,0} = (\mathbf{e}_1 \mp \mathrm{i}\mathbf{e}_2)E_{1,0}/\sqrt{2}$ with real unit vectors $\mathbf{e}_1$ und $\mathbf{e}_2$, where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{k}_0$ form an orthogonal right-handed basis. The upper sign applies for a right-, the lower for a left-polarized wave.

16.c.γ Time averages and time integrals

Energy-density and Poynting vector are quantities bilinear in the fields. In case of a monochromatic wave as in (16.43) and (16.44) these quantities oscillate. One is often interested in the averages of these quantities. Thus if we have two quantities

$$a = \Re\Big(a_0\mathrm{e}^{-\mathrm{i}\omega t}\Big), \quad b = \Re\Big(b_0\mathrm{e}^{-\mathrm{i}\omega t}\Big), \tag{16.45}$$

then one has

$$ab = \frac{1}{4}a_0b_0\mathrm{e}^{-2\mathrm{i}\omega t} + \frac{1}{4}(a_0b_0^* + a_0^*b_0) + \frac{1}{4}a_0^*b_0^*\mathrm{e}^{2\mathrm{i}\omega t}. \tag{16.46}$$

The first and the last term oscillate (we assume $\omega \neq 0$). They cancel in the time average. Thus one obtains in the time average

$$\overline{ab} = \frac{1}{4}(a_0b_0^* + a_0^*b_0) = \frac{1}{2}\Re(a_0^*b_0). \tag{16.47}$$

Please note that a_0 and b_0 are in general complex and that the time average depends essentially on the relative phase between both quantities and not only on the moduli $|a_0|$ and $|b_0|$.
If a and b are given by Fourier integrals

$$a(t) = \Re\Big(\int d\omega a_0(\omega)e^{-i\omega t}\Big) \tag{16.48}$$

and analogously for $b(t)$, then often the time integrals of these quantities and their products over all times will be finite. For this purpose the time integral $\int_{-\infty}^{\infty} dt e^{-i\omega t}$ has to be determined. This integral is not well defined. In practice it has often to be multiplied with a function continuous in ω. Thus it is sufficient to find out how the time-integral of this frequency-integral behaves. For this purpose we go back to the insertion on Fourier series with x und k and find that

$$\int_{-L/2}^{L/2} dx e^{-2\pi i n x/L} = L\delta_{0,n}, \tag{16.49}$$

thus

$$\sum_{n=n_-}^{n_+} \int_{-L/2}^{L/2} dx e^{-2\pi i n x/L} = L, \tag{16.50}$$

if $n_- \le 0$ and $n_+ \ge 0$. Otherwise the sum vanishes. Now we perform again the limit $L \to \infty$ and obtain

$$\sum_{k_-}^{k_+} \Delta k \int_{-L/2}^{L/2} dx e^{-ikx} = \Delta k L \to \int_{k_-}^{k_+} dk \int_{-\infty}^{\infty} dx e^{-ikx} = 2\pi, \tag{16.51}$$

if k_- is negative and k_+ positive, otherwise it vanishes. Thus we obtain

$$\int_{-\infty}^{\infty} dx e^{-ikx} = 2\pi\delta(k). \tag{16.52}$$

With this result we obtain

$$\int_{-\infty}^{\infty} dt a(t)b(t) = \frac{\pi}{2}\int_{-\infty}^{\infty} d\omega\Big(a_0(\omega) + a_0^*(-\omega))(b_0(-\omega) + b_0^*(\omega)\Big). \tag{16.53}$$

If there are only positive frequencies ω under the integral then one obtains

$$\int_{-\infty}^{\infty} dt a(t)b(t) = \frac{\pi}{2}\int_0^{\infty} d\omega\Big(a_0(\omega)b_0^*(\omega) + a_0^*(\omega)b_0(\omega)\Big) = \pi\Re\Big(\int_0^{\infty} d\omega a_0^*(\omega)b_0(\omega)\Big). \tag{16.54}$$

17 Electromagnetic Waves in Homogeneous Conductors

17.a Transverse Oscillations at Low Frequencies

We investigate the transverse oscillations in a homogeneous conductor. We put $\mu = 1$. From

$$\mathbf{j}_\mathrm{f} = \sigma \mathbf{E} \tag{17.1}$$

one obtains

$$\operatorname{curl} \mathbf{B} - \frac{1}{c}\epsilon\dot{\mathbf{E}} = \frac{4\pi}{c}\sigma\mathbf{E}. \tag{17.2}$$

For periodic fields of frequency ω,

$$\mathbf{E} = \mathbf{E}_0(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}, \quad \mathbf{B} = \mathbf{B}_0(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}, \tag{17.3}$$

and similarly for ρ_f and $\mathbf{j}_\mathrm{f}$, one obtains

$$\operatorname{curl} \mathbf{B}_0 + (\frac{\mathrm{i}\omega}{c}\epsilon - \frac{4\pi}{c}\sigma)\mathbf{E}_0 = \mathbf{0}. \tag{17.4}$$

This can be written

$$\operatorname{curl} \mathbf{B}_0 + \frac{\mathrm{i}\omega}{c}\epsilon(\omega)\mathbf{E}_0 = \mathbf{0}, \quad \epsilon(\omega) = \epsilon - \frac{4\pi\sigma}{\mathrm{i}\omega}. \tag{17.5}$$

From the equation of continuity

$$\dot{\rho}_\mathrm{f} + \operatorname{div} \mathbf{j}_\mathrm{f} = 0 \tag{17.6}$$

one obtains

$$-\mathrm{i}\omega\rho_{\mathrm{f},0} + \operatorname{div} \mathbf{j}_{\mathrm{f},0} = 0 \tag{17.7}$$

and thus

$$\operatorname{div} \mathbf{D}_0 = 4\pi\rho_{\mathrm{f},0} = \frac{4\pi}{\mathrm{i}\omega}\operatorname{div} \mathbf{j}_{\mathrm{f},0} = \frac{4\pi\sigma}{\mathrm{i}\omega}\operatorname{div} \mathbf{E}_0. \tag{17.8}$$

Thus we have

$$\epsilon(\omega)\operatorname{div} \mathbf{E}_0 = 0 \tag{17.9}$$

because of $\operatorname{div} \mathbf{D}_0 = \epsilon \operatorname{div} \mathbf{E}_0$. We may thus transfer our results from insulators to conductors, if we replace ϵ by $\epsilon(\omega)$. Thus we obtain

$$k^2 = \epsilon(\omega)\frac{\omega^2}{c^2}. \tag{17.10}$$

Since $\epsilon(\omega)$ is complex, one obtains for real ω a complex wave-vector $\mathbf{k}$. We put

$$\sqrt{\epsilon(\omega)} = n + \mathrm{i}\kappa, \quad k = \frac{\omega}{c}(n + \mathrm{i}\kappa) \tag{17.11}$$

and obtain a damped wave with

$$\mathrm{e}^{\mathrm{i}kz} = \mathrm{e}^{\mathrm{i}\omega nz/c - \omega\kappa z/c}. \tag{17.12}$$

For the fields we obtain

$$\mathbf{E} = \Re(\mathbf{E}_0\mathrm{e}^{\mathrm{i}\omega(nz/c-t)})\mathrm{e}^{-\omega\kappa z/c}, \tag{17.13}$$

$$\mathbf{B} = \Re(\sqrt{\epsilon(\omega)}\mathbf{e}_z \times \mathbf{E}_0\mathrm{e}^{\mathrm{i}\omega(nz/c-t)})\mathrm{e}^{-\omega\kappa z/c}. \tag{17.14}$$

The amplitude decays in a distance $d = \frac{c}{\omega\kappa}$ by a factor 1/e. This distance is called penetration depth or skin depth. For small frequencies one can approximate

$$\sqrt{\epsilon(\omega)} \approx \sqrt{-\frac{4\pi\sigma}{\mathrm{i}\omega}} = (1+\mathrm{i})\sqrt{\frac{2\pi\sigma}{\omega}}, \quad n = \kappa = \sqrt{\frac{2\pi\sigma}{\omega}}, \quad d = \frac{c}{\sqrt{2\pi\sigma\omega}}. \tag{17.15}$$

For copper one has $\sigma = 5.8 \cdot 10^{17}\ \mathrm{s}^{-1}$, for $\omega = 2\pi \cdot 50\ \mathrm{s}^{-1}$ one obtains $d = 9$ mm. This effect is called the skin-effect. The alternating current decays exponentially inside the conductor. For larger frequencies the decay is more rapidly.

17.b Transverse Oscillations at High Frequencies

In reality ϵ and σ depend on ω. We will now consider the frequency dependence of the conductivity within a simple model and start out from the equation of motion of a charge (for example an electron in a metal)

$$m_0\ddot{\mathbf{r}} = e_0\mathbf{E} - \frac{m_0}{\tau}\dot{\mathbf{r}}, \tag{17.16}$$

where m_0 and e_0 are mass and charge of the carrier. The last term is a friction term which takes the collisions with other particles in a rough way into account. There τ is the relaxation time, which describes how fast the velocity decays in the absense of an electric field. One obtains with $\mathbf{j}_\mathrm{f} = \rho_\mathrm{f}\dot{\mathbf{r}} = n_0 e_0\dot{\mathbf{r}}$, where n_0 is the density of the freely moving carriers

$$\frac{m_0}{n_0 e_0}\frac{\partial \mathbf{j}_\mathrm{f}}{\partial t} = e_0\mathbf{E} - \frac{m_0}{n_0\tau e_0}\mathbf{j}_\mathrm{f}. \tag{17.17}$$

In the stationary case $\partial\mathbf{j}_\mathrm{f}/\partial t = \mathbf{0}$ one obtains the static conductivity $\sigma_0 = \frac{n_0\tau e_0^2}{m_0}$. Thus we can write

$$\tau\frac{\partial \mathbf{j}_\mathrm{f}}{\partial t} = \sigma_0\mathbf{E} - \mathbf{j}_\mathrm{f}. \tag{17.18}$$

With the time dependence $\propto \mathrm{e}^{-\mathrm{i}\omega\tau}$ one obtains

$$(1 - \mathrm{i}\omega\tau)\mathbf{j}_{\mathrm{f},0} = \sigma_0\mathbf{E}_0, \tag{17.19}$$

which can be rewritten

$$\mathbf{j}_{\mathrm{f},0} = \sigma(\omega)\mathbf{E}_0 \tag{17.20}$$

$$\sigma(\omega) = \frac{\sigma_0}{1 - \mathrm{i}\omega\tau} \tag{17.21}$$

$$\epsilon(\omega) = \epsilon - \frac{4\pi\sigma_0}{\mathrm{i}\omega(1 - \mathrm{i}\omega\tau)}. \tag{17.22}$$

For large frequencies, $\omega\tau \gg 1$ one obtains

$$\epsilon(\omega) = \epsilon - \frac{4\pi\sigma_0}{\tau\omega^2} = \epsilon - \frac{4\pi n_0 e_0^2}{m_0\omega^2} = \epsilon\Big(1 - \frac{\omega_\mathrm{P}^2}{\omega^2}\Big) \tag{17.23}$$

with the plasma frequency

$$\omega_\mathrm{P} = \sqrt{\frac{4\pi n_0 e_0^2}{\epsilon m_0}}. \tag{17.24}$$

For $\omega < \omega_\mathrm{P}$ one obtains a negative $\epsilon(\omega)$, that is

$$n = 0, \quad \kappa = \sqrt{\epsilon\Big(\frac{\omega_\mathrm{P}^2}{\omega^2} - 1\Big)} \tag{17.25}$$

with an exponential decay of the wave. However, for $\omega > \omega_\mathrm{P}$ one obtains a positive ϵ

$$n = \sqrt{\epsilon\Big(1 - \frac{\omega_\mathrm{P}^2}{\omega^2}\Big)}, \quad \kappa = 0. \tag{17.26}$$

For such large frequencies the conductor becomes transparent. For copper one has $1/\tau = 3.7\cdot 10^{13}\ \mathrm{s}^{-1}$, $\sigma_0 = 5.8\cdot 10^{17}\ \mathrm{s}^{-1}$ and $\omega_\mathrm{P} = 1.6\cdot 10^{16}\ \mathrm{s}^{-1}$. For visible light one has the frequency-region $\omega = 2.4...5.2\cdot 10^{15}\ \mathrm{s}^{-1}$, so that copper is non-transparent in the visible range. In electrolytes, however, the carrier density is less, the mass is bigger, so that the plasma-frequency is smaller. Thus electrolytes are normally transparent.

17.c Longitudinal = Plasma Oscilations

One has $\epsilon(\omega) = 0$ for $\omega = \omega_\mathrm{P}$. Then (17.9) allows for longitudinal electric waves

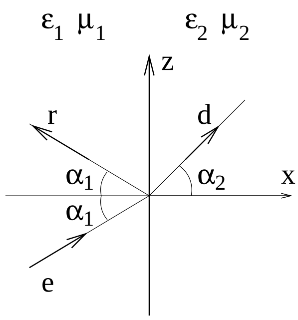
$$\mathbf{E} = E_0\mathbf{e}_z\mathrm{e}^{\mathrm{i}(k_z z - \omega_\mathrm{P} t)}, \quad \mathbf{B} = \mathbf{0}. \tag{17.27}$$

These go along with longitudinal oscillations of the charge carriers, which are obtained by neglecting the friction term in (17.17).

18 Reflection and Refraction at a Planar Surface

18.a Problem and Direction of Propagation

We consider an incident plane wave $\propto e^{i(\mathbf{k}_e\cdot\mathbf{r}-\omega t)}$ for $x < 0$, $\mathbf{k}_e = (k', 0, k_z)$, which hits the plane boundary $x = 0$. For $x < 0$ the dielectric constant and the permeability be ϵ_1 and μ_1, resp., for $x > 0$ these constants are ϵ_2 and μ_2. At the boundary $x = 0$ the wave oscillates $\propto e^{i(k_z z-\omega t)}$. The reflected and the refracted wave show the same behaviour at the boundary, i.e. all three waves have k_z, k_y and ω in common and differ only in k_x. From



$$\mathbf{k}_i^2 = \frac{\epsilon_i\mu_i\omega^2}{c^2} = \frac{n_i^2\omega^2}{c^2}, \quad n_i = \sqrt{\epsilon_i\mu_i} \tag{18.1}$$

one obtains

$$k_1^2 = k_z^2 + k'^2 = \frac{n_1^2\omega^2}{c^2} \qquad \mathbf{k}_r = (-k', 0, k_z) \tag{18.2}$$

$$k_2^2 = k_z^2 + k''^2 = \frac{n_2^2\omega^2}{c^2} \qquad \mathbf{k}_d = (k'', 0, k_z). \tag{18.3}$$

Here $n_{1,2}$ are the indices of refraction of both media. The x-component of the wave-vector of the reflected wave $\mathbf{k}_r$ is the negative of that of the incident wave. Thus the angle of the incident wave α_1 and of the reflected wave are equal. If k'' is real then $k'' > 0$ has to be chosen so that the wave is outgoing and not incoming. If k'' is imaginary then $\Im k'' > 0$ has to be chosen, so that the wave decays exponentially in the medium 2 and does grow exponentially. For real k'' one has

$$k_z = k_1 \sin\alpha_1 = k_2 \sin\alpha_2. \tag{18.4}$$

Thus SNELL's law follows from (18.1)

$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2. \tag{18.5}$$

If $\sin\alpha_2 > 1$ results, then this corresponds to an imaginary k''. We finally remark

$$\frac{k'}{k''} = \frac{k_1\cos\alpha_1}{k_2\cos\alpha_2} = \frac{\tan\alpha_2}{\tan\alpha_1}. \tag{18.6}$$

18.b Boundary Conditions, Amplitudes

In the following we have to distinguish two polarizations. They are referred to the plane of incidence. The plane of incidence is spaned by the direction of the incident wave and by the normal to the boundary (in our coordinates the x-z-plane). The polarization 1 is perpendicular to the plane of incidence, i.e. **E** is polarized in y-direction. The polarization 2 lies in the plane of incidence, **H** points in y-direction. One obtains the following conditions on polarization and continuity

	polarization 1	polarization 2	
E	$\perp$ plane of inc.	in plane of inc.	
H	in plane of inc.	$\perp$ plane of inc.	
$\mathbf{E}_t$	$E_{1,y} = E_{2,y}$	$E_{1,z} = E_{2,z}$	(18.7)
$D_n = \epsilon E_n$		$\epsilon_1 E_{1,x} = \epsilon_2 E_{2,x}$	(18.8)
$\mathbf{H}_t$	$H_{1,z} = H_{2,z}$	$H_{1,y} = H_{2,y}$	(18.9)
$B_n = \mu H_n$	$\mu_1 H_{1,x} = \mu_2 H_{2,x}$		(18.10)

Thus the ansatz for the electric field of polarization 1 is

$$\mathbf{E}(\mathbf{r},t) = e^{i(k_z z-\omega t)}\mathbf{e}_y \cdot \begin{cases} (E_e e^{ik'x} + E_r e^{-ik'x}) & x<0 \\ E_d e^{ik''x} & x>0 \end{cases}. \tag{18.11}$$

From MAXWELL's equations one obtains for the magnetic field

$$\operatorname{curl}\mathbf{E} = -\frac{\mu}{c}\dot{\mathbf{H}} = \frac{i\omega\mu}{c}\mathbf{H}, \tag{18.12}$$

$$\mu H_x = \frac{c}{i\omega}(\operatorname{curl}\mathbf{E})_x = -\frac{c}{i\omega}\frac{\partial E_y}{\partial z} = -\frac{ck_z}{\omega}E_y \tag{18.13}$$

$$H_z = \frac{c}{i\mu\omega}\frac{\partial E_y}{\partial x} = e^{i(k_z z-\omega t)}\frac{c}{\omega}\cdot\begin{cases} \frac{k'}{\mu_1}(E_e e^{ik'x} - E_r e^{-ik'x}) & x<0 \\ \frac{k''}{\mu_2}E_d e^{ik''x} & x>0. \end{cases} \tag{18.14}$$

The boundary conditions come from the continuity of E_y, which is identical to the continuity of μH_x, and from the continuity of H_z,

$$E_e + E_r = E_d, \quad \frac{k'}{\mu_1}(E_e - E_r) = \frac{k''}{\mu_2}E_d, \tag{18.15}$$

from which one obtains the amplitudes

$$E_r = \frac{\mu_2 k' - \mu_1 k''}{\mu_2 k' + \mu_1 k''}E_e, \quad E_d = \frac{2\mu_2 k'}{\mu_2 k' + \mu_1 k''}E_e. \tag{18.16}$$

One comes from polarization 1 to polarization 2 by the transformation

$$\mathbf{E} \to \mathbf{H}, \quad \mathbf{H} \to -\mathbf{E}, \quad \epsilon \leftrightarrow \mu. \tag{18.17}$$

Thus one obtains for the amplitudes

$$H_r = \frac{\epsilon_2 k' - \epsilon_1 k''}{\epsilon_2 k' + \epsilon_1 k''}H_e, \quad H_d = \frac{2\epsilon_2 k'}{\epsilon_2 k' + \epsilon_1 k''}H_e. \tag{18.18}$$

18.c Discussion for $\mu_1 = \mu_2$

Now we discuss the results for $\mu = 1 = \mu_2$, since for many media the permeability is practically equal to 1.

18.c.α Insulator, $|\sin\alpha_2| < 1$: Refraction

Now we determine the amplitude of the reflected wave from that of the incident wave. The reflection coefficient R, i.e. the percentage of the incident power which is reflected is given by

$$R = \left(\frac{E_r}{E_e}\right)^2 = \left(\frac{H_r}{H_e}\right)^2, \tag{18.19}$$

since the modulus of the time averaged POYNTING vector $\mathbf{S} = c\mathbf{E}\times\mathbf{H}/(4\pi)$ for vectors $\mathbf{E}$ and $\mathbf{H}$ which are orthogonal to each other yields

$$\overline{|\mathbf{S}|} = \frac{c}{8\pi}|\mathbf{E}|\cdot|\mathbf{H}| = \frac{c'}{8\pi}\epsilon E^2 = \frac{c'}{8\pi}\mu H^2. \tag{18.20}$$

For the polarisation 1 one obtains with (18.6)

$$E_r = \frac{k'-k''}{k'+k''}E_e = \frac{\tan\alpha_2 - \tan\alpha_1}{\tan\alpha_2 + \tan\alpha_1}E_e = \frac{\sin(\alpha_2-\alpha_1)}{\sin(\alpha_2+\alpha_1)}E_e. \tag{18.21}$$

For polarization 2 one has

$$H_r = \frac{n_2^2 k' - n_1^2 k''}{n_2^2 k' + n_1^2 k''}H_e = \frac{\sin^2\alpha_1\tan\alpha_2 - \sin^2\alpha_2\tan\alpha_1}{\sin^2\alpha_1\tan\alpha_2 + \sin^2\alpha_2\tan\alpha_1}H_e = \frac{\tan(\alpha_1-\alpha_2)}{\tan(\alpha_1+\alpha_2)}H_e. \tag{18.22}$$

One finds that the reflection vanishes for polarization 2 for $\alpha_1+\alpha_2 = 90^0$, from which one obtains $\tan\alpha_1 = n_2/n_1$ because of $\sin\alpha_2 = \cos\alpha_1$ and (18.5). This angle is called Brewster's angle. By incidence of light under this angle only light of polarization 1 is reflected. This can be used to generate linearly polarized light. In the limit α approaching zero, i.e. by incidence of the light perpendicular to the surface one obtains for both polarizations (which can no longer be distinguished)

$$R = \Big(\frac{n_2 - n_1}{n_2 + n_1}\Big)^2, \quad \alpha = 0 \tag{18.23}$$

18.c.β Insulator, $|sin\alpha_2| > 1$: Total Reflection

In this case k'' is imaginary. The wave penetrates only exponentially decaying into the second medium. From the first expressions of (18.21) and (18.22) one finds since the numerator of the fraction is the conjugate complex of the denominator that

$$|E_r| = |E_e|, \quad |H_r| = |H_e|. \quad R = 1, \tag{18.24}$$

Thus one has total reflection.

18.c.γ Metallic Reflection, $\alpha = 0$

In the case of metallic reflection we set $n_1 = 1$ (vacuum or air) and $n_2 = n + \mathrm{i}\kappa$ (17.11). Then one obtains from (18.23) for $\alpha = 0$ the reflection coefficient

$$R = \left|\frac{n + \mathrm{i}\kappa - 1}{n + \mathrm{i}\kappa + 1}\right|^2 = \frac{(n-1)^2 + \kappa^2}{(n+1)^2 + \kappa^2} = 1 - \frac{4n}{(n+1)^2 + \kappa^2}. \tag{18.25}$$

For $\omega\epsilon \ll 2\pi\sigma$ one obtains from (17.5) and (17.15)

$$n \approx \kappa \approx \sqrt{\frac{2\pi\sigma}{\omega}}, \quad R \approx 1 - \frac{2}{n} \approx 1 - \sqrt{\frac{2\omega}{\pi\sigma}}, \tag{18.26}$$

a result named after Hagen and Rubens.

18.c.δ Surface Waves along a Conductor

Finally we consider waves, which run along the boundary of a conductor with the vacuum. Thus we set $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon(\omega)$ from (17.5). Then we need one wave on each side of the boundary. We obtain this by looking for a solution, where no wave is reflected. Formally this means that we choose a wave of polarization 2 for which H_r in (18.18) vanishes, thus

$$\epsilon(\omega)k' = k'' \tag{18.27}$$

has to hold. With (18.2) and (18.3)

$$k_z^2 + k'^2 = \frac{\omega^2}{c^2}, \quad k_z^2 + k''^2 = \frac{\epsilon(\omega)\omega^2}{c^2} \tag{18.28}$$

one obtains the solution

$$k_z = \frac{\omega}{c}\sqrt{\frac{\epsilon(\omega)}{1+\epsilon(\omega)}}, \quad k' = \frac{k_z}{\sqrt{\epsilon(\omega)}}, \quad k'' = \sqrt{\epsilon(\omega)}k_z. \tag{18.29}$$

Using approximation (17.15) one obtains for frequencies which are not too large

$$k_z = \frac{\omega}{c}\Big(1 + \frac{\mathrm{i}\omega}{8\pi\sigma}\Big) \tag{18.30}$$

$$k' = \frac{(1-\mathrm{i})\omega^{3/2}}{2c\sqrt{2\pi\sigma}} \tag{18.31}$$

$$k'' = \frac{(1+\mathrm{i})\omega^{1/2}\sqrt{2\pi\sigma}}{c}. \tag{18.32}$$

Thus for small frequencies, $\omega < \sigma$, the exponential decay in direction of propagation (k_z) is smallest, into the vacuum it is faster (k') and into the metal it is fastest (k'').

19 Wave Guides

There are various kinds of wave guides. They may consist for example of two conductors, which run in parallel (two wires) or which are coaxial conductors. But one may also guide an electro-magnetic wave in a dielectric wave guide (for light e.g.) or in a hollow metallic cylinder.
In all cases we assume translational invariance in z-direction, so that material properties ϵ, μ, and σ are only functions of x and y. Then the electromagnetic fields can be written

$$\mathbf{E} = \mathbf{E}_0(x,y)\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}, \quad \mathbf{B} = \mathbf{B}_0(x,y)\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}. \tag{19.1}$$

Then the functions $\mathbf{E}_0$, $\mathbf{B}_0$ and $\omega(k_z)$ have to be determined.

19.a Wave Guides

We will carry through this program for a wave guide which is a hollow metallic cylinder (not necessarily with circular cross-section). We start out from the boundary conditions, where we assume that the cylinder surface is an ideal metal $\sigma = \infty$. Then one has at the surface

$$\mathbf{E}_\mathrm{t} = \mathbf{0}, \tag{19.2}$$

since a tangential component would yield an infinite current density at the surface. Further from $\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}/c$ it follows that

$$\mathrm{i}kB_\mathrm{n} = (\operatorname{curl}\mathbf{E})_\mathrm{n} = (\operatorname{curl}\mathbf{E}_\mathrm{t})\cdot\mathbf{e}_\mathrm{n} \quad k = \omega/c, \tag{19.3}$$

from which one obtains

$$B_\mathrm{n} = 0. \tag{19.4}$$

Inside the wave guide one has

$$(\operatorname{curl}\mathbf{E})_y = -\frac{1}{c}\dot{B}_y \quad\rightarrow\quad \mathrm{i}k_zE_{0,x} - \nabla_xE_{0,z} = \mathrm{i}kB_{0,y} \tag{19.5}$$

$$(\operatorname{curl}\mathbf{B})_x = \frac{1}{c}\dot{E}_x \quad\rightarrow\quad \nabla_yB_{0,z} - \mathrm{i}k_zB_{0,y} = -\mathrm{i}kE_{0,x}. \tag{19.6}$$

By use of

$$k_\perp^2 = k^2 - k_z^2 \tag{19.7}$$

one can express the transverse components by the longitudinal components

$$k_\perp^2 E_{0,x} = \mathrm{i}k_z\nabla_xE_{0,z} + \mathrm{i}k\nabla_yB_{0,z} \tag{19.8}$$

$$k_\perp^2 B_{0,y} = \mathrm{i}k\nabla_xE_{0,z} + \mathrm{i}k_z\nabla_yB_{0,z}. \tag{19.9}$$

Similar equations hold for $E_{0,y}$ and $B_{0,x}$. In order to determine the longitudinal components we use the wave equation

$$(\triangle - \frac{\partial^2}{c^2\partial t^2})(E_{0,z}\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}) = 0, \tag{19.10}$$

from which one obtains

$$(\nabla_x^2 + \nabla_y^2 + k_\perp^2)E_{0,z}(x,y) = 0 \tag{19.11}$$

and similarly

$$(\nabla_x^2 + \nabla_y^2 + k_\perp^2)B_{0,z}(x,y) = 0. \tag{19.12}$$

One can show that the other equations of Maxwell are fulfilled for $k_\perp \neq 0$, since

$$\left.\begin{array}{r} k_\perp^2 \operatorname{div}\mathbf{E} = \mathrm{i}k_z \\ k_\perp^2(\operatorname{curl}\mathbf{B} - \dot{\mathbf{E}}/c)_z = \mathrm{i}k \end{array}\right\}\cdot(\nabla_x^2 + \nabla_y^2 + k_\perp^2)E_{0,z}\mathrm{e}^{\mathrm{i}(k_z z-\omega t)} \tag{19.13}$$

$$\left.\begin{array}{r} k_\perp^2 \operatorname{div}\mathbf{B} = \mathrm{i}k_z \\ k_\perp^2(\operatorname{curl}\mathbf{E} + \dot{\mathbf{B}}/c)_z = -\mathrm{i}k \end{array}\right\}\cdot(\nabla_x^2 + \nabla_y^2 + k_\perp^2)B_{0,z}\mathrm{e}^{\mathrm{i}(k_z z-\omega t)}. \tag{19.14}$$

Thus it is sufficient to fulfill the wave equations. We further note that $E_{0,z}$ and $B_{0,z}$ are independent from each other. Correspondingly one distinguishes TE-modes (transverse electric) with $E_{0,z} = 0$ and TM-modes (transverse magnetic) with $B_{0,z} = 0$.
We return to the boundary conditions. The components perpendicular to the direction of propagation z read

$$k_\perp^2(\mathbf{e}_x E_{0,x} + \mathbf{e}_y E_{0,y}) = \mathrm{i}k_z \,\mathrm{grad}\, E_{0,z} - \mathrm{i}k\mathbf{e}_z \times \,\mathrm{grad}\, B_{0,z} \tag{19.15}$$
$$k_\perp^2(\mathbf{e}_x B_{0,x} + \mathbf{e}_y B_{0,y}) = \mathrm{i}k_z \,\mathrm{grad}\, B_{0,z} + \mathrm{i}k\mathbf{e}_z \times \,\mathrm{grad}\, E_{0,z}. \tag{19.16}$$

If we introduce besides the normal vector $\mathbf{e}_n$ and the vector $\mathbf{e}_z$ a third unit vector $\mathbf{e}_c = \mathbf{e}_z \times \mathbf{e}_n$ at the surface of the waveguide then the tangential plain of the surface is spanned by $\mathbf{e}_z$ and $\mathbf{e}_c$. $\mathbf{e}_c$ itself lies in the xy-plain. Since $\mathbf{e}_n$ lies in the xy-plain too, we may transform to n and c components

$$\mathbf{e}_x E_{0,x} + \mathbf{e}_y E_{0,y} = \mathbf{e}_c E_{0,c} + \mathbf{e}_n E_{0,n}. \tag{19.17}$$

Then (19.15, 19.16) can be brought into the form

$$k_\perp^2(\mathbf{e}_n E_{0,n} + \mathbf{e}_c E_{0,c}) = \mathrm{i}k_z(\mathbf{e}_n \partial_n E_{0,z} + \mathbf{e}_c \partial_c E_{0,z}) - \mathrm{i}k(\mathbf{e}_c \partial_n B_{0,z} - \mathbf{e}_n \partial_c B_{0,z}), \tag{19.18}$$
$$k_\perp^2(\mathbf{e}_n B_{0,n} + \mathbf{e}_c B_{0,c}) = \mathrm{i}k_z(\mathbf{e}_n \partial_n B_{0,z} + \mathbf{e}_c \partial_c B_{0,z}) + \mathrm{i}k(\mathbf{e}_c \partial_n E_{0,z} - \mathbf{e}_n \partial_c E_{0,z}). \tag{19.19}$$

At the surface one has

$$E_{0,z} = E_{0,c} = B_{0,\mathrm{n}} = 0 \tag{19.20}$$

according to (19.2, 19.4). From (19.18, 19.19) one obtains

$$k_\perp^2 E_{0,c} = \mathrm{i}k_z \partial_c E_{0,z} - \mathrm{i}k\partial_n B_{0,z}, \tag{19.21}$$
$$k_\perp^2 B_{0,\mathrm{n}} = \mathrm{i}k_z \partial_n B_{0,z} + \mathrm{i}k\partial_c E_{0,z}. \tag{19.22}$$

Since $E_{0,z} = 0$ holds at the surface one has $\partial_c E_{0,z} = 0$ at the surface too. Apparently the second condition is $\partial_n B_{0,z} = 0$.
Then the following eigenvalue problem has to be solved

$$\text{TM-Mode:} \quad (\nabla_x^2 + \nabla_y^2 + k_\perp^2)E_{0,z} = 0, \quad E_{0,z} = 0 \text{ at the surface}, \tag{19.23}$$
$$\text{TE-Mode:} \quad (\nabla_x^2 + \nabla_y^2 + k_\perp^2)B_{0,z} = 0, \quad (\,\mathrm{grad}\, B_{0,z})_\mathrm{n} = 0 \text{ at the surface}. \tag{19.24}$$

Then one obtains the dispersion law

$$\omega = c\sqrt{k_z^2 + k_\perp^2}. \tag{19.25}$$

TEM-modes By now we did not discuss the case $k_\perp = 0$. We will not do this in all details. One can show that for these modes both longitudinal components vanish, $E_{0,z} = B_{0,z} = 0$. Thus one calls them TEM-modes. Using $k_z = \pm k$ from (19.5) and similarly after a rotation of $\mathbf{E}$ and $\mathbf{B}$ around the z-axis by 90^0 $E_{0,x} \to E_{0,y}$, $B_{0,y} \to -B_{0,x}$ one obtains

$$B_{0,y} = \pm E_{0,x}, \quad B_{0,x} = \mp E_{0,y}. \tag{19.26}$$

From $(\,\mathrm{curl}\,\mathbf{E})_z = 0$ it follows that $\mathbf{E}_0$ can be expressed by the gradient of a potential

$$\mathbf{E}_0 = -\,\mathrm{grad}\,\Phi(x,y), \tag{19.27}$$

which due to $\mathrm{div}\,\mathbf{E}_0 = 0$ fulfills Laplace's equation

$$(\nabla_x^2 + \nabla_y^2)\Phi(x,y) = 0. \tag{19.28}$$

Thus Laplace's homogeneous equation in two dimensions has to be solved. Because of $\mathbf{E}_{0,\mathrm{t}} = \mathbf{0}$ the potential on the surface has to be constant. Thus one obtains a non-trivial solution only in multiply connected regions, i.e. not inside a circular or rectangular cross-section, but outside such a region or in a coaxial wire or outside two wires.

19.b Solution for a Rectangular Cross Section

We determine the waves in a wave guide of rectangular cross-section with sides a and b. For the TM-wave we start with the factorization ansatz

$$E_{0,z}(x,y) = f(x)g(y) \tag{19.29}$$

Insertion into (19.11) yields

$$f''g + fg'' + k_\perp^2 fg = 0 \tag{19.30}$$

and equivalently

$$\frac{f''}{f} + \frac{g''}{g} = -k_\perp^2, \tag{19.31}$$

from which one concludes that f''/f and g''/g have to be constant. Since $E_{0,z}$ has to vanish at the boundary, one obtains

$$E_{0,z}(x,y) = E_0 \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{b}), \quad k_\perp^2 = \left(\frac{nu\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2, \quad n \geq 1, m \geq 1. \tag{19.32}$$

For the TE-wave one obtains with the corresponding ansatz

$$B_{0,z}(x,y) = f(x)g(y) \tag{19.33}$$

and the boundary condition $(\operatorname{grad} B_{0,z})_{\mathrm{n}} = 0$ the solutions

$$B_{0,z}(x,y) = B_0 \cos(\frac{n\pi x}{a}) \cos(\frac{m\pi y}{b}), \quad k_\perp^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2, \quad n \geq 0, \quad m \geq 0, \quad n+m \geq 1. \tag{19.34}$$

19.c Wave Packets

Often one does not deal with monochromatic waves, but with wave packets, which consist of Fourier components with $k_z \approx k_{z,0}$

$$\mathbf{E} = \mathbf{E}_0(x,y) \int dk_z f_0(k_z) \mathrm{e}^{\mathrm{i}(k_z z - \omega(k_z)t)}, \tag{19.35}$$

where $f_0(k_z)$ has a maximum at $k_z = k_{z,0}$ and decays rapidly for other values of k_z. Then one expands $\omega(k_z)$ around $k_{z,0}$

$$\omega(k_z) = \omega(k_{z,0}) + v_{\mathrm{gr}}(k_z - k_{z,0}) + ... \tag{19.36}$$

$$v_{\mathrm{gr}} = \left.\frac{d\omega(k_z)}{dk_z}\right|_{k_z = k_{z,0}}. \tag{19.37}$$

In linear approximation of this expansion one obtains

$$\mathbf{E} = \mathbf{E}_0(x,y) \mathrm{e}^{\mathrm{i}(k_{z,0}z - \omega(k_{z,0})t)} f(z - v_{\mathrm{gr}}t), \quad f(z - v_{\mathrm{gr}}t) = \int \mathrm{d}k_z f_0(k_z) \mathrm{e}^{\mathrm{i}(k_z - k_{z,0})(z - v_{\mathrm{gr}}t)}. \tag{19.38}$$

The factor in front contains the phase $\phi = k_{z,0}z - \omega(k_{z,0})t$. Thus the wave packet oscillates with the phase velocity

$$v_{\mathrm{ph}} = \left.\frac{\partial z}{\partial t}\right|_\phi = \frac{\omega(k_{z,0})}{k_{z,0}}. \tag{19.39}$$

On the other hand the local dependence of the amplitude is contained in the function $f(z - v_{\mathrm{gr}}t)$. Thus the wave packet moves with the group velocity (signal velocity) v_{gr}, (19.37).
For the waves in the wave-guide we obtain from (19.25)

$$v_{\mathrm{ph}} = c\frac{\sqrt{k_\perp^2 + k_{z,0}^2}}{k_{z,0}}, \tag{19.40}$$

$$v_{\mathrm{gr}} = c\frac{k_{z,0}}{\sqrt{k_\perp^2 + k_{z,0}^2}}. \tag{19.41}$$

The phase velocity is larger than the velocity of light in vacuum c, the group velocity (velocity of a signal) less than c. If one performs the expansion (19.36) beyond the linear term, then one finds that the wave packets spread in time.
Exercise Determine $\omega(k)$ for transverse oscillations in a conductor above the plasma frequency (section 17.b) for $\epsilon = 1$ and the resulting phase- and group-velocities, resp.

G Electrodynamic Potentials

20 Electrodynamic Potentials, Gauge Transformations

We already know the electric potential Φ from electrostatics and the vector potential $\mathbf{A}$ from magnetostatics. Both can also be used for time-dependent problems and allow the determination of $\mathbf{B}$ and $\mathbf{E}$.

20.a Potentials

MAXWELL's third and fourth equations are homogeneous equations, i.e. they do not contain charges and currents explicitly. They allow to express the fields $\mathbf{B}$ and $\mathbf{E}$ by means of potentials. One obtains from $\operatorname{div}\mathbf{B} = 0$

$$\mathbf{B}(\mathbf{r}, t) = \operatorname{curl}\mathbf{A}(\mathbf{r}, t). \tag{20.1}$$

Proof: Due to $\operatorname{div}\mathbf{B} = 0$ one has $\triangle\mathbf{B} = -\operatorname{curl}\operatorname{curl}\mathbf{B}$ (B.26), from which one concludes similarly as in (9.16) and (9.17)

$$\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r' \Big(\operatorname{curl}'\operatorname{curl}'\mathbf{B}(\mathbf{r}')\Big)\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi}\operatorname{curl}\int d^3r' \frac{\operatorname{curl}'\mathbf{B}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{20.2}$$

when the vector potential was introduced in the magnetostatics. An elementary proof is left as exercise.
From $\operatorname{curl}\mathbf{E} + \dot{\mathbf{B}}/c = \mathbf{0}$ one obtains

$$\operatorname{curl}\Big(\mathbf{E} + \frac{1}{c}\dot{\mathbf{A}}\Big) = \mathbf{0}, \tag{20.3}$$

so that the argument under the curl can be expressed as a gradient. Conventionally one calls it $-\operatorname{grad}\Phi$, so that

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \operatorname{grad}\Phi \tag{20.4}$$

follows. The second term is already known from electrostatics. The time derivative of $\mathbf{A}$ contains the law of induction. One sees contrarily that the representations of the potentials in (20.4) and (20.1) fulfill the homogeneous MAXWELL equations.

20.b Gauge Transformations

The potentials $\mathbf{A}$ and Φ are not uniquely determined by the fields $\mathbf{B}$ and $\mathbf{E}$. We may replace $\mathbf{A}$ by

$$\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \operatorname{grad}\Lambda(\mathbf{r}, t) \tag{20.5}$$

without changing $\mathbf{B}$

$$\mathbf{B} = \operatorname{curl}\mathbf{A} = \operatorname{curl}\mathbf{A}', \tag{20.6}$$

since $\operatorname{curl}\operatorname{grad}\Lambda = \mathbf{0}$. It follows that

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}}' - \operatorname{grad}(\Phi - \frac{1}{c}\dot{\Lambda}). \tag{20.7}$$

If we replace simultaneously Φ by

$$\Phi'(\mathbf{r}, t) = \Phi(\mathbf{r}, t) - \frac{1}{c}\dot{\Lambda}(\mathbf{r}, t), \tag{20.8}$$

then $\mathbf{E}$ and $\mathbf{B}$ remain unchanged. One calls the transformations (20.5) and (20.8) gauge transformations.

The arbitrariness in the gauge allows to impose restrictions on the potentials Φ and $\mathbf{A}$

$$\text{Lorenz gauge} \qquad \operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\Phi} = 0, \tag{20.9}$$

$$\text{Coulomb gauge} \qquad \operatorname{div}\mathbf{A} = 0. \tag{20.10}$$

If potentials Φ' and $\mathbf{A}'$ do not obey the desired gauge, the potentials Φ and $\mathbf{A}$ are obtained by an appropriate choice of Λ

$$\text{Lorenz gauge} \qquad \operatorname{div}\mathbf{A}' + \frac{1}{c}\dot{\Phi}' = \Box\Lambda, \tag{20.11}$$

$$\text{Coulomb gauge} \qquad \operatorname{div}\mathbf{A}' = \triangle\Lambda, \tag{20.12}$$

where

$$\Box := \triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{20.13}$$

is d'Alembert's operator. The Lorenz gauge traces back to the Danish physicist Ludvig V. Lorenz (1867) in contrast to the Lorentz transformation (section 23) attributed to the Dutch physicist Hendrik A. Lorentz. Insertion of the expressions (20.4) and (20.1) for $\mathbf{E}$ and $\mathbf{B}$ into Maxwell's first equation yields

$$\operatorname{curl}\operatorname{curl}\mathbf{A} + \frac{1}{c^2}\ddot{\mathbf{A}} + \frac{1}{c}\operatorname{grad}\dot{\Phi} = \frac{4\pi}{c}\mathbf{j}, \tag{20.14}$$

that is

$$-\Box\mathbf{A} + \operatorname{grad}\left(\operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\Phi}\right) = \frac{4\pi}{c}\mathbf{j}, \tag{20.15}$$

whereas Maxwell's second equation reads

$$-\triangle\Phi - \frac{1}{c}\operatorname{div}\dot{\mathbf{A}} = 4\pi\rho. \tag{20.16}$$

From this one obtains for both gauges

$$\text{Lorenz gauge} \qquad \begin{cases} \Box\mathbf{A} = -\frac{4\pi}{c}\mathbf{j} \\ \Box\Phi = -4\pi\rho \end{cases} \tag{20.17}$$

$$\text{Coulomb gauge} \qquad \begin{cases} \Box\mathbf{A} = -\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\operatorname{grad}\dot{\Phi} \\ \triangle\Phi = -4\pi\rho. \end{cases} \tag{20.18}$$

Exercise Show that a vector field $\mathbf{B}(\mathbf{r})$, which obeys $\operatorname{div}\mathbf{B} = 0$ can be represented by $\operatorname{curl}\mathbf{A}(\mathbf{r})$. Therefore put $A_z(\mathbf{r}) = 0$ and express $A_y(\mathbf{r})$ by $A_y(x, y, 0)$ and B_x, similarly $A_x(\mathbf{r})$ by $A_x(x, y, 0)$ and B_y. Insert this in $B_z = (\operatorname{curl}\mathbf{A})_z$ and show by use of $\operatorname{div}\mathbf{B} = 0$ that one can find fitting components of $\mathbf{A}$ at $\mathbf{r} = (x, y, 0)$.

21 Electromagnetic Potentials of a general Charge and Current Distribution

21.a Calculation of the Potentials

Using the LORENZ gauge we had

$$\Box\Phi(\mathbf{r},t) = -4\pi\rho(\mathbf{r},t), \tag{21.1}$$

$$\Box\mathbf{A}(\mathbf{r},t) = -\frac{4\pi}{c}\mathbf{j}(\mathbf{r},t) \tag{21.2}$$

with D'ALEMBERT's operator

$$\Box = \triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{21.3}$$

and the gauge condition

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\Phi} = 0. \tag{21.4}$$

We perform the FOURIER transform with respect to time

$$\Phi(\mathbf{r},t) = \int_{-\infty}^{\infty} \mathrm{d}\omega\hat{\Phi}(\mathbf{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{21.5}$$

analogously for $\mathbf{A}$, ρ, $\mathbf{j}$. Then one obtains

$$\Box\Phi(\mathbf{r},t) = \int \mathrm{d}\omega(\triangle + \frac{\omega^2}{c^2})\hat{\Phi}(\mathbf{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t} = \int \mathrm{d}\omega(-4\pi\hat{\rho}(\mathbf{r},\omega))\mathrm{e}^{-\mathrm{i}\omega t}, \tag{21.6}$$

from which by comparison of the integrands

$$\Big(\triangle + \frac{\omega^2}{c^2}\Big)\hat{\Phi}(\mathbf{r},\omega) = -4\pi\hat{\rho}(\mathbf{r},\omega) \tag{21.7}$$

is obtained. We now introduce the GREEN's function G, i.e. we write the solution of the linear differential equation as

$$\hat{\Phi}(\mathbf{r},\omega) = \int \mathrm{d}^3r' G(\mathbf{r},\mathbf{r}',\omega)\hat{\rho}(\mathbf{r}',\omega). \tag{21.8}$$

Insertion of this ansatz into the differential equation (21.7) yields

$$\Big(\triangle + \frac{\omega^2}{c^2}\Big)G(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \tag{21.9}$$

Since there is no preferred direction and moreover the equation is invariant against displacement of the vectors $\mathbf{r}$ and $\mathbf{r}'$ by the same constant vector, we may assume that the solution depends only on the distance between $\mathbf{r}$ and $\mathbf{r}'$ and additionally of course on ω

$$G = g(a,\omega), \quad a = |\mathbf{r}-\mathbf{r}'|. \tag{21.10}$$

Then one obtains

$$(\triangle + \frac{\omega^2}{c^2})g(a,\omega) = \frac{1}{a}\frac{d^2(ag)}{da^2} + \frac{\omega^2}{c^2}g = 0 \text{ for } a \neq 0. \tag{21.11}$$

Here we use the Laplacian in the form (5.15), where $\triangle_\Omega g = 0$, since g does not depend on the direction of $\mathbf{a} = \mathbf{r}-\mathbf{r}'$, but on the modulus a. This yields the equation of a harmonic oscillation for ag with the solution

$$G = g(a,\omega) = \frac{1}{a}\Big(c_1\mathrm{e}^{\mathrm{i}\omega a/c} + c_2\mathrm{e}^{-\mathrm{i}\omega a/c}\Big). \tag{21.12}$$

At short distances the solution diverges like $(c_1 + c_2)/a$. In order to obtain the δ-function in (21.9) as an inhomogeneity with the appropriate factor in front, one requires $c_1 + c_2 = 1$. We now insert

$$\begin{aligned}
\Phi(\mathbf{r}, t) &= \int d\omega \hat{\Phi}(\mathbf{r}, \omega) e^{-i\omega t} \\
&= \int d\omega \int d^3r' e^{-i\omega t} G(\mathbf{r}, \mathbf{r}', \omega) \hat{\rho}(\mathbf{r}', \omega) \\
&= \int d\omega \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \left(c_1 e^{i\omega|\mathbf{r}-\mathbf{r}'|/c} + c_2 e^{-i\omega|\mathbf{r}-\mathbf{r}'|/c} \right) e^{-i\omega t} \hat{\rho}(\mathbf{r}', \omega) \\
&= \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \left(c_1 \rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) + c_2 \rho(\mathbf{r}', t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}) \right). \qquad (21.13)
\end{aligned}$$

Going from the second to the third line we have inserted G. Then we perform the ω-integration, compare (21.5). However, ω in the exponent in (21.13) is not multiplied by t, but by $t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}$. The solution in the last line contains a contribution of Φ at time t which depends on ρ at an earlier time (with factor c_1), and one which depends on ρ at a later time (with factor c_2). The solution which contains only the first contribution ($c_1 = 1$, $c_2 = 0$) is called the retarded solution, and the one which contains only the second contribution ($c_1 = 0$, $c_2 = 1$) the advanced solution.

$$\Phi_{r,a}(\mathbf{r}, t) = \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \rho(\mathbf{r}', t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}). \qquad (21.14)$$

Normally the retarded solution (upper sign) is the physical solution, since the potential is considered to be created by the charges, but not the charges by the potentials. Analogously, one obtains the retarded and advanced solutions for the vector potential

$$\mathbf{A}_{r,a}(\mathbf{r}, t) = \frac{1}{c} \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathbf{j}(\mathbf{r}', t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}). \qquad (21.15)$$

21.b Gauge Condition

It remains to be shown that the condition for Lorenz gauge (20.9) is fulfilled

$$\dot{\Phi} + c \operatorname{div} \mathbf{A} = \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} (\dot{\rho} + \operatorname{div} \mathbf{j}) + \int d^3r' \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathbf{j}. \qquad (21.16)$$

The arguments of ρ and $\mathbf{j}$ are as above $\mathbf{r}'$ and $t' = t \mp |\mathbf{r}-\mathbf{r}'|/c$. In the second integral one can replace ∇ by $-\nabla'$ and perform a partial integration. This yields

$$\dot{\Phi} + c \operatorname{div} \mathbf{A} = \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} (\dot{\rho} + (\nabla + \nabla')\mathbf{j}). \qquad (21.17)$$

Since $(\nabla + \nabla')t'(t, \mathbf{r}, \mathbf{r}') = \mathbf{0}$, one obtains from the equation of continuity

$$\dot{\rho}(\mathbf{r}', t'(t, \mathbf{r}, \mathbf{r}')) + (\nabla + \nabla')\mathbf{j}(\mathbf{r}', t'(t, \mathbf{r}, \mathbf{r}')) = \frac{\partial \rho}{\partial t'} + \nabla' \mathbf{j}(\mathbf{r}', t')|_{t'} = 0, \qquad (21.18)$$

so that the gauge condition (20.9) is fulfilled, since the integrand in (21.17) vanishes because of the equation of continuity.

22 Radiation from Harmonic Oscillations

In this section we consider the radiation of oscillating charges and currents.

22.a Radiation Field

We consider harmonic oscillations, i.e. the time dependence of ρ and $\mathbf{j}$ is proportional to $e^{-i\omega t}$

$$\rho(\mathbf{r},t) = \Re\Big(\rho_0(\mathbf{r})e^{-i\omega t}\Big) \tag{22.1}$$

$$\mathbf{j}(\mathbf{r},t) = \Re\Big(\mathbf{j}_0(\mathbf{r})e^{-i\omega t}\Big), \tag{22.2}$$

analogously for Φ, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$. One obtains

$$\Phi(\mathbf{r},t) = \Re\int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho_0(\mathbf{r}')e^{-i\omega(t-|\mathbf{r}-\mathbf{r}'|/c)}. \tag{22.3}$$

With the notation $k = \omega/c$ it follows that

$$\Phi_0(\mathbf{r}) = \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho_0(\mathbf{r}')e^{ik|\mathbf{r}-\mathbf{r}'|}, \tag{22.4}$$

analogously

$$\mathbf{A}_0(\mathbf{r}) = \frac{1}{c}\int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}_0(\mathbf{r}')e^{ik|\mathbf{r}-\mathbf{r}'|}. \tag{22.5}$$

22.a.α Near Zone (Static Zone)

In the near zone, i.e. for $k|\mathbf{r}-\mathbf{r}'| \ll 2\pi$ which is equivalent to $|\mathbf{r}-\mathbf{r}'| \ll \lambda$, where λ is the wave-length of the electromagnetic wave, the expression $e^{ik|\mathbf{r}-\mathbf{r}'|}$ can be approximated by 1. Then the potentials Φ_0, (22.4) and $\mathbf{A}_0$, (22.5) reduce to the potentials of electrostatics (3.14) and of magnetostatics (9.17).

22.a.β Far Zone (Radiation Zone)

At large distances one expands the expression in the exponent

$$|\mathbf{r}-\mathbf{r}'| = r\sqrt{1-2\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}+\frac{r'^2}{r^2}} = r - \mathbf{n}\cdot\mathbf{r}' + O(\frac{r'^2}{r}), \quad \mathbf{n} = \frac{\mathbf{r}}{r}. \tag{22.6}$$

This is justified for $r \gg kR^2$, where R is an estimate of the extension of the charge and current distribution, $r' < R$ for $\rho(\mathbf{r}') \neq 0$ and $\mathbf{j}(\mathbf{r}') \neq \mathbf{0}$, resp. We approximate in the denominator $|\mathbf{r}-\mathbf{r}'| \approx r$ which is reasonable for $r \gg R$. Then one obtains

$$\mathbf{A}_0(\mathbf{r}) = \frac{e^{ikr}}{cr}\mathbf{g}(k\mathbf{n}) + O(\frac{1}{r^2}) \tag{22.7}$$

with the Fourier transform of the current distribution

$$\mathbf{g}(k\mathbf{n}) = \int d^3r' \mathbf{j}_0(\mathbf{r}')e^{-ik\mathbf{n}\cdot\mathbf{r}'}. \tag{22.8}$$

From this one deduces the magnetic field

$$\mathbf{B}_0(\mathbf{r}) = \operatorname{curl}\mathbf{A}_0(\mathbf{r}) = \frac{\operatorname{grad} e^{ikr}}{cr}\times\mathbf{g}(k\mathbf{n}) + O(\frac{1}{r^2}) = ik\frac{e^{ikr}}{cr}\mathbf{n}\times\mathbf{g} + O(\frac{1}{r^2}). \tag{22.9}$$

The electric field is obtained from

$$\operatorname{curl}\mathbf{B} = \frac{1}{c}\dot{\mathbf{E}} \to \operatorname{curl}\mathbf{B}_0 = -\frac{i\omega}{c}\mathbf{E}_0 \tag{22.10}$$

as

$$\mathbf{E}_0 = \frac{\mathrm{i}}{k}\,\mathrm{curl}\,\mathbf{B}_0 = -\mathbf{n}\times\mathbf{B}_0 + O(\frac{1}{r^2}). \tag{22.11}$$

$\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{n}$ are orthogonal to each other. The moduli of $\mathbf{E}_0$ and $\mathbf{B}_0$ are equal and both decay like $1/r$. The Poynting vector yields in the time average

$$\bar{\mathbf{S}} = \frac{1}{T}\int_0^T \mathbf{S}(t)\mathrm{d}t, \quad T = \frac{2\pi}{\omega} \tag{22.12}$$

$$\begin{aligned}\bar{\mathbf{S}} &= \frac{c}{4\pi}\overline{\Re\mathbf{E}\times\Re\mathbf{B}} = \frac{c}{8\pi}\Re(\mathbf{E}_0^*\times\mathbf{B}_0) \\ &= -\frac{c}{8\pi}\Re\Big((\mathbf{n}\times\mathbf{B}_0^*)\times\mathbf{B}_0\Big) = -\frac{c}{8\pi}\Re(\mathbf{n}\cdot\mathbf{B}_0)\mathbf{B}_0^* + \frac{c}{8\pi}\mathbf{n}(\mathbf{B}_0^*\cdot\mathbf{B}_0). \end{aligned}\tag{22.13}$$

The first term after the last equals sign vanishes, since $\mathbf{B}_0 \perp \mathbf{n}$. Thus there remains

$$\bar{\mathbf{S}}(\mathbf{r}) = \frac{c}{8\pi}\mathbf{n}(\mathbf{B}_0^*\cdot\mathbf{B}_0) = \frac{k^2\mathbf{n}}{8\pi c r^2}|\mathbf{n}\times\mathbf{g}(k\mathbf{n})|^2 + O(\frac{1}{r^3}). \tag{22.14}$$

The average power radiated is

$$\dot{U}_{\mathrm{s}} = \frac{k^2}{8\pi c}\int |\mathbf{n}\times\mathbf{g}(k\mathbf{n})|^2\mathrm{d}\Omega_n, \tag{22.15}$$

where the integral is performed over the solid angle Ω_n of $\mathbf{n}$.

22.b Electric Dipole Radiation (Hertz Dipole)

If the charge and current distribution is within a range R small in comparison to the wave length λ, then it is reasonable to expand $\mathrm{e}^{-\mathrm{i}k\mathbf{n}\cdot\mathbf{r}'}$

$$\mathbf{g}(k\mathbf{n}) = \mathbf{g}^{(0)} - \mathrm{i}k\mathbf{g}^{(1)} + ..., \quad \mathbf{g}^{(0)} = \int \mathrm{d}^3r'\mathbf{j}_0(\mathbf{r}'), \quad \mathbf{g}^{(1)} = \int \mathrm{d}^3r'(\mathbf{n}\cdot\mathbf{r}')\mathbf{j}_0(\mathbf{r}') \tag{22.16}$$

This expansion is sufficient to investigate the radiation field in the far zone. If one is interested to consider it also in the near zone and the intermediate zone, one has to expand

$$\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathrm{e}^{\mathrm{i}kr}}{r} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r}(-\mathrm{i}k + \frac{1}{r})(\mathbf{n}\cdot\mathbf{r}') + O(r'^2) \tag{22.17}$$

in the expression for $\mathbf{A}_0$, which yields

$$\mathbf{A}_0(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{cr}\mathbf{g}^{(0)} + (-\mathrm{i}k + \frac{1}{r})\frac{\mathrm{e}^{\mathrm{i}kr}}{cr}\mathbf{g}^{(1)} + ... \tag{22.18}$$

We first consider the contribution from $\mathbf{g}^{(0)}$. We use that

$$\mathrm{div}'\mathbf{j}(\mathbf{r}') = -\dot{\rho}(\mathbf{r}') = \mathrm{i}\omega\rho(\mathbf{r}') \rightarrow \mathrm{div}'\mathbf{j}_0(\mathbf{r}') = \mathrm{i}\omega\rho_0(\mathbf{r}'). \tag{22.19}$$

Then we obtain from

$$\int \mathrm{d}^3r'\,\mathrm{div}'\Big(f(\mathbf{r}')\mathbf{j}_0(\mathbf{r}')\Big) = 0 \tag{22.20}$$

the relation

$$\int \mathrm{d}^3r'\,\mathrm{grad}'f(\mathbf{r}')\cdot\mathbf{j}_0(\mathbf{r}') = -\mathrm{i}\omega\int \mathrm{d}^3r'f(\mathbf{r}')\rho_0(\mathbf{r}'). \tag{22.21}$$

One obtains with $f(\mathbf{r}') = x'_\alpha$

$$g_\alpha^{(0)} = \int \mathrm{d}^3r'\,j_{0,\alpha}(\mathbf{r}') = -\mathrm{i}\omega\int \mathrm{d}^3r'x'_\alpha\rho_0(\mathbf{r}') = -\mathrm{i}\omega p_{0,\alpha}, \tag{22.22}$$

that is $\mathbf{g}^{(0)}$ can be expressed by the amplitude of the electric dipole moment

$$\mathbf{g}^{(0)} = -\mathrm{i}\omega\mathbf{p}_0. \tag{22.23}$$

Thus one calls this contribution electric dipole radiation. One finds

$$\mathbf{A}_0(\mathbf{r}) = -\mathrm{i}k\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathbf{p}_0, \tag{22.24}$$

thus

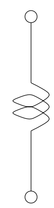
$$\mathbf{B}_0(\mathbf{r}) = \Big(\frac{k^2}{r} + \frac{\mathrm{i}k}{r^2}\Big)\mathrm{e}^{\mathrm{i}kr}\mathbf{n}\times\mathbf{p}_0 \tag{22.25}$$

$$\mathbf{E}_0(\mathbf{r}) = -\frac{k^2}{r}\mathrm{e}^{\mathrm{i}kr}\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0) + \Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_0) - \mathbf{p}_0\Big)\Big(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\Big)\mathrm{e}^{\mathrm{i}kr}. \tag{22.26}$$

The first term is leading in the far zone ($1/r \ll k$), the second one in the near zone ($1/r \gg k$). One obtains the time-averaged Poynting vector from the expression for the far zone

$$\bar{\mathbf{S}} = \frac{ck^4\mathbf{n}}{8\pi r^2}|\mathbf{n}\times\mathbf{p}_0|^2 = \frac{ck^4|\mathbf{p}_0|^2\mathbf{n}}{8\pi r^2}\sin^2\theta, \tag{22.27}$$

In the second expression it is assumed that real and imaginary part of $\mathbf{p}_0$ point into the same direction. Then θ is the angle between $\mathbf{p}_0$ and $\mathbf{n}$. The radiated power is then



$$\dot{U}_\mathrm{s} = \frac{ck^4|\mathbf{p}_0|^2}{3}. \tag{22.28}$$

The radiation increases with the fourth power of the frequency ($\omega = ck$) (Rayleigh radiation). As an example one may consider two capacitor spheres at distance l with $I(t) = \Re(I_0\mathrm{e}^{-\mathrm{i}\omega t})$. Then one has

$$|\mathbf{g}^{(0)}| = |\int \mathrm{d}^3r'\mathbf{j}_0(\mathbf{r}')| = |\int \mathrm{d}lI_0| = |I_0 l|, \quad p_0 = \frac{\mathrm{i}I_0 l}{\omega}, \quad \dot{U}_\mathrm{s} = \frac{(klI_0)^2}{3c} \tag{22.29}$$

This power release yields a radiation resistance R_s

$$\dot{U}_\mathrm{s} = \frac{1}{2}R_\mathrm{s}I_0^2, \quad R_\mathrm{s} = \frac{2}{3c}(kl)^2 \,\hat{=}\, 20\Omega\cdot(kl)^2 \tag{22.30}$$

in addition to an Ohmic resistance. Note that $\frac{1}{c}\,\hat{=}\,30\Omega$, compare (A.4).

22.c Magnetic Dipole radiation and Electric Quadrupole Radiation

Now we consider the second term in (22.16)

$$\begin{aligned} g_\alpha^{(1)} &= n_\beta\int \mathrm{d}^3r' x'_\beta j_{0,\alpha}(\mathbf{r}') \\ &= \frac{n_\beta}{2}\int \mathrm{d}^3r'(x'_\beta j_{0,\alpha} - x'_\alpha j_{0,\beta}) + \frac{n_\beta}{2}\int \mathrm{d}^3r'(x'_\beta j_{0,\alpha} + x'_\alpha j_{0,\beta}). \end{aligned} \tag{22.31}$$

The first term yields the magnetic dipole moment (10.7)

$$n_\beta c\epsilon_{\beta,\alpha,\gamma}m_{0,\gamma} = -c(\mathbf{n}\times\mathbf{m}_0)_\alpha. \tag{22.32}$$

The second term can be expressed by the electric quadrupole moment (4.10). With $f = \frac{1}{2}x'_\alpha x'_\beta$ one obtains from (22.21)

$$-\mathrm{i}\omega\frac{n_\beta}{2}\int \mathrm{d}^3r' x'_\alpha x'_\beta\rho_0(\mathbf{r}') = -\mathrm{i}\omega\frac{n_\beta}{2}\Big(Q_{0,\alpha,\beta} + \frac{1}{3}\delta_{\alpha,\beta}\int \mathrm{d}^3r' r'^2\rho_0(\mathbf{r}')\Big). \tag{22.33}$$

Thus we have

$$\mathbf{g}^{(1)} = -c\mathbf{n}\times\mathbf{m}_0 - \frac{\mathrm{i}\omega}{2}Q_{0,\alpha,\beta}n_\beta\mathbf{e}_\alpha + \text{ const. }\mathbf{n}. \tag{22.34}$$

We observe that the third term proportional $\mathbf{n}$ does neither contribute to $\mathbf{B}_0$ (22.9) nor to $\mathbf{E}_0$ (22.11).

22.c.α Magnetic Dipole Radiation

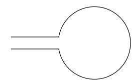
The first contribution in (22.34) yields the magnetic dipole radiation. We obtain

$$\mathbf{A}_0(\mathbf{r}) = \Big(\mathrm{i}k - \frac{1}{r}\Big)\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathbf{n}\times\mathbf{m}_0 \tag{22.35}$$

$$\mathbf{B}_0(\mathbf{r}) = -k^2\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathbf{n}\times(\mathbf{n}\times\mathbf{m}_0) + \Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}_0) - \mathbf{m}_0\Big)\Big(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\Big)\mathrm{e}^{\mathrm{i}kr} \tag{22.36}$$

$$\mathbf{E}_0(\mathbf{r}) = (-\frac{k^2}{r} - \frac{\mathrm{i}k}{r^2}\Big)(\mathbf{n}\times\mathbf{m}_0)\mathrm{e}^{\mathrm{i}kr}. \tag{22.37}$$

As an example we consider a current along a loop which includes the area f,



$$m_0 = I_0 f/c, \quad \dot{U}_\mathrm{s} = \frac{ck^4m_0^2}{3} = \frac{k^4I_0^2f^2}{3c}, \tag{22.38}$$

which corresponds to a radiation resistance

$$R_\mathrm{s} = \frac{2}{3c}k^4f^2 \mathrel{\hat{=}} 20\Omega\,(k^2f)^2. \tag{22.39}$$

22.c.β Electric Quadrupole Radiation

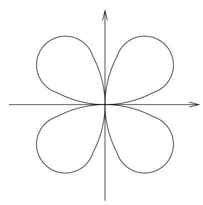
We finally consider the second term in (22.34) in the far zone. It yields

$$\mathbf{g} = -\mathrm{i}k\mathbf{g}^{(1)} = -\frac{k^2c}{2}Q_{0,\alpha,\beta}n_\beta\mathbf{e}_\alpha. \tag{22.40}$$

As special case we investigate the symmetric quadrupole (4.27), $Q_{0,x,x} = Q_{0,y,y} = -\frac{1}{3}Q_0$, $Q_{0,z,z} = \frac{2}{3}Q_0$, whereas the off-diagonal elements vanish. Then one has

$$Q_{0,\alpha,\beta} = -\frac{1}{3}Q_0\delta_{\alpha,\beta} + Q_0\delta_{\alpha,3}\delta_{\beta,3}, \tag{22.41}$$

from which



$$\mathbf{g} = -\frac{k^2c}{2}Q_0n_3\mathbf{e}_3 + \frac{k^2c}{6}Q_0\mathbf{n}, \quad n_3 = \cos\theta \tag{22.42}$$

$$\mathbf{B}_0 = -\mathrm{i}k^3\frac{\mathrm{e}^{\mathrm{i}kr}}{2r}Q_0\mathbf{n}\times\mathbf{e}_3\cos\theta \tag{22.43}$$

$$\mathbf{E}_0 = \mathrm{i}k^3\frac{\mathrm{e}^{\mathrm{i}kr}}{2r}Q_0\mathbf{n}\times(\mathbf{n}\times\mathbf{e}_3)\cos\theta \tag{22.44}$$

$$\bar{\mathbf{S}} = \frac{ck^6\mathbf{n}}{32\pi r^2}|Q_0|^2\sin^2\theta\cos^2\theta \tag{22.45}$$

$$\dot{U}_\mathrm{s} = \frac{ck^6}{60}|Q_0|^2 \tag{22.46}$$

follows. The intensity of the quadrupole-radiation is radially sketched as function of the angle θ.

H Lorentz Invariance of Electrodynamics

23 Lorentz Transformation

23.a Galilei and Lorentz Transformation

The equations of Newton's mechanics are invariant under the Galilei transformation (Galilei invariance)

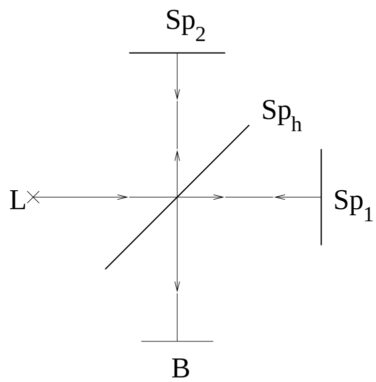
$$x' = x, \quad y' = y, \quad z' = z - vt, \quad t' = t. \tag{23.1}$$

We will see in the following that Maxwell's equations are invariant under appropriate transformations of fields, currents and charges against linear transformations of the coordinates x, y, z, and t, which leave the velocity of light invariant (Lorentz invariance). Such a transformation reads

$$x' = x, \quad y' = y, \quad z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{vz}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{23.2}$$

Consider two charges q und $-q$, which are for $t \le 0$ at the same point and which are also for $t \ge \Delta t$ at the same point, which move however in the time interval $0 < t < \Delta t$ against each other. They separate at time 0 at $\mathbf{r}_0$ and they meet again at time Δt at $\mathbf{r}_1$. They generate according to (21.14) and (21.15) a field, which propagates with light-velocity. It is different from zero at point $\mathbf{r}$ at time t only, if $t > |\mathbf{r} - \mathbf{r}_0|/c$ and $t < \Delta t + |\mathbf{r} - \mathbf{r}_1|/c$ holds. This should hold independently of the system of inertia in which we consider the wave. (We need only assume that the charges do not move faster than with light-velocity.) If we choose an infinitesimal Δt then the light flash arrives at time $t = |\mathbf{r} - \mathbf{r}_0|/c$, since it propagates with light-velocity. Since the Lorentz transformation is not in agreement with the laws of Newton's mechanics and the Galilei transformation not with Maxwell's equations (in a moving inertial frame light would have a velocity dependent on the direction of light-propagation) the question arises which of the three following possibilities is realized in nature:
(i) there is a preferred system of inertia for electrodynamics, in which Maxwell's equations hold (ether-hypothesis),
(ii) Newton's mechanics has to be modified
(iii) Maxwell's equations have to be modified.

The decision can only be made experimentally: An essential experiment to refute (i) is the Michelson-Morley experiment: A light beam hits a half-transparent mirror $\mathrm{Sp_h}$, is split into two beams, which are reflected at mirror Sp_1 and Sp_2, resp. at distance l and combined again at the half-transparent mirror. One observes the interference fringes of both beams at B. If the apparatus moves with velocity v in the direction of the mirror Sp_1, then the time t_1 the light needs to propagate from the half-transparent mirror to Sp_1 and back is



$$t_1 = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2} = \frac{2l}{c}(1 + \frac{v^2}{c^2} + ...). \tag{23.3}$$

The time t_2 the light needs to the mirror Sp_2 is

$$t_2 = \frac{2l}{\sqrt{c^2 - v^2}} = \frac{2l}{c}(1 + \frac{v^2}{2c^2} + ...), \tag{23.4}$$

since the light velocity c has to be separated into the two components v and $\sqrt{c^2 - v^2}$. Thus there remains the time difference

$$t_1 - t_2 = \frac{lv^2}{c^3}, \tag{23.5}$$

which would be measurable by a displacement of the interference fringes, if for example the velocity v is the velocity of the earth against the sun. This displacement has not been observed. One may object that this is due to a drag of the ether by the earth. There are however many other experiments, which are all in agreement with LORENTZ invariance, i.e. the constancy of the velocity of light in vacuum independent of the system of inertia. The consequences in mechanics for particles with velocities comparable to the velocity of light in particular for elementary particles have confirmed LORENTZ invariance very well.

Development of the Theory of Relativity

In order to determine the velocity of the earth against the postulated ether MICHELSON and MORLEY performed their experiment initially in 1887 with the negative result: No motion against the ether was detected. In order to explain this FITZGERALD (1889) and LORENTZ (1892) postulated that all material objects are contracted in their direction of motion against the ether (compare LORENTZ contraction, subsection 23.b.β).

In the following we will develop the idea of a four-dimensional space-time, in which one may perform transformations similar to orthogonal transformations in three-dimensional space, to which we are used. However this space is not a EUCLIDean space, i.e. a space with definite metric. Instead space and time have a metric with different sign (see the metric tensor g, eq. 23.10). This space is also called MINKOWSKI space. We use the modern four-dimensional notation introduced by MINKOWSKI in 1908.

Starting from the basic ideas of special relativity

The laws of nature and the results of experiments in a system of inertia are independent of the motion of such a system as whole.

The velocity of light is the same in each system of inertia and independent of the velocity of the source

we will introduce the LORENTZ-invariant formulation of MAXWELL's equations and of relativistic mechanics.

23.b LORENTZ Transformation

We introduce the notation

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \tag{23.6}$$

or shortly

$$(x^\mu) = (ct, \mathbf{r}) \tag{23.7}$$

and denotes them as the contravariant components of the vector. Further one introduces

$$(x_\mu) = (ct, -\mathbf{r}). \tag{23.8}$$

which are called covariant components of the vector. Then we may write

$$x^\mu = g^{\mu\nu} x_\nu, \quad x_\mu = g_{\mu\nu} x^\nu \tag{23.9}$$

(summation convention)

$$(g^{\cdot\cdot}) = (g_{\cdot\cdot}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{23.10}$$

One calls g the metric tensor. Generally one has the rules for lifting and lowering of indices

$$C\cdot\cdot^\mu\cdot\cdot = g^{\mu\nu} C\cdot\cdot_\nu\cdot\cdot, \quad C\cdot\cdot_\mu\cdot\cdot = g_{\mu\nu} C\cdot\cdot^\nu\cdot\cdot \tag{23.11}$$

We introduce the convention: Indices κ, λ, μ, ν run from 0 to 3, indices α, β, γ, ... from 1 to 3. One observes that according to (23.11) $g_\mu{}^\nu = g_{\mu\kappa} g^{\kappa\nu} = \delta_\mu{}^\nu$, $g^\mu{}_\nu = g^{\mu\kappa} g_{\kappa\nu} = \delta^\mu{}_\nu$ with the KRONECKER delta.

If a light-flash is generated at time $t = 0$ at $\mathbf{r} = \mathbf{0}$, then its wave front is described by

$$s^2 = c^2 t^2 - \mathbf{r}^2 = x^\mu x_\mu = 0. \tag{23.12}$$

We denote the system described by the coordinates x^μ by S. Now we postulate with EINSTEIN: Light in vacuum propagates in each inertial system with the same velocity c. (principle of the constance of light velocity) Then the propagation of the light flash in the uniformly moving system S' whose origin agrees at $t = t' = 0$ with that of S is given by

$$s'^2 = x'^\mu x'_\mu = 0. \tag{23.13}$$

Requiring a homogeneous space-time continuum the transformation between x' and x has to be linear

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{23.14}$$

and $s'^2 = f s^2$ with some constant f has to hold. If we require that space is isotropic and no system of inertia is preferred, then $f = 1$ has to hold. The condition $s'^2 = s^2$ implies

$$s'^2 = x'^\mu x'_\mu = \Lambda^\mu{}_\nu x^\nu \Lambda_\mu{}^\kappa x_\kappa = s^2 = x^\nu \delta_\nu{}^\kappa x_\kappa, \tag{23.15}$$

which is fulfilled for arbitrary x, if

$$\Lambda^\mu{}_\nu \Lambda_\mu{}^\kappa = \delta_\nu{}^\kappa \tag{23.16}$$

holds. The inverse transformation of (23.14) follows from

$$x^\kappa = \delta_\nu{}^\kappa x^\nu = \Lambda_\mu{}^\kappa \Lambda^\mu{}_\nu x^\nu = \Lambda_\mu{}^\kappa x'^\mu. \tag{23.17}$$

From (23.16) one obtains in particular for $\nu = \kappa = 0$ the relation $(\Lambda^{00})^2 - \sum_\alpha (\Lambda^{\alpha 0})^2 = 1$. Note that $\Lambda^\alpha{}_0 = +\Lambda^{\alpha 0}$, $\Lambda_\alpha{}^0 = -\Lambda^{\alpha 0}$. Thus one has $|\Lambda^{00}| > 1$. One distinguishes between transformations with positive and negative Λ^{00}, since there is no continuous transition between these two classes. The condition $\Lambda^{00} > 0$ means that $\Lambda^{00} = \frac{\mathrm{d}t}{\mathrm{d}t'}|_{\mathbf{r}'} > 0$, that is a clock which is at rest in S' changes its time seen from S with the same direction as the clock at rest in S (and not backwards).
Finally we can make a statement on $\det(\Lambda^\mu{}_\nu)$. From (23.16) it follows that

$$\Lambda^\mu{}_\nu g_{\mu\lambda} \Lambda^\lambda{}_\rho g^{\rho\kappa} = \delta_\nu{}^\kappa. \tag{23.18}$$

Using the theorem on the multiplication of determinants we obtain

$$\det(\Lambda^\mu{}_\nu)^2 \det(g_{\mu\lambda}) \det(g^{\rho\kappa}) = 1. \tag{23.19}$$

Since $\det(g_{\mu\lambda}) = \det(g^{\rho\kappa}) = -1$ one obtains

$$\det(\Lambda^\mu{}_\nu) = \pm 1. \tag{23.20}$$

If we consider only a right-basis-system then we have $\det(\Lambda^\mu{}_\nu) = +1$. Transformations which fulfill

$$\Lambda^{00} > 0, \quad \det(\Lambda^\mu{}_\nu) = 1 \tag{23.21}$$

are called proper LORENTZ transformations.
Eq. (23.21) has the consequence that the fourdimensional space time volume is invariant

$$\mathrm{d}t'\mathrm{d}^3r' = \frac{1}{c}\mathrm{d}^4x' = \frac{1}{c}\frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)}\mathrm{d}^4x = \frac{1}{c}\det(\Lambda^\mu{}_\nu)\mathrm{d}^4x = \frac{1}{c}\mathrm{d}^4x = \mathrm{d}t\mathrm{d}^3r. \tag{23.22}$$

If the direction of the z- and the z'-axes point into the direction of the relative velocity between both inertial systems and $x' = x$, $y' = y$, then the special transformation (23.2) follows. The corresponding matrix Λ reads

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{23.23}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \beta = \frac{v}{c}. \tag{23.24}$$

23.b.α Time Dilatation

We consider now a clock in the system S which is at rest in S'. From

$$t = \gamma(t' + \frac{vz'}{c^2}) \tag{23.25}$$

we find that

$$\Lambda_0{}^0 = \left.\frac{\partial t}{\partial t'}\right|_{\mathbf{r}'} = \gamma. \tag{23.26}$$

Thus the clock at rest in S' runs slower when seen from S

$$\Delta t' = \left.\frac{\partial t'}{\partial t}\right|_{\mathbf{r}'} \Delta t = \frac{1}{\gamma}\Delta t = \sqrt{1 - \frac{v^2}{c^2}}\Delta t. \tag{23.27}$$

This phenomenon is called time dilatation.

23.b.β LORENTZ Contraction

From

$$z' = \gamma(z - vt) \tag{23.28}$$

one obtains

$$\Lambda^3{}_3 = \left.\frac{\partial z'}{\partial z}\right|_t = \gamma \tag{23.29}$$

and therefore

$$\Delta z = \left.\frac{\partial z}{\partial z'}\right|_t \Delta z' = \frac{1}{\gamma}\Delta z' = \sqrt{1 - \frac{v^2}{c^2}}\Delta z'. \tag{23.30}$$

A length-meter which is at rest in S' and is extended in the direction of the relative movement, appears consequently contracted in S. This is called LORENTZ contraction or FITZGERALD-LORENTZ contraction. However, the distances perpendicular to the velocity are unaltered: $\Delta x' = \Delta x$, $\Delta y' = \Delta y$.

This contraction has the effect that in (23.3) the length l has to be replaced by $l\sqrt{1 - \frac{v^2}{c^2}}$. Then the two times the light has to travel agree independent of the velocity v, $t_1 = t_2$.

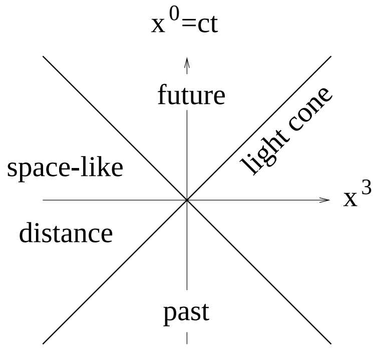
24 Four-Scalars and Four-Vectors

24.a Distance and Proper Time as Four-Scalars

A quantity which is invariant under Lorentz transformations is called four-scalar.
Example: Given two points in space-time (events) (x^μ), $(\bar{x}^\mu)$. The quantity

$$s^2 = (x^\mu - \bar{x}^\mu)(x_\mu - \bar{x}_\mu) \tag{24.1}$$

is a four-scalar. It assumes the same number in all systems of inertia. Especially for $\bar{x}^\mu = 0$ (origin) it is $s^2 = x^\mu x_\mu$.



24.a.α Space-like distance $s^2 < 0$

If $s^2 < 0$, then there are systems of inertia, in which both events occur at the same time $x'^0 = 0$. If for example $(x^\mu) = (ct, 0, 0, z)$. Then one obtains from 23.2)

$$t' = \frac{t - \frac{vz}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{24.2}$$

with $v = tc^2/z$

$$t' = 0, \quad z' = \frac{z(1 - \frac{v^2}{c^2})}{\sqrt{1 - \frac{v^2}{c^2}}} = z\sqrt{1 - \frac{v^2}{c^2}} = \pm\sqrt{z^2 - c^2t^2} = \pm\sqrt{-s^2}. \tag{24.3}$$

Thus one calls such two events space-like separated.

24.a.β Time-like distance $s^2 > 0$

In this case there exists a system of inertia in which both events take place at the same point in space ($\mathbf{x}' = \mathbf{0}$). We choose $v = z/t$ in the transformation (23.2) and obtain

$$t' = \frac{t(1 - \frac{v^2}{c^2})}{\sqrt{1 - \frac{v^2}{c^2}}} = t\sqrt{1 - \frac{v^2}{c^2}} = \operatorname{sign}(t)\sqrt{t^2 - \frac{z^2}{c^2}} = \operatorname{sign}(t)\frac{s}{c}, \quad z' = 0. \tag{24.4}$$

One event takes place before the other that is the sign of t' agrees with that of t.
Proper Time τ
The proper time τ is the time which passes in the rest system under consideration. If a point moves with velocity $\mathbf{v}(t)$ its proper time varies as

$$\mathrm{d}\tau = \frac{\mathrm{d}s}{c} = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\mathrm{d}t, \tag{24.5}$$

that is

$$\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{\mathbf{v}^2(t)}{c^2}}\mathrm{d}t. \tag{24.6}$$

The proper time is independent of the system of inertia, thus it is a four-scalar.

24.a.γ Light-like distance $s^2 = 0$

If a light flash propagates directly from one event to another, then the distance of these two events $s = 0$. The time measured in a system of inertia depends on the system of inertia and may be arbitrarily long or short, however, the sequence of the events (under proper Lorentz transformation) cannot be reversed.
Another four-scalar is the charge.

24.b World Velocity as Four-Vector

If a four-component quantity (A^μ) transforms by the transition from one system of inertia to another as the space-time coordinates (x^μ), then it is a four-vector

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \tag{24.7}$$

An example is the world velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma v^\mu \text{ with } v^0 = \frac{dx^0}{dt} = c\frac{dt}{dt} = c. \tag{24.8}$$

The world velocity $(u^\mu) = (c\gamma, \mathbf{v}\gamma)$ is a four-vector. Since τ is invariant under LORENTZ transformations, its components transform like (x^μ). However, $(c, \mathbf{v})$ is not a four-vector. One has

$$u^\mu u_\mu = (c^2 - \mathbf{v}^2)\gamma^2 = c^2. \tag{24.9}$$

Quite generally the scalar product of two four-vectors (A^μ) and (B^μ) is a four-scalar

$$A'^\mu B'_\mu = \Lambda^\mu{}_\nu \Lambda_\mu{}^\kappa A^\nu B_\kappa = \delta^\kappa_\nu A^\nu B_\kappa = A^\nu B_\nu. \tag{24.10}$$

We show the following lemma: If (a^μ) is an arbitrary four-vector (or one has a complete set of four-vectors) and $a^\mu b_\mu$ is a four-scalar then (b^μ) is a four-vector too. Proof:

$$a^\mu b_\mu = a'^\kappa b'_\kappa = \Lambda^\kappa{}_\mu a^\mu b'_\kappa. \tag{24.11}$$

Since this holds for all (a^μ) or for a complete set, one has $b_\mu = \Lambda^\kappa{}_\mu b'_\kappa$. This, however, is the transformation formula (23.17) for four-vectors.

Addition theorem for velocities

The system of inertia S' moves with velocity v in z-direction with respect to S. A point in S' moves with velocity w' also in z-direction. With which velocity does the point move in S? We have

$$z = \frac{z' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t = \frac{t' + \frac{vz'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{24.12}$$

With $z' = w't'$ one obtains

$$z = \frac{(v + w')t'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t = \frac{(1 + \frac{vw'}{c^2})t'}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{24.13}$$

From this one obtains the velocity of the point in S

$$w = \frac{z}{t} = \frac{w' + v}{1 + \frac{w'v}{c^2}}. \tag{24.14}$$

We observe

$$1 - \frac{w^2}{c^2} = 1 - \left(\frac{\frac{w'}{c} + \frac{v}{c}}{1 + \frac{w'v}{c^2}}\right)^2 = \frac{(1 - \frac{w'^2}{c^2})(1 - \frac{v^2}{c^2})}{(1 + \frac{w'v}{c^2})^2}. \tag{24.15}$$

If $|w'| < c$ and $|v| < c$, then this expression is positive. Then one obtains also $|w| < c$. Example: $w' = v = 0.5c$, then one obtains $w = 0.8c$.

24.c Current Density Four-Vector

We combine charge- and current-density in the charge-current density

$$(j^\mu) = (c\rho, \mathbf{j}) \tag{24.16}$$

and convince us that j^μ is a four-vector. For charges of velocity $\mathbf{v}$ one has (the contributions of charges of different velocities can be superimposed)

$$j^\mu = \rho v^\mu, \quad (v^0 = c), \quad j^\mu = \rho\sqrt{1-\beta^2}u^\mu \tag{24.17}$$

If $\rho\sqrt{1-\beta^2}$ is a four-scalar then indeed j^μ is a four-vector. Now one has

$$\rho = \frac{q}{V} = \frac{q}{V_0\sqrt{1-\beta^2}} \tag{24.18}$$

with the volume V_0 in the rest system and the LORENTZ contraction $V = V_0\sqrt{1-\beta^2}$. Since the charge q and V_0 are four-scalars this holds also $\rho\sqrt{1-\beta^2}$.
We bring the equation of continuity in LORENTZ-invariant form. From $\dot\rho + \operatorname{div}\mathbf{j} = 0$ one obtains

$$\frac{\partial j^\mu}{\partial x^\mu} = 0, \tag{24.19}$$

since $\partial j^0/\partial x^0 = \partial\rho/\partial t$. We consider now the transformation properties of the derivatives $\partial/\partial x^\mu$

$$\frac{\partial f}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial f}{\partial x^\nu} = \Lambda_\mu{}^\nu\frac{\partial f}{\partial x^\nu}, \tag{24.20}$$

that is the derivatives transform according to

$$\frac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu\frac{\partial}{\partial x^\nu} \tag{24.21}$$

as $x'_\mu = \Lambda_\mu{}^\nu x_\nu$. Thus one writes

$$\frac{\partial}{\partial x^\mu} = \partial_\mu, \quad (\partial_\mu) = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla). \tag{24.22}$$

Watch the positions of the indices. Similarly one has

$$\frac{\partial}{\partial x_\mu} = \partial^\mu, \quad (\partial^\mu) = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla). \tag{24.23}$$

Then the equation of continuity can be written

$$\partial_\mu j^\mu = 0. \tag{24.24}$$

Generally the four-divergency $\partial_\mu P^\mu = \partial^\mu P_\mu$ of a four-vector P is a four-scalar.

24.d Four-Potential

We combine the potentials $\mathbf{A}$ and Φ in the four-potential

$$(A^\mu) = (\Phi, \mathbf{A}), \tag{24.25}$$

then one has

$$\Box A^\mu = -\frac{4\pi}{c}j^\mu \tag{24.26}$$

in the LORENZ gauge with the gauge condition

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\dot\Phi = 0 \to \partial_\mu A^\mu = 0. \tag{24.27}$$

There the D'ALEMBERT operator

$$\Box = \triangle - \frac{1}{c^2}\partial_t^2 = -\partial_\mu\partial^\mu \tag{24.28}$$

is a four-scalar $\Box' = \Box$.

We now show that the retarded solution A^μ_r is manifestly LORENTZ-invariant. We claim

$$A^\mu_\mathrm{r}(x) = \frac{1}{c}\int d^4y\, j^\mu(y)\delta(\frac{1}{2}s^2)\theta(x^0 - y^0) \tag{24.29}$$

$$s^2 = (x^\mu - y^\mu)(x_\mu - y_\mu) = c^2(t_y - t_x)^2 - (\mathbf{x} - \mathbf{y})^2 \tag{24.30}$$

$$\theta(x^0) = \begin{cases} 1 & x^0 > 0 \\ 0 & x^0 < 0 \end{cases} \tag{24.31}$$

We consider now generally the integration over a δ-function, which depends on a function f. Apparently only the zeroes t_i of f contribute,

$$\int g(t)\delta(f(t))dt = \sum_i \int_{t_i-\epsilon}^{t_i+\epsilon} g(t)\delta(f(t))dt \quad \text{with} \quad f(t_i) = 0. \tag{24.32}$$

With $z = f(t)$, $dz = f'(t)dt$ one obtains

$$\int g(t)\delta(f(t))dt = \sum_i \int_{-\epsilon f'(t_i)}^{\epsilon f'(t_i)} g(t_i)\delta(z)\frac{dz}{f'(t_i)} = \sum_i \frac{g(t_i)}{|f'(t_i)|}. \tag{24.33}$$

Thus the zeroes in the δ-function of (24.29) are $t_y = t_x \pm |\mathbf{x} - \mathbf{y}|/c$ and their derivatives are given by $f'(t_y) = c^2(t_y - t_x) = \pm c|\mathbf{x} - \mathbf{y}|$, which yields

$$A^\mu_r(x) = \frac{1}{c}\int d^4y\, j^\mu\delta(\frac{1}{2}s^2)\theta(t_x - t_y) = \int d^3y \frac{1}{c|\mathbf{x} - \mathbf{y}|} j^\mu(\mathbf{y}, t_x - \frac{|\mathbf{x} - \mathbf{y}|}{c}). \tag{24.34}$$

The factor $\theta(t_x - t_y)$ yields the retarded solution. The solution is in agreement with (21.14) and (21.15). If we replace the θ-function by $\theta(t_y - t_x)$, then we obtain the advanced solution. Remember that the sign of the time difference does not change under proper LORENTZ transformations.

25 Electromagnetic Field Tensor

25.a Field Tensor

We obtain the fields **E** and **B** from the four-potential

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \quad \mathbf{E} = -\operatorname{grad}\Phi - \frac{1}{c}\dot{\mathbf{A}}, \tag{25.1}$$

for example

$$B_1 = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \partial^3 A^2 - \partial^2 A^3, \quad E_1 = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = \partial^1 A^0 - \partial^0 A^1. \tag{25.2}$$

Thus we introduce the electromagnetic field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad F^{\mu\nu} = -F^{\nu\mu}. \tag{25.3}$$

It is an antisymmetric four-tensor. It reads explicitely

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{25.4}$$

25.b Maxwell's Equations

25.b.α The Inhomogeneous Equations

The equation $\operatorname{div}\mathbf{E} = 4\pi\rho$ reads

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{4\pi}{c} j^0. \tag{25.5}$$

From the 1-component of $\operatorname{curl}\mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{j}$ one obtains

$$\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3} - \frac{\partial E_1}{\partial x^0} = \frac{4\pi}{c} j^1 \rightarrow \partial_2 F^{21} + \partial_3 F^{31} + \partial_0 F^{01} = \frac{4\pi}{c} j^1, \tag{25.6}$$

similarly for the other components. These four component-equations can be combined to

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu. \tag{25.7}$$

If we insert the representation of the fields by the potentials, (25.3), we obtain

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{4\pi}{c} j^\nu. \tag{25.8}$$

Together with the condition for the Lorenz gauge $\partial_\mu A^\mu = 0$, (24.27) one obtains

$$\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} j^\nu \tag{25.9}$$

in agreement with (24.26) and (24.28).

25.b.β The Homogeneous Equations

Similarly the homogeneous MAXWELL's equations can be written. From div $\mathbf{B} = 0$ one obtains

$$\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0 \tag{25.10}$$

and (curl $\mathbf{E} + \frac{1}{c}\dot{\mathbf{B}})_x = 0$ reads

$$-\partial^2 F^{30} - \partial^3 F^{02} - \partial^0 F^{23} = 0. \tag{25.11}$$

These equations can be combined to

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \tag{25.12}$$

One observes that these equations are only non-trivial for $\lambda \neq \mu \neq \nu \neq \lambda$. If two indices are equal, then the left-hand side vanishes identically. One may represent the equations equally well by the dual field tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}. \tag{25.13}$$

Here $\epsilon^{\kappa\lambda\mu\nu}$ is completely antisymmetric against interchange of the four indices. Thus it changes sign, if two of the indices are exchanged. This implies that it vanishes, if two indices are equal. It is only different from zero, if all four indices are different. It is normalized to $\epsilon^{0123} = 1$. Then one obtains explicitely

$$(\tilde{F}^{\mu\nu}) = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}. \tag{25.14}$$

and (25.12) can be written

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{25.15}$$

One should convince oneself that ϵ is an invariant pseudo-tensor of fourth order, i.e.

$$\epsilon'^{\mu\nu\kappa\lambda} = \det(\Lambda)\epsilon^{\mu\nu\kappa\lambda}, \tag{25.16}$$

where $\det(\Lambda)$ takes only the values ± 1 according to the discussion after (23.19). For proper LORENTZ transformations it equals +1 (23.21).

25.c Transformation of the Electric and Magnetic Fields

Since (∂^μ) and (A^ν) transform like four-vectors, one has

$$F'^{\mu\nu} = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda F^{\kappa\lambda} \tag{25.17}$$

for the transformation of the electromagnetic field. If we choose in particular

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{25.18}$$

then one obtains

$$E'_1 = F'^{10} = \Lambda^1{}_\kappa \Lambda^0{}_\lambda F^{\kappa\lambda} = \gamma F^{10} - \beta\gamma F^{13} = \gamma(E_1 - \beta B_2), \tag{25.19}$$

thus

$$E'_1 = \gamma(E_1 - \frac{v}{c}B_2), \tag{25.20}$$

similarly

$$B'_1 = \gamma(B_1 + \frac{v}{c}E_2) \tag{25.21}$$

$$E'_2 = \gamma(E_2 + \frac{v}{c}B_1), \qquad B'_2 = \gamma(B_2 - \frac{v}{c}E_1) \tag{25.22}$$

$$E'_3 = E_3, \qquad B'_3 = B_3, \tag{25.23}$$

which can be combined to

$$E'_{\|} = E_{\|}, \qquad B'_{\|} = B_{\|}, \qquad \text{component } \| \mathbf{v} \tag{25.24}$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \frac{\mathbf{v}}{c} \times \mathbf{B}), \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c} \times \mathbf{E}), \qquad \text{components } \perp \mathbf{v}. \tag{25.25}$$

25.d Fields of a Point Charge in Uniform Motion

From this we can determine the fields of a charge which moves with constant velocity $\mathbf{v} = v\mathbf{e}_z$. In the rest system S' of the charge, which is supposed to be in the origin of S', one has

$$\mathbf{E}' = q\frac{\mathbf{r}'}{r'^3}, \quad \mathbf{B}' = \mathbf{0}. \tag{25.26}$$

In the system S the coordinates of the charge are $x_q = y_q = 0$, $z_q = vt$. Now we express $\mathbf{r}'$ by $\mathbf{r}$ and t and obtain

$$\mathbf{E}' = \left(\frac{qx}{N}, \frac{qy}{N}, \frac{q\gamma(z - vt)}{N}\right), \tag{25.27}$$

$$\mathbf{B}' = \mathbf{0}, \tag{25.28}$$

$$N = r'^3 = (x^2 + y^2 + \gamma^2(z - vt)^2)^{3/2}. \tag{25.29}$$

It follows that

$$\left.\begin{array}{l} E_1 = \gamma(E'_1 + \frac{v}{c}B'_2) = \frac{q\gamma x}{N} \\ E_2 = \gamma(E'_2 - \frac{v}{c}B'_1) = \frac{q\gamma y}{N} \\ E_3 = E'_3 = \frac{q\gamma(z-vt)}{N} \end{array}\right\} \mathbf{E} = \frac{q\gamma(\mathbf{r} - \mathbf{v}t)}{N} \tag{25.30}$$

$$\left.\begin{array}{l} B_1 = \gamma(B'_1 - \frac{v}{c}E'_2) = -\frac{q\gamma\beta y}{N} \\ B_2 = \gamma(B'_2 + \frac{v}{c}E'_1) = \frac{q\gamma\beta x}{N} \\ B_3 = B'_3 = 0 \end{array}\right\} \mathbf{B} = \frac{q\gamma(\mathbf{v} \times \mathbf{r})}{cN}. \tag{25.31}$$

Areas of constant N are oblate rotational ellipsoids. The ratio short half-axis / long half-axis is given by $1/\gamma = \sqrt{1 - \frac{v^2}{c^2}}$, thus the same contraction as for the Lorentz contraction.

25.e Doppler Effect

We consider a monochromatic plane wave

$$\mathbf{E} = \mathbf{E}_0 e^{i\phi}, \quad \mathbf{B} = \mathbf{B}_0 e^{i\phi} \quad \text{with } \phi = \mathbf{k} \cdot \mathbf{r} - \omega t. \tag{25.32}$$

We know, how $\mathbf{E}$ and $\mathbf{B}$ and thus $\mathbf{E}_0$ and $\mathbf{B}_0$ transform. Thus we are left with considering the phase ϕ which is a four-scalar. If we write

$$(k^\mu) = (\frac{\omega}{c}, \mathbf{k}), \tag{25.33}$$

then

$$\phi = -k_\mu x^\mu \tag{25.34}$$

follows. Since (x^μ) is an arbitrary four-vector and ϕ is a four-scalar, it follows that (k^μ) has to be a four-vector. Thus one obtains for the special Lorentz transformation (25.18)

$$\omega' = ck'^0 = c\gamma(k^0 - \beta k^3) = \gamma(\omega - \beta ck^3), \quad k'^1 = k^1, \quad k'^2 = k^2, \quad k'^3 = \gamma(k^3 - \beta\frac{\omega}{c}). \tag{25.35}$$

If the angle between z-axis and direction of propagation is θ, then $k^3 = \frac{\omega}{c}\cos\theta$ holds, and one obtains

$$\omega' = \omega\gamma(1 - \beta\cos\theta). \tag{25.36}$$

Thus if **v** is parallel and antiparallel to the direction of propagation, resp., then one deals with the longitudinal DOPPLER shift

$$\theta = 0 : \quad \omega' = \omega \sqrt{\tfrac{1-\beta}{1+\beta}} \tag{25.37}$$

$$\theta = \pi : \quad \omega' = \omega \sqrt{\tfrac{1+\beta}{1-\beta}}. \tag{25.38}$$

If however $\theta = \pi/2$ and $\theta' = \pi/2$, resp., then one deals with the transverse DOPPLER shift.

$$\theta = \frac{\pi}{2} : \quad \omega' = \frac{\omega}{\sqrt{1-\beta^2}} \tag{25.39}$$

$$\theta' = \frac{\pi}{2} : \quad \omega' = \omega \sqrt{1-\beta^2}. \tag{25.40}$$

Here θ' is the angle between the z'-axis and the direction of propagation in S'.

26 Relativistic Mechanics

Einstein realized that the constance of light velocity in vacuum and the resulting Lorentz transformation is not restricted to electrodynamics, but is generally valid in physics. Here we consider its application to mechanics starting from the force on charges.

26.a Lorentz Force Density

The force density on moving charges reads

$$\mathbf{k} = \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B}, \tag{26.1}$$

that is e.g. for the first component

$$k^1 = \rho E_1 + \frac{1}{c}(j^2 B_3 - j^3 B_2) = \frac{1}{c}(j^0 F^{10} - j^2 F^{12} - j^3 F^{13}) = \frac{1}{c} j_\nu F^{1\nu}. \tag{26.2}$$

Thus one introduces the four-vector of the Lorentz force density

$$k^\mu = \frac{1}{c} j_\nu F^{\mu\nu}. \tag{26.3}$$

We consider the time-like component

$$k^0 = \frac{1}{c} j_\nu F^{0\nu} = \frac{1}{c} \mathbf{j} \cdot \mathbf{E}. \tag{26.4}$$

The time-like component gives the mechanical energy acquired per time and volume, whereas the space-like components give the rate of change of mechanic momentum per time and volume

$$(k^\mu) = (\frac{1}{c} \mathbf{j} \cdot \mathbf{E}, \mathbf{k}). \tag{26.5}$$

26.b Lorentz Force Acting on a Point Charge

The four-current-density at $\mathbf{x}$ of a point charge q at $\mathbf{x}_\mathrm{q}$ reads

$$j^\nu(\mathbf{x}, t) = q \delta^3(\mathbf{x} - \mathbf{x}_\mathrm{q}(t)) v^\nu. \tag{26.6}$$

Thus the force acting on the point charge is given by

$$K^\mu = \frac{q}{c} v_\nu F^{\mu\nu}. \tag{26.7}$$

This is not a four-vector, since (v^μ) is not a four-vector. If we multiply it by γ then we obtain a four-vector, the Minkowski force

$$\gamma K^\mu = \frac{q}{c} u_\nu F^{\mu\nu}. \tag{26.8}$$

$\mathbf{K}$ is the momentum which is fed into the point charge per time unit, cK^0 is the power fed into it. The Minkowski force is the momentum and the energy divided by c, resp., fed into it per proper time.

26.c Energy and Momentum of a Mass Point

We assume that also mechanical momentum and energy/c combine to a four-vector, since the change of momentum and energy divided by c are components of a four-vector

$$(G^\mu) = (\frac{1}{c} E, \mathbf{G}). \tag{26.9}$$

In the rest system S' we expect $\mathbf{G}' = \mathbf{0}$ to hold, i.e.

$$(G'^{\mu}) = (\frac{1}{c}E_0, \mathbf{0}). \tag{26.10}$$

In the system S the special transformation (23.23) yields for $\mathbf{v} = v\mathbf{e}_z$

$$\mathbf{G} = \gamma \tfrac{v}{c^2} E_0 \mathbf{e}_z = \gamma \mathbf{v} \frac{E_0}{c^2}, \tag{26.11}$$

$$E = cG^0 = c\gamma G'^0 = \gamma E_0. \tag{26.12}$$

For velocities small in comparison to light-velocity one obtains

$$\mathbf{G} = \frac{E_0}{c^2}\mathbf{v}\Big(1 + \frac{v^2}{2c^2} + ...\Big). \tag{26.13}$$

In NEWTON's mechanics we have

$$\mathbf{G}_{\text{Newton}} = m\mathbf{v} \tag{26.14}$$

for a mass point of mass m. For velocities $v \ll c$ the momentum of NEWTON's and of the relativistic mechanics should agree. From this one obtains

$$m = \frac{E_0}{c^2} \rightarrow E_0 = mc^2, \quad \mathbf{G} = m\gamma\mathbf{v}. \tag{26.15}$$

Then one obtains for the energy E

$$E = mc^2\gamma = mc^2 + \frac{m}{2}v^2 + O(v^4/c^2). \tag{26.16}$$

One associates a rest energy $E_0 = mc^2$ with the masses. At small velocities the contribution $\frac{m}{2}v^2$ known from NEWTONian mechanics has to be added

$$G^{\mu} = mu^{\mu}. \tag{26.17}$$

This G is called four-momentum. We finally observe

$$G^{\mu}G_{\mu} = m^2 u^{\mu} u_{\mu} = m^2 c^2, \tag{26.18}$$

from which one obtains

$$-\mathbf{G}^2 + \frac{1}{c^2}E^2 = m^2c^2, \quad E^2 = m^2c^4 + \mathbf{G}^2c^2. \tag{26.19}$$

One does not observe the rest energy $E_0 = mc^2$ as long as the particles are conserved. However they are observed when the particles are converted, for example, when a particle decays into two other ones

$$\Lambda^0 \rightarrow \pi^- + p^+. \tag{26.20}$$

With the masses

$$m_\Lambda = 2182 m_e, \quad m_\pi = 273 m_e, \quad m_p = 1836 m_e \tag{26.21}$$

one obtains the momentum and energy bilance for the Λ which is at rest before the decay

$$m_\Lambda c^2 = \sqrt{m_\pi^2 c^4 + \mathbf{G}_\pi^2 c^2} + \sqrt{m_p^2 c^4 + \mathbf{G}_p^2 c^2} \tag{26.22}$$

$$\mathbf{0} = \mathbf{G}_\pi + \mathbf{G}_p. \tag{26.23}$$

The solution of the system of equations yields

$$|\mathbf{G}| = 4c\sqrt{M(m_\Lambda - M)(M - m_\pi)(M - m_p)}/m_\Lambda, \quad 2M = m_\Lambda + m_\pi + m_p. \tag{26.24}$$

By means of the four-vectors one may solve

$$G_\Lambda{}^{\mu} = G_\pi{}^{\mu} + G_p{}^{\mu} \tag{26.25}$$

with respect to G_p and take the square

$$G_p{}^{\mu} G_{p\mu} = (G^{\mu}_{\Lambda} - G^{\mu}_{\pi})(G_{\Lambda\mu} - G_{\pi\mu}) = G^{\mu}_{\Lambda} G_{\Lambda\mu} + G^{\mu}_{\pi} G_{\pi\mu} - 2G^{\mu}_{\Lambda} G_{\pi\mu}. \tag{26.26}$$

This yields

$$m_p^2 c^2 = m_\Lambda^2 c^2 + m_\pi^2 c^2 - 2m_\Lambda E_\pi \tag{26.27}$$

and therefore

$$E_\pi = \frac{c^2}{2m_\Lambda}\left(m_\Lambda^2 + m_\pi^2 - m_p^2\right) \tag{26.28}$$

and analogously

$$E_p = \frac{c^2}{2m_\Lambda}\left(m_\Lambda^2 - m_\pi^2 + m_p^2\right). \tag{26.29}$$

26.d Equation of Motion

Finally we write down explicitely the equations of motion for point masses

$$\frac{\mathrm{d}G^\mu}{\mathrm{d}t} = K^\mu. \tag{26.30}$$

As mentioned before these equations are not manifestly Lorentz-invariant. We have, however,

$$\frac{\mathrm{d}G^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}G^\mu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}G^\mu}{\mathrm{d}t} = \gamma K^\mu, \tag{26.31}$$

where the right-hand side is the Minkowski force. In this form the equations of motion are manifestly Lorentz invariant.
If the force does not change the rest energy of a particle, one obtains from

$$G^\mu G_\mu = m^2c^2 \to \frac{\mathrm{d}}{\mathrm{d}\tau}(G^\mu G_\mu) = 0 \to G^\mu \gamma K_\mu = 0 \to u^\mu K_\mu = 0. \tag{26.32}$$

The force is orthogonal on the world velocity. An example is the Lorentz force

$$u_\mu K^\mu = \frac{q}{c}\gamma v_\mu v_\nu F^{\mu\nu} = 0, \tag{26.33}$$

since $F^{\mu\nu}$ is antisymmetric. We observe

$$v^\mu K_\mu = -\mathbf{v}\cdot\mathbf{K} + \frac{c}{c}\frac{\mathrm{d}E}{\mathrm{d}t} = 0. \tag{26.34}$$

Thus equation (26.32) is equivalent to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{v}\cdot\mathbf{K}, \tag{26.35}$$

which yields the power fed into the kinetic energy of the mass.

27 Lagrangian Formulation

27.a Lagrangian of a Massive Charge in the Electromagnetic Field

We claim that the Lagrangian $\mathcal{L}$ of a point charge q of mass m in an electromagnetic field can be written

$$\begin{aligned}\mathcal{L} &= -mc^2\sqrt{1-\frac{\dot{\mathbf{r}}^2}{c^2}} - q\Phi(\mathbf{r},t) + \frac{q}{c}\mathbf{A}(\mathbf{r},t)\cdot\dot{\mathbf{r}} \\ &= -mc^2\sqrt{1+\frac{\dot{x}^\alpha\dot{x}_\alpha}{c^2}} - \frac{q}{c}A^\mu(x)\dot{x}_\mu. \end{aligned} \tag{27.1}$$

Then the action I can be written

$$I = \int \mathrm{d}t\mathcal{L} = -mc^2\int \mathrm{d}\tau - \frac{q}{c}\int \mathrm{d}tA^\mu\frac{\mathrm{d}x_\mu}{\mathrm{d}t} = \int \mathrm{d}\tau(-mc^2 - \frac{q}{c}A^\mu u_\mu), \tag{27.2}$$

that is as a four-scalar.
Now we show that this Lagrangian yields the correct equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}_\alpha} - \frac{\partial\mathcal{L}}{\partial x_\alpha} = 0, \tag{27.3}$$

from which by use of

$$-\frac{\partial\mathcal{L}}{\partial\dot{x}_\alpha} = \frac{m\dot{x}^\alpha}{\sqrt{1-\frac{\dot{\mathbf{r}}^2}{c^2}}} + \frac{q}{c}A^\alpha(\mathbf{r}(t),t) = G^\alpha + \frac{q}{c}A^\alpha \tag{27.4}$$

one finally obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} + \frac{q}{c}\dot{\mathbf{A}} + \frac{q}{c}(\mathbf{v}\cdot\nabla)\mathbf{A} + q\nabla\Phi - \frac{q}{c}\nabla(\mathbf{v}\cdot\mathbf{A}) = 0. \tag{27.5}$$

Note that $\dot{\mathbf{A}}$ contains only the partial time-derivative of $\mathbf{A}$, thus we have $\mathrm{d}\mathbf{A}/\mathrm{d}t = \dot{\mathbf{A}} + (\mathbf{v}\cdot\nabla)\mathbf{A}$. By suitable combination of the contributions one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} + q(\nabla\Phi + \frac{1}{c}\dot{\mathbf{A}}) - \frac{q}{c}\mathbf{v}\times(\nabla\times\mathbf{A}) = 0 \tag{27.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} - q\mathbf{E} - \frac{q}{c}\mathbf{v}\times\mathbf{B} = 0. \tag{27.7}$$

Thus the Lagrangian given above yields the correct equation of motion.

27.b Lagrangian Density of the Electromagnetic Field

The Lagrangian density L of the electromagnetic field of a system of charges consists of three contributions

$$L = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}A^\mu j_\mu + L_{\mathrm{mech}}. \tag{27.8}$$

The mechanical part for point charges of mass m_i reads

$$L_{\mathrm{mech}} = -\sum_i m_i c^3\int d\tau\delta^4(x - x_i(\tau)), \tag{27.9}$$

which yields after integration over d^4x the corresponding contribution to the action I given in (27.1). The second contribution in (27.8) describes the interaction between field and charge. Integration of this contribution for point charges using

$$j_\mu(\mathbf{r},t) = \sum_i q_i\frac{dx_{i,\mu}}{dt}\delta^3(\mathbf{r}-\mathbf{r}_i) \tag{27.10}$$

yields the corresponding contribution in (27.1). The first contribution is that of the free field. Below we will see that it yields MAXWELL's equations correctly. The action itself reads

$$I = \frac{1}{c}\int d^4x L(x) = \int dt \int d^3x L(\mathbf{x},t) = \int dt \mathcal{L}(t), \quad \mathcal{L}(t) = \int d^3x L(\mathbf{x},t). \tag{27.11}$$

The action has to be extremal if the fields A are varied. There we have to consider F as function of A (25.3), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then the variation with respect to A yields

$$\delta L = -\frac{1}{8\pi} F_{\mu\nu}\delta F^{\mu\nu} - \frac{1}{c} j_\nu \delta A^\nu \tag{27.12}$$

$$\delta F^{\mu\nu} = \delta(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu \tag{27.13}$$

$$F_{\mu\nu}\delta F^{\mu\nu} = F_{\mu\nu}\partial^\mu \delta A^\nu - F_{\mu\nu}\partial^\nu \delta A^\mu = 2F_{\mu\nu}\partial^\mu \delta A^\nu \tag{27.14}$$

$$\delta L = -\frac{1}{4\pi} F_{\mu\nu}\partial^\mu \delta A^\nu - \frac{1}{c} j_\nu \delta A^\nu. \tag{27.15}$$

Thus the variation of the action with respect to A is

$$\begin{aligned} \delta I &= \int d^4x \Big(-\frac{1}{4\pi c} F_{\mu\nu}\partial^\mu \delta A^\nu - \frac{1}{c^2} j_\nu \delta A^\nu \Big) \\ &= -\int d^4x \frac{1}{4\pi c}\partial^\mu (F_{\mu\nu}\delta A^\nu) + \int d^4x \Big(\frac{1}{4\pi c}\partial^\mu F_{\mu\nu} - \frac{1}{c^2} j_\nu \Big)\delta A^\nu. \end{aligned} \tag{27.16}$$

The first term of the second line is a surface-term (in four dimensions). From the second term one concludes MAXWELL's inhomogeneous equations (25.7)

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu. \tag{27.17}$$

MAXWELL's homogeneous equations are already fulfilled due to the representation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$. Generally one obtains for a Lagrangian density, which depends on a field (A^μ) and its derivatives by variation

$$\begin{aligned} c\delta I &= \int d^4x \delta L(x) \\ &= \int d^4x \left(\frac{\delta L}{\delta A^\nu(x)}\delta A^\nu(x) + \frac{\delta L}{\delta \partial^\mu A^\nu(x)}\partial^\mu \delta A^\nu(x) \right) \\ &= \int d^4x \partial^\mu \left(\frac{\delta L}{\delta \partial^\mu A^\nu(x)}\delta A^\nu(x) \right) + \int d^4x \left(\frac{\delta L}{\delta A^\nu(x)} - \partial^\mu \left(\frac{\delta L}{\delta \partial^\mu A^\nu(x)} \right) \right) \delta A^\nu(x). \end{aligned} \tag{27.18}$$

Usually one denotes the partial derivatives of L with respect to A and ∂A by $\delta L/\delta ...$. Since the variation has to vanish, one obtains in general the equations of motion

$$\partial^\mu \left(\frac{\delta L}{\delta \partial^\mu A^\nu(x)} \right) - \frac{\delta L}{\delta A^\nu(x)} = 0. \tag{27.19}$$

This is the generalization of LAGRANGE's equations of motion (27.3) for fields. There appear derivatives of $\delta L/\delta \nabla A^\nu$ with respect to the space variables besides the time-derivatives of $\delta L/\delta \dot{A}^\nu$.

28 Energy Momentum Tensor and Conserved Quantities

28.a The Tensor

In section 15.b we have calculated the conservation law for momentum from the density of the LORENTZ force in vacuo that is without considering additional contributions due to matter

$$-\mathbf{k} = \frac{\partial}{\partial t}\mathbf{g}_s - \frac{\partial}{\partial x^\beta} T^{\alpha\beta}\mathbf{e}^\alpha, \tag{28.1}$$

$$\mathbf{g}_s = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}, \tag{28.2}$$

$$T^{\alpha\beta} = \frac{1}{4\pi}\Big(E_\alpha E_\beta + B_\alpha B_\beta\Big) - \frac{\delta_{\alpha\beta}}{8\pi}\Big(E^2 + B^2\Big). \tag{28.3}$$

The zeroth component is the energy-density. For this density we have obtainded in section 15.a

$$-k^0 = -\frac{1}{c}\mathbf{j}\cdot\mathbf{E} = \frac{1}{c}\,\mathrm{div}\,\mathbf{S} + \frac{1}{c}\dot{u} \tag{28.4}$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \tag{28.5}$$

$$u = \frac{1}{8\pi}\Big(E^2 + B^2\Big). \tag{28.6}$$

We summarize

$$-k^\mu = -\partial_\nu T^{\mu\nu} \tag{28.7}$$

with the electromagnetic energy-momentum tensor

$$(T^{\mu\nu}) = \begin{pmatrix} -u & -\frac{1}{c}S_1 & -\frac{1}{c}S_2 & -\frac{1}{c}S_3 \\ -cg_{s1} & T_{11} & T_{12} & T_{13} \\ -cg_{s2} & T_{21} & T_{22} & T_{23} \\ -cg_{s3} & T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{28.8}$$

This energy-momentum tensor is built up from the energy density u, the POYNTING vector (density of energy current) $\mathbf{S}$, the momentum density $\mathbf{g}$, and the stress tensor T. One observes that $T^{\mu\nu}$ is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, since $T_{\alpha\beta}$ is symmetric and $c\mathbf{g}_s = \frac{1}{c}\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{B}$ holds. One easily checks that

$$T^{\mu\nu} = \frac{1}{4\pi}\Big(- F^\mu{}_\lambda F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^\kappa{}_\lambda F^\lambda{}_\kappa\Big) \tag{28.9}$$

holds either by explicit calculation and comparison or from

$$k^\mu = \frac{1}{c}j_\lambda F^{\mu\lambda} = \frac{1}{4\pi}(\partial^\nu F_{\nu\lambda})F^{\mu\lambda} = \frac{1}{4\pi}\partial^\nu(F_{\nu\lambda}F^{\mu\lambda}) - \frac{1}{4\pi}F_{\nu\lambda}\partial^\nu F^{\mu\lambda}. \tag{28.10}$$

From

$$F_{\nu\lambda}\Big(\partial^\nu F^{\mu\lambda} + \partial^\mu F^{\lambda\nu} + \partial^\lambda F^{\nu\mu}\Big) = 0 \tag{28.11}$$

one obtains the relation

$$\frac{1}{2}\partial^\mu\Big(F_{\nu\lambda}F^{\lambda\nu}\Big) + 2F_{\nu\lambda}\partial^\nu F^{\mu\lambda} = 0, \tag{28.12}$$

so that finally we obtain

$$\begin{aligned} k^\mu &= \frac{1}{4\pi}\partial^\nu\Big(F_{\nu\lambda}F^{\mu\lambda}\Big) + \frac{1}{16\pi}\partial^\mu\Big(F_{\nu\lambda}F^{\lambda\nu}\Big) \\ &= \frac{1}{4\pi}\partial_\nu\Big(- F^\mu{}_\lambda F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^\kappa{}_\lambda F^\lambda{}_\kappa\Big). \end{aligned} \tag{28.13}$$

$T^{\mu\nu}$ is a symmetric four-tensor, i.e. it transforms according to

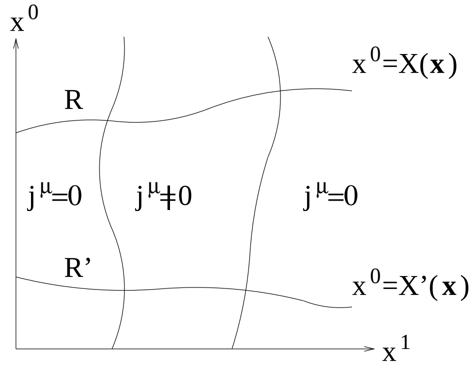
$$T'^{\mu\nu} = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda T^{\kappa\lambda}. \tag{28.14}$$

28.b Conservation Laws

We start out from a four-vector field $(j^\mu(x)$. In any three-dimensional space-like subspace R of the four-dimensional space be (j^μ) different from zero only in a finite region. We call a space space-like if any two points in this space have a space-like distance. A world-line, i.e. a line which everywhere has a velocity below light-speed hits a space-like subspace in exactly one point. If one plots the subspace as a function $x^0(\mathbf{r})$ then its slope is everywhere less then 1. The slope of the world-line is everywhere larger than 1. For example, the points of constant time in an inertial frame constitute such a space-like space. We now integrate the divergence $\partial_\mu j^\mu$ over the four-dimensional volume Ω, which is bounded by two space-like spaces R and R' and obtain

$$\int_\Omega \mathrm{d}^4x \frac{\partial j^\mu}{\partial x^\mu} = \int_R \mathrm{d}^3x\Big(j^0 - \frac{\partial X}{\partial x^\alpha} j^\alpha\Big) - \int_{R'} \mathrm{d}^3x\Big(j^0 - \frac{\partial X'}{\partial x^\alpha} j^\alpha\Big). \tag{28.15}$$

The contribution $\partial_\mu j^\mu$ is integrated in x^μ-direction until the boundary R or R' or until j^μ vanishes. This yields immediately the contribution given for the 0-component. For the 1-component one obtains initially the integral $\pm\int \mathrm{d}x^0\mathrm{d}x^2\mathrm{d}x^3 j^1$ at the boundary.The $\mathrm{d}x^0$-integration may be transformed into an $\mathrm{d}x^1\frac{\partial X}{\partial x^1}$-integration. If $X = x^0$ increases (decreases) at the boundary with x^1, then this is the lower (upper) limit of the integration. Thus we have a minus-sign in front of $\frac{\partial X}{\partial x^1}$, similarly for the other space-components.We may convince ourselves that



$$\int_R \mathrm{d}^3x\Big(j^0 - \frac{\partial X}{\partial x^\alpha} j^\alpha\Big) = \int_R \mathrm{d}V_\mu j^\mu \tag{28.16}$$

with $(\mathrm{d}V_\mu) = (1, -\nabla X)\mathrm{d}^3x$ is a four-scalar. If we introduce a four-vector $(\bar{j}^\mu)$, so that

$$\bar{j}^\mu = \begin{cases} j^\mu & \text{in } R \\ 0 & \text{in } R' \end{cases}, \tag{28.17}$$

then it follows that

$$\int_R \mathrm{d}V_\mu j^\mu = \int_R \mathrm{d}V_\mu \bar{j}^\mu = \int_\Omega \mathrm{d}^4x \frac{\partial \bar{j}^\mu}{\partial x^\mu}, \tag{28.18}$$

where the last integral is obviously a four-scalar, since both d^4x and the four-divergence of $\bar{j}$ is a four-scalar. Since the field (j^μ) is arbitrary, we find that $\mathrm{d}V_\mu j^\mu$ has to be a four-scalar for each infinitesimal $(\mathrm{d}V_\mu)$ in R. Since (j^μ) is a four-vector, $(\mathrm{d}V^\mu)$ must be a four-vector, too. Then (28.16) reads

$$\int_\Omega \mathrm{d}^4x\partial_\mu j^\mu = \int_R \mathrm{d}V_\mu j^\mu - \int_{R'} \mathrm{d}V_\mu j^\mu. \tag{28.19}$$

This is the divergence theorem in four dimensions.
From this we conclude:

28.b.α Charge

(j^μ) be the four-vector of the current density. One obtains from the equation of continuity $\partial_\mu j^\mu = 0$ for each space-like R the same result

$$q = \frac{1}{c}\int_R \mathrm{d}V_\mu j^\mu \tag{28.20}$$

for the charge, since the integral of the divergence in Ω in (28.19) vanishes, (since the integrand vanishes) and since one may always choose the same R'. Thus the charge is a conserved quantity, more precisely we have found a consistent behaviour, since we already have assumed in subsection 24.c that charge is conserved. New is that it can be determined in an arbitrary space-like three-dimensional space.

28.b.β Energy and Momentum

From

$$k^\mu = \partial_\nu T^{\mu\nu} \tag{28.21}$$

one obtains

$$\int_\Omega d^4 x k^\mu = \int_R dV_\nu T^{\mu\nu} - \int_{R'} dV_\nu T^{\mu\nu}. \tag{28.22}$$

In a charge-free space ($k^\mu = 0$), i.e. for free electromagnetic waves one finds that the components of the momentum of radiation

$$G_s^\mu = -\frac{1}{c}\int_R dV_\nu T^{\mu\nu} \tag{28.23}$$

are independent of R. Thus they are conserved. Now be (b_μ) an arbitrary but constant four-vector. Then $b_\mu T^{\mu\nu}$ is a four-vector and $\partial_\nu(b_\mu T^{\mu\nu}) = 0$. Then $b_\mu G_s^\mu$ is a four-scalar and G_s^μ is a four-vector.
If there are charges in the four-volume Ω, then one obtains.

$$G_s^\mu(R) = -\frac{1}{c}\int_\Omega d^4 x k^\mu + G_s^\mu(R'). \tag{28.24}$$

For point-charges q_i one has (26.7, 26.30)

$$\frac{1}{c}\int d^4 x k^\mu = \sum_i \int dt K_i^\mu = \sum_i \int dt \dot{G}_i^\mu = \sum_i (G_i^\mu(R) - G_i^\mu(R')). \tag{28.25}$$

Here $G_i^\mu(R) = m_i u_i^\mu(R)$ is the four-momentum of the charge #i at the point where its worldline hits the three-dimensional space R. Then

$$G^\mu = G_s^\mu(R) + \sum_i G_i^\mu(R) \tag{28.26}$$

is the conserved four-momentum.

28.b.γ Angular Momentum and Movement of Center of Mass

Eq. (28.7) yields

$$\partial_\nu\Big(x^\lambda T^{\mu\nu} - x^\mu T^{\lambda\nu}\Big) = x^\lambda k^\mu - x^\mu k^\lambda + T^{\mu\lambda} - T^{\lambda\mu}. \tag{28.27}$$

Since the tensor T is symmetric, the last two terms cancel. We introduce the tensor

$$M_s^{\lambda\mu}(R) = -\frac{1}{c}\int_R dV_\nu (x^\lambda T^{\mu\nu} - x^\mu T^{\lambda\nu}\Big). \tag{28.28}$$

It is antisymmetric $M_s^{\lambda\mu} = -M_s^{\mu\lambda}$. Due to (28.19) one has

$$M_s^{\lambda\mu}(R) = -\frac{1}{c}\int_\Omega d^4 x\Big(x^\lambda k^\mu - x^\mu k^\lambda\Big) + M_s^{\lambda\mu}(R'). \tag{28.29}$$

For point-charges one obtains

$$\frac{1}{c}\int_\Omega d^4 x\Big(x^\lambda k^\mu - x^\mu k^\lambda\Big) = \sum_i \int dt\Big(x_i^\lambda K_i^\mu - x_i^\mu K_i^\lambda\Big) = \sum_i \int dt \frac{d}{dt}\Big(x_i^\lambda G_i^\mu - x_i^\mu G_i^\lambda\Big), \tag{28.30}$$

since $\dot{x}^\lambda G^\mu = \dot{x}^\mu G^\lambda$. Therefore

$$M^{\lambda\mu}(R) = M_s^{\lambda\mu}(R) + M_m^{\lambda\mu}(R) \tag{28.31}$$

including the mechanical contribution

$$M_m^{\lambda\mu}(R) = \sum_i \Big(x_i^\lambda G_i^\mu - x_i^\mu G_i^\lambda\Big)\Big|_R \tag{28.32}$$

is a conserved quantity, i.e. $M^{\lambda\mu}(R)$ is independent of the choice of R. Simultaneously $(M^{\lambda\mu})$ is a four-tensor.

Finally we have to determine the meaning of M. For this purpose we consider M in the three-dimensional space R given by constant time t for a system of inertia S. Then we have

$$M^{\lambda\mu} = -\frac{1}{c}\int d^3x\Big(x^\lambda T^{\mu 0} - x^\mu T^{\lambda 0}\Big) + \sum_i \Big(x_i^\lambda G_i^\mu - x_i^\mu G_i^\lambda\Big) \tag{28.33}$$

First we consider the space-like components

$$M^{\alpha\beta} = \int d^3x\Big(x^\alpha g_s^\beta - x^\beta g_s^\alpha\Big) + \sum_i \Big(x_i^\alpha G_i^\beta - x_i^\beta G_i^\alpha\Big). \tag{28.34}$$

This is for $\alpha \neq \beta$ a component of the angular momentum **L**, namely $\epsilon_{\alpha\beta\gamma}L_\gamma$. Thus we have found the conservation of angular momentum.
If one component is time-like then one finds

$$M^{0\alpha} = ct\Big(\int d^3x g_s^\alpha + \sum_i G_i^\alpha\Big) - \frac{1}{c}\Big(\int d^3x x^\alpha u + \sum_i x_i^\alpha E_i\Big). \tag{28.35}$$

The first contribution is ct multiplied by the total momentum. The second contribution is the sum of all energies times their space-coordinates x^α divided by c. This second contribution can be considered as the center of energy (actually its α-component) multiplied by the total energy divided by c. Since total momentum and total energy are constant, one concludes that the center of energy moves with the constant velocity $c^2\frac{\text{total momentum}}{\text{total energy}}$.
For non-relativistic velocities the mechanical part of the energy reduces to

$$M_m^{0\alpha} = c\left(t\sum_i G_i^\alpha - \sum_i m_i x_i^\alpha\right). \tag{28.36}$$

Then the conservation of this quantity comprises the uniform movement of the center of mass with the velocity total momentum divided by total mass. In the theory of relativity this transforms into a uniform moving center of energy. Lorentz invariance combines this conservation with the conservation of angular momentum to the antisymmetric tensor M.

29 Field of an arbitrarily Moving Point-Charge

29.a LIÉNARD-WIECHERT Potential

First we determine the potential at a point (x^μ) of a point-charge q which moves along a world-line $\mathbf{r}_q(t)$. Its four-current density reads

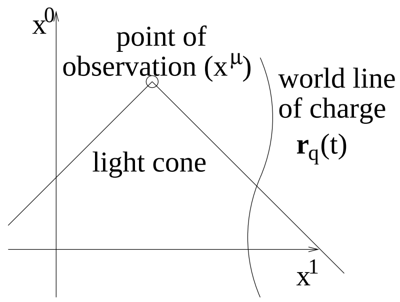
$$j^\mu(x') = qv^\mu\delta^3(\mathbf{x}' - \mathbf{r}_q(t)), \quad v^\mu = (c, \dot{\mathbf{r}}_q(t)). \tag{29.1}$$

According to (24.29) the four-potential reads

$$A^\mu(x) = \frac{1}{c}\int d^4x' j^\mu(x')\delta(\frac{1}{2}s^2)\theta(t-t') = q\int dt' v^\mu(t')\delta(\frac{1}{2}s^2)\theta(t-t') \tag{29.2}$$

with

$$s^2 = a^\nu a_\nu, \quad a^\nu = x^\nu - x_q^\nu(t'). \tag{29.3}$$



(a^ν) is a function of (x^ν) and t'. The differential of $\frac{1}{2}s^2$ is given by

$$d(\frac{1}{2}s^2) = a_\nu da^\nu = a_\nu dx^\nu - a_\nu v^\nu dt'. \tag{29.4}$$

Thus one obtains the LIÉNARD-WIECHERT potential

$$A^\mu(x) = qv^\mu(t')\frac{1}{|\frac{\partial\frac{1}{2}s^2}{\partial t'}|} = \left.\frac{qv^\mu}{a_\nu v^\nu}\right|_r = \left.\frac{qu^\mu}{a_\nu u^\nu}\right|_r. \tag{29.5}$$

Here the two expressions with the index $_r$ are to be evaluated at the time t' at which $s^2 = 0$ and $t > t'$.
We note that $a_\nu v^\nu = ac - \mathbf{a}\cdot\mathbf{v} > 0$, since $a = c(t-t') = |\mathbf{a}|$. $a_\nu u^\nu/c$ is the distance between point of observation and charge in the momentary rest system of the charge.

29.b The Fields

Starting from the potentials we calculate the fields

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{29.6}$$

In order to do this we have to determine the derivatives of v, a and t'

$$\partial^\mu v^\nu = \frac{\partial v^\nu}{\partial t'}\frac{\partial t'}{\partial x_\mu} \tag{29.7}$$

$$\partial^\mu a^\nu = \partial^\mu(x^\nu - x_q^\nu(t')) = g^{\mu\nu} - v^\nu\frac{\partial t'}{\partial x_\mu} \tag{29.8}$$

$$\frac{\partial t'}{\partial x_\mu} = \frac{a^\mu}{(a\cdot v)}, \tag{29.9}$$

where the last expression has been obtained from $s^2 = 0$ by means of (29.4). Here and in the following we use

$$(a\cdot v) = a^\nu v_\nu = ac - \mathbf{a}\cdot\mathbf{v} = c(a - \mathbf{a}\cdot\boldsymbol{\beta}) \tag{29.10}$$

$$(v\cdot v) = v^\nu v_\nu = c^2 - v^2 = c^2(1-\beta^2) \tag{29.11}$$

$$(a\cdot\dot{v}) = a^\nu\dot{v}_\nu = -\mathbf{a}\cdot\dot{\mathbf{v}}. \tag{29.12}$$

One evaluates

$$\partial^\mu v^\nu = \frac{\dot{v}^\nu a^\mu}{(a\cdot v)} \tag{29.13}$$

$$\partial^\mu a^\nu = g^{\mu\nu} - \frac{v^\nu a^\mu}{(a\cdot v)} \tag{29.14}$$

$$\begin{aligned}\partial^\mu (a\cdot v) &= (\partial^\mu a^\kappa) v_\kappa + a_\kappa (\partial^\mu v^\kappa)\\ &= g^{\mu\kappa} v_\kappa - \frac{v^\kappa a^\mu}{(a\cdot v)} v_\kappa + a_\kappa \frac{\dot v^\kappa a^\mu}{(a\cdot v)}\\ &= v^\mu - a^\mu \frac{(v\cdot v)}{(a\cdot v)} + a^\mu \frac{(a\cdot \dot v)}{(a\cdot v)}.\end{aligned} \tag{29.15}$$

Then one obtains

$$\begin{aligned}\partial^\mu A^\nu &= \partial^\mu \left(q\frac{v^\nu}{(a\cdot v)}\right) = q\frac{\partial^\mu v^\nu}{(a\cdot v)} - q\frac{v^\nu \partial^\mu (a\cdot v)}{(a\cdot v)^2}\\ &= a^\mu b^\nu - q\frac{v^\mu v^\nu}{(a\cdot v)^2},\end{aligned} \tag{29.16}$$

$$b^\nu = q\frac{v^\nu (v\cdot v) - v^\nu (a\cdot \dot v) + \dot v^\nu (a\cdot v)}{(a\cdot v)^3}. \tag{29.17}$$

Therefore

$$(b^\nu) = \frac{q}{(a - \mathbf{a}\cdot\boldsymbol{\beta})^3}\left(1-\beta^2 + \frac{\mathbf{a}\cdot\dot{\boldsymbol{\beta}}}{c}, \boldsymbol{\beta}(1-\beta^2) + \frac{1}{c}\boldsymbol{\beta}(\mathbf{a}\cdot\dot{\boldsymbol{\beta}}) + \frac{1}{c}(a - \mathbf{a}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}}\right) \tag{29.18}$$

and the fields read

$$F^{\mu\nu} = a^\mu b^\nu - a^\nu b^\mu \tag{29.19}$$

$$\mathbf{E} = \mathbf{a} b^0 - a\mathbf{b} = \frac{q(1-\beta^2)(\mathbf{a} - \boldsymbol{\beta} a)}{(a - \mathbf{a}\cdot\boldsymbol{\beta})^3} + \frac{q\mathbf{a}\times((\mathbf{a} - \boldsymbol{\beta} a)\times\dot{\boldsymbol{\beta}})}{c(a - \mathbf{a}\cdot\boldsymbol{\beta})^3} \tag{29.20}$$

$$\mathbf{B} = -\mathbf{a}\times\mathbf{b} = \frac{\mathbf{a}\times\mathbf{E}}{a} \tag{29.21}$$

The contribution proportional to the acceleration $\dot{\boldsymbol{\beta}}$ decreases like $1/a$; $\mathbf{a}$, $\mathbf{E}$, and $\mathbf{B}$ constiute an orthogonal system for this contribution. The contribution independent of $\dot{\boldsymbol{\beta}}$ falls off like $1/a^2$.

29.c Uniform Motion

(compare section 25.d). The scalar $\gamma a^\lambda v_\lambda / c$ is the distance between the point of observation and the point of the charge in the rest-system of the charge. Thus one has

$$a - \mathbf{a}\cdot\boldsymbol{\beta} = \frac{1}{\gamma}|\mathbf{r}'|, \quad (a - \mathbf{a}\cdot\boldsymbol{\beta})^3 = N/\gamma^3. \tag{29.22}$$

Considering that $\mathbf{a} = \mathbf{r} - \mathbf{v}t'$, $a = c(t-t')$, one obtains

$$\mathbf{a} - \boldsymbol{\beta} a = \mathbf{r} - \mathbf{v}t' - \mathbf{v}t + \mathbf{v}t' = \mathbf{r} - \mathbf{v}t \tag{29.23}$$

and thus

$$\mathbf{E} = \frac{q\gamma(\mathbf{r} - \mathbf{v}t)}{N}, \quad \mathbf{B} = \frac{(\mathbf{r} - \mathbf{v}t')\times(\mathbf{r} - \mathbf{v}t)q\gamma}{c(t-t')N} = \frac{q\gamma\mathbf{v}\times\mathbf{r}}{cN} \tag{29.24}$$

in accordance with (25.30) and (25.31).

29.d Accelerated Charge Momentarily at Rest

The equations (29.20) and (29.21) simplify for $\beta = 0$ to

$$\mathbf{E} = \frac{q\mathbf{a}}{a^3} + \frac{q}{ca^3}\mathbf{a}\times(\mathbf{a}\times\dot{\boldsymbol{\beta}}) \tag{29.25}$$

$$\mathbf{B} = -\frac{q}{ca^2}(\mathbf{a}\times\dot{\boldsymbol{\beta}}), \tag{29.26}$$

from which the power radiated into the solid angle $d\Omega$ can be determined with the energy-current density $\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}$

$$\frac{\mathrm{d}\dot{U}_\mathrm{s}}{\mathrm{d}\Omega} = a^2\mathbf{S}\cdot\mathbf{n} = \frac{ca}{4\pi}[\mathbf{a},\mathbf{E},\mathbf{B}] = \frac{q^2}{4\pi ca^2}(\mathbf{a}\times\dot{\boldsymbol{\beta}})^2 = \frac{q^2}{4\pi c^3}(\mathbf{n}\times\dot{\mathbf{v}})^2 \tag{29.27}$$

and the total radiated power

$$\dot{U}_\mathrm{s} = \frac{2}{3}\frac{q^2}{c^3}\dot{\mathbf{v}}^2 \tag{29.28}$$

(Larmor-formula) follows.
For a harmonic motion $\mathbf{r}_\mathrm{q} = \mathbf{r}_{0\mathrm{q}}\cos(\omega t)$ and $\dot{\mathbf{v}} = -\mathbf{r}_{0\mathrm{q}}\omega^2\cos(\omega t)$ one obtains

$$\dot{U}_\mathrm{s} = \frac{2}{3}\frac{q^2\mathbf{r}_{0\mathrm{q}}^2}{c^3}\omega^4(\cos(\omega t))^2, \quad \overline{\dot{U}_\mathrm{s}} = \frac{1}{3}\frac{p_0^2}{c^3}\omega^4 \tag{29.29}$$

in agreement with section 22.b. This applies for $\beta \ll 1$. Otherwise one has to take into account quadrupole and higher multipole contributions in 22.b, and here that β cannot be neglected anymore, which yields additional contributions in order ω^6 and higher orders.

29.e Emitted Radiation $\beta \neq 0$

We had seen that in the system momentarily at rest the charge emits the power $\dot{U}_\mathrm{s} = \frac{2}{3}\frac{q^2}{c^3}\dot{\mathbf{v}}^2$. The emitted momentum vanishes because of the symmetry of the radiation (without consideration of the static contribution of $\mathbf{E}$, which, however, decays that fast that it does not contribute for sufficiently large a)

$$\mathbf{E}(-\mathbf{a}) = \mathbf{E}(\mathbf{a}), \quad \mathbf{B}(-\mathbf{a}) = -\mathbf{B}(\mathbf{a}), \quad T_{\alpha\beta}(-\mathbf{a}) = T_{\alpha\beta}(\mathbf{a}). \tag{29.30}$$

Thus we may write the energy-momentum-vector emitted per proper time

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\frac{1}{c}U_\mathrm{s},\mathbf{G}_\mathrm{s}\Big) = \frac{u^\mu}{c}\frac{2q^2}{3c^3}\left(-\frac{\mathrm{d}u^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}u_\lambda}{\mathrm{d}\tau}\right), \tag{29.31}$$

since $\dot{u}^0 = c\dot{\gamma} \propto \mathbf{v}\cdot\dot{\mathbf{v}} = 0$. Since the formula is written in a lorentz-invariant way, it holds in each inertial frame, i.e.

$$\begin{aligned}\frac{\mathrm{d}U_\mathrm{s}}{\mathrm{d}t} &= \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{u^0}{c}\frac{2q^2}{3c^3}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\left(-\frac{\mathrm{d}(\gamma v^\lambda)}{\mathrm{d}t}\frac{\mathrm{d}(\gamma v_\lambda)}{\mathrm{d}t}\right)\\ &= \frac{2q^2}{3c^3}\gamma^2\left((\gamma\mathbf{v})^{\cdot}(\gamma\mathbf{v})^{\cdot} - c^2\dot{\gamma}^2\right)\\ &= \frac{2q^2}{3c^3}\gamma^2\left(\gamma^2\dot{\mathbf{v}}^2 + 2\gamma\dot{\gamma}(\mathbf{v}\cdot\dot{\mathbf{v}}) + \dot{\gamma}^2(\mathbf{v}^2 - c^2)\right).\end{aligned} \tag{29.32}$$

With $\mathrm{d}\tau/\mathrm{d}t\cdot u^0/c = 1$ and

$$\dot{\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \gamma^3\frac{\mathbf{v}\cdot\dot{\mathbf{v}}}{c^2} \tag{29.33}$$

one obtains finally

$$\dot{U}_\mathrm{s} = \frac{2}{3}\frac{q^2}{c^3}\left(\gamma^4\dot{\mathbf{v}}^2 + \gamma^6\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right). \tag{29.34}$$

Orbiting in a synchrotron of radius r a charge undergoes the acceleration $\dot{v} = v^2/r$ perpendicular to its velocity. Thus one has

$$\dot{U}_\mathrm{s} = \frac{2}{3}q^2c\beta^4\gamma^4/r^2 = \frac{2}{3}q^2c(\gamma^2-1)^2/r^2. \tag{29.35}$$

The radiated energy per circulation is

$$\Delta U_\mathrm{s} = \frac{2\pi r}{v}\dot{U}_\mathrm{s} = \frac{4\pi}{3}q^2\beta^3\gamma^4/r. \tag{29.36}$$

At Desy one obtains for an orbiting electron of energy E =7.5 GeV and mass m_0c^2 =0.5 MeV a value $\gamma = E/(m_0c^2) = 15000$. For $r = 32$m one obtains $\Delta U = 9.5$MeV. Petra yields with $E = 19$GeV a $\gamma = 38000$ and with $r = 367$m a radiation $\Delta U = 34$MeV per circulation.

Exercise Hera at Desy has $r = 1008$m and uses electrons of $E_\mathrm{e} = 30$GeV and protons of $E\mathrm{p} = 820$GeV. Calculate the energy radiated per circulation.

I Review and Outlook

In this last chapter a review in the form of a short account of the historic development of electrodynamics is given. Additionally we will catch a glimpse of the general theory of relativity by showing that clocks run differently in the presence of gravitation.

30 Short History of Electrodynamics

I conclude with a short history of electrodynamics. For this purpose I have mainly used the following literature:
Sir Edmund Whittaker, A History of the Theories of Aether and Electricity
Emilio Segrè, Die großen Physiker und ihre Entdeckungen, Teil 1, Piper Band 1174
Wilhelm H. Westphal, Physik, Springer-Verlag
Wilhelm H. Westphal, Anhang I, Physikalisches Wörterbuch
Max Born, Emil Wolf, Principles of Optics, Historical Introduction
Edmund Hoppe, Geschichte der Physik
Encyclopedia Brittanica: Article 'Electromagnetic Waves' and 'Magnetism'
Woldemar Voigt, Theoretische Physik
J.D. Jackson and L.B. Okun, Historical roots of gauge invariance, Rev. Mod. Phys. 73 (2001) 663.

It is not easy to redraw a historical development. First of all, there is the question whether one has sufficiently complete sources. Secondly, it often happens that several persons are named for some discovery or explanation, sometimes even at separate times. One reason might be that one person did not know of the other's discovery. But it may also be that they observed or explained the phenomenon differently well. Sometimes they published the result particularly well so that their paper has become rather popular and they were considered alleged authors.

Who for example has explained how the rainbow comes about? Dietrich von Freiberg, Mahmud al Schirazi and Kamal al-Din who in the beginning of the 14th century found that sunlight is refracted twice and reflected once or twice inside the raindrop, or Descartes who found around 1625 that the total angle of reflection has an extremum so that a high intensity of light is reflected under a certain angle, or Fresnel and Airy who took around 1820 and 1836 the wave character of light into account? All of them contributed a piece to our knowledge.

Initially there were three different phenomena of electrodynamics observed by man without forseeing their connection. The most obvious one was light which offered him excellent orientation and which sometimes appeared to him both frightening and agreeable as in a flash of lightning and a rainbow.

Two other phenomena already known in antiquity were much rarer observed, the curious properties possessed by two minerals, amber ($\eta\lambda\epsilon\kappa\tau\rho o\nu$) and magnetic iron ore (η $\lambda\iota\theta o\zeta$ $M\alpha\gamma\nu\eta\tau\iota\zeta$). The former, when rubbed, attracts light bodies; the latter has the power of attracting iron and has its name from Magnesia in Thessalia, where this stone is found. Thales of Milet (600 BC) is said to have known the properties of these minerals.

Accordingly, the investigation of these phenomena developed in parallel into a theory of light, of electrostatics, and of magnetostatics, before one realized that they are connected.

30.a Theory of Light to Fresnel

Hero of Alexandria argued that reflected light uses the shortest path and thus for light reflected at a mirror the angle of incidence equals the angle of reflexion. In antiquity and the middle ages one assumed that nature has final causes and thus asked *why*, not *how* does nature proceed?

Hero and Ptolemy held the opinion that men saw by means of rays of light issuing from the eye and reflecting from the objects seen. Alhazen held the correct view that light was issued from the sun or from some other luminous source and was reflected from the object seen into the eye. Alhazen made important discoveries in optics (1030): pinhole camera and parabolic mirror. Kepler learnt a lot from his work. Alhazen already knew that in refraction the incident, the reflected and the refracted beam lie in one plane.

Eye glasses were invented in the 13th century.

The explanation for the occurence of the rainbow by two refractions and one or two reflections of sunlight in the rain drop was given by Dietrich von Freiberg, Al-Shirazi, and Kamal al-Din at the beginning of the 14th century.

In 1621 Snell of Royen found experimentally the law of refraction. Descartes gave a theoretical derivation assuming that the velocity of the beams have given values in both media and that the component of the velocity vector parallel to the boundary is conserved. This derivation becomes correct if the vector of velocity is replaced by the wave-vector. However, Fermat introduced the principle of least time (1657) and derived from this the law of refraction (1661).

Hooke was probably the first who described in his *Micrographia* light as a wave, since he had observed diffraction. Considering theoretically the progression of the wave-front he derived the law of refraction. However, Huygens, developed in his *Traité de la lumière* (1678-1690) a wave theory of light. Important for the theory of diffraction but also refraction became his principle, which says: Each point of a wave-front may be regarded as the source of a secondary wave.' Newton is considered having put forward the theory of emanation, i.e. the idea that light is of corpuscular nature. This is not completely correct. Newton disliked to introduce imaginative hypotheses, which could not be proven experimentally. 'To avoid dispute, and make this hypothesis general, let every man here take his fancy; only whatever light be, I suppose it consists of rays differing from one another in contingent circumstances, as bigness, form or vigour.' Later however, he was in favour of the corpuscular nature of light.

Newton devoted considerable attention to the colours of thin plates. He supposed (*Opticks*) that 'every ray of light, in its passage through any refracting surface, is put into a certain transient state, which, in the progress of the ray, returns at equal intervals, and disposes the ray, at every return, to be easily transmitted through the next refracting surface, and, between the returns, to be easily reflected by it.'He found that the intervals between easy transmission vary with colour, being greatest for red and least for violet. If he had accepted the wave picture, he could have determined the wavelengths of visible light.

In 1717 the meanwhile known phenomenon of double-refraction was explained by Newton by light corpuscles of different shape which comes close the idea of a transversal polarization. Huygens' wave theory of light assigned elastic properties to the aether; However, he considered only longitudinal waves, and was forced to introduce two different kinds of these waves for double-refraction, one of which propagated isotropically, the other spheroidally. At that time Newton's explanation was generally accepted.

In this course we did not consider double-refraction and diffraction. They played an important role in the development of the theory of light. It should be remarked that double-refraction appears in anisotropic crystals where the dielectric constant is a tensor.

In 1675 Römer was able to determine the time light needs to transverse the distance from sun to earth by observing the eclipses of the moons of Jupiter. Until that time it was not clear whether light propagates instantly or at a finite velocity.

In 1728 James Bradley found the aberration of light, i.e. a change in the direction of the light from a star due to the perpendicular motion of the observer to the direction of the star. This was considered a proof of the corpuscular nature of light. In 1677 Römer already had conjectured such a phenomenon in a letter to Huygens.

In 1744 Maupertuis took up the old controversy between Descartes and Fermat. Convinced of the corpuscular nature of light but wishing to retain Fermat's method, he supposed that 'the path described is that by which the quantity of action is the least' and required that instead Fermat's $\int \mathrm{d}t = \int \mathrm{d}s/v$ the action $\int v\mathrm{d}s$ should be extremal. In this way he introduced for the first time the principle of least action which was soon taken up by Euler and Lagrange, and which today is considered the principle governing all dynamics of nature.

In 1801 Thomas Young introduced the concept of the interference of two waves and brought Huygens' concept anew in play. He is able to explain Newton's rings with this concept. In 1808 Malus found that reflected light is normally partially polarized and found the angle of total polarization, now known as Brewster's angle (after eq. 18.22). The problem to explain the extraordinary beam in double-refracting crystals continued with explanations from both sides, in 1808 Laplace argued for corpuscles, in 1809 Young argued for waves, both agreeing that the medium has to be anisotropic. In 1815 when Brewster discovered crystals with two extraordinary beams (the case of three different eigenvalues of the dielectric tensor) the situation became even more complex.

In 1818 the French Academy announced a prize for the explanation of diffraction. The followers of the theory of emission (Laplace, Poisson, Biot) were confident of victory but Fresnel submitted a paper at the basis of the papers by Huygens and Young, in which he explained this phenomenon for several arrangements by means

of the wave theory. Poisson who studied the paper carefully found that in the centre of the shadow behind a circular disc there had to be a bright spot and asked for an experimental test. Arago found the bright spot and Fresnel received the prize. Since in 1818 Young was also able to explain aberration by means of the wave theory it became the leading theory.
In 1817 Young proposed for the first time, light might consist of transversal waves. This was supported by the observation that two light beams polarized perpendicular to each other do not show interferences. Fresnel picked up this idea and developed a successful theory of double-refraction, although Maxwell's equations were not yet available. Clever experiments by Airy (1831) showing that light irradiated under Brewster's angle suppresses Newton's rings, and that light propagated slower in water than air, proved the wave nature of light. (Wave-theory predicts in a medium with larger index of refraction a smaller velocity of light, in the corpuscular theory a larger one.)
Fresnel derived an expression for the change of velocity of light in moving matter which was confirmed experimentally by Fizeau (1851). However, there were several different theories on this subject among others by Stokes (1846). Different ideas competed on the question, to which extend matter would drag the aether.
It may be remarked that the aether as an elastic solid occupied many excellent scientists in the following time and made the theory of elasticity flourishing in the following years. Applied to the theory of light there remained the problem to suppress longitudinal waves..
There remained the puzzle whether the space above the earth is a plenum, which provides the necessary properties of elasticity for the propagation of light, or a vacuum, which allows the planets to move freely. This discussion existed already centuries before. Space was, in Descartes' view, a plenum (in contrast to a vacuum), being occupied by a medium which, though imperceptible to the senses, is capable of transmitting force, and exerting effects on material bodies immersed in it - the aether, as it was called. Gassendi, a follower of Copernicus and Galileo, re-introduced the doctrine of the ancient atomists that the universe is formed of material atoms, eternal and unchangeable, moving about in a space which except for them is empty, thus he re-introduced the vacuum. His doctrine was accepted not long afterwards by Newton and in fact became the departure point for all subsequent natural philosophy.

30.b Electrostatics

Thales of Milet (600 BC) is said to have known that rubbed amber (Greek 'elektron') attracts light bodies. Around 1600 Gilbert discovered that many other materials assume the same property by rubbing. He coined the word 'electric' for this property. The word 'electricity' was introduced by Browne in 1646. Gilbert remarked essential differences between magnetic and electric forces. (Magnets are permanent in contrast to electrified bodies. Magnetic forces are not shielded by other substances. Magnets attract only magnetizable substances, electrified ones all substances.)
Otto of Guericke known for the preparation of the vacuum in the Magdebourgous spheres made pretty early a number of important electric discoveries - his *Experimanta nova magdeburgica* appeared in 1672 - For the first time he introduced the distinction between conductors and non-conductors, he observed electric attraction and repulsion, the phenomenon of influence. He designed the first reasonably working electrostatic generator. It seems that his discoveries did not receive general attention.
In 1708 Wall compared the spark which flashes over rubbed amber with thunder and flash, an indication that a flash is an electrostatic discharge.
In 1729 Gray found that electricity is transferred by certain substances which Desaguliers called non-electrics or conductors. Gray found that electricity is assembled at the surface of bodies. In 1734 DuFay observed that there are two kinds of electricity, vitreous and resinous electricity; similar ones repel each other, whereas dissimilar ones attract each other.
Improved electrostatic generators were designed between 1744 and 1746 by Johann Heinrich Winkler, George Matthias Bose and Benjamin Wilson.
The capacitor in form of a Leyden jar was invented in 1745 by Pieter van Musschenbroek, and independently probably a bit earlier by Ewald von Kleist, described by J. G. Krüger in 1746.
In 1746 William Watson concluded that 'in charging or discharging of a Leyden jar electricity is transferred, but it is not created or destroyed.' 'Under certain circumstances, it was possible to render the electricity in some bodies more rare than it naturally is, and, by communicating this to other bodies, to give them an additional quantity, and make their electricity more dense.' This was a first indication of the conservation of charge.

Similar experiments conducted by Benjamin Franklin after a talk by Dr. Spence who had come from Scotland to America, brought him in 1747 to the conclusion that 'the total amount of electricity in an insulated system is invariable.' Franklin became popular by the introduction of the lightning rod. He realized that lightning was an electric discharge.

The introduction of the signs for charges is ascribed to both Franklin and Lichtenberg (1777): 'I call that electricity positive, which, stimulated by blank glass, is transferred to conductiong bodies; the opposite one I call negative.'

Aepinus and Wilcke found that 'ordinary matter' (this is approximately what we nowadays call matter without outer electrons) repels itself, particles of the 'electric fluid' (nowadays called outer electrons) are repelling themselves, too, and ordinary matter and the electric fluid attract each other. Further, they realized that glass and even air is impermeable for the electric fluid despite the fact that the electric interaction acts over larger distances.

Aepinus explained in 1757 the phenomenon of influence (or electric induction), which had already been observed by Guericke, Canton, and Wilcke, by the electrostatic forces and the free mobility of the electric fluid. Wilcke described in 1762 many experiments in connection with influence and argues that dielectric media are polarized in an electric field.

Joseph Priestley communicates in his work *The History and present State of Electricity ...* which did not receive much attention an experiment conducted by Franklin and repeated by him that inside a metallic box there is no electric force and the interior sides do not carry any charges. He concludes that charges of equal sign repel each other with a force inversely proportional to the square of the distance. 'May we not infer from this experiment that the attraction of electricity is subject to the same laws with that of gravitation, ... since it is easily demonstrated that were the earth in the form of a shell, a body in the inside of it would not be attracted to one side more than another?'

In 1760 Daniel Bernoulli conjectured that there might be a $1/r^2$-law for the electrostatic interaction. In 1769 John Robison was presumably the first to measure a $1/r^n$-dependence with $n = 2 \pm 0.06$. In 1771 Cavendish declared that the interaction falls off with an inverse power less than 3. It took many years until Robison's and Cavendish's results were published. In 1775 Cavendish gave comparative results for the conductances of various materials. (iron, sea water, etc.)

In 1785 Coulomb verified by means of the torsion balance invented by Michell and independently by himself the $1/r^2$-law with high precision. This torsion balance served also for the determination of the gravitional constant (Cavendish).

In 1813 Poisson showed that the electrostatic potential obeys the equation, now called after him. In 1777 Laplace had shown that the operator, now called Laplacian, applied to the gravitational potential in matter free space yields zero. Poisson had included the regions filled with matter, and explicitely stated that one has an analog equation in electrostatics. Thus he has introduced the electrostatic potential and has stated that it is constant over the surface of a conductor. In 1828 George Green continued the calculations of Poisson. We know Green's theorem (B.67). Green's functions are named after him.

William Thomson (Lord Kelvin) (1845) and Mossotti (1847) formulated on the basis of Faraday's considerations the relation between electric field and polarisation, $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$, $\rho_P = -4\pi \operatorname{div} \mathbf{P}$, which we showed in sect. 6.

30.c Magnetostatics

Magnets were already known in antiquity. Their name is derived from the city of Magnesia in Thessalia, where load stone (magnetite Fe_3O_4) occurs naturally, which has the property to attract other load stone and iron. Already about the year 1000 in China magnetic needles were known to have directive properties. The English encyclopedist Alexander Neckam reports on the compass.

In 1269 The crusader Petrus Peregrinus de Maricourt gave a precise description of magnetic stones in his *Epistola de magnete*. He laid an iron needle on a round magnetic stone and marked the directions which was assumed by the needle. He found that these lines formed circles like the meridians of the earth which passed through two points, which he called poles. He observed that a magnet broken into two pieces constitute again magnets with North and South poles; thus no magnetic monopoles exist.

In 1588 the idea of two magnetic poles of the earth was first noted by Livio Sanuto. In 1600 William Gilbert gave a comprehensive review in his work *De magnete*. He emphasizes that the earth is a large magnet.

Similarly to the force law between charges the force between poles of a magnet was investigated. Newton found a law close to $1/r^3$. In 1750 Michell found the $1/r^2$-law based on own measurements and on those of Brook Taylor and Musschenbroek, similarly in 1760 Tobias Mayer and in 1766 Lambert. This led soon to the idea of a 'magnetic fluid' in the sense of magnetic charges similarly to electric ones. Coulomb put forward the thesis that magnetism is captured in molecules and only inside molecules both magnetic fluids can be separated and yield magnetization. (Taylor series are named after Brook Taylor, although they were known before.)
In 1824 Poisson introduced a magnetic potential besides the electric one similarly to the one in subsection 11.b and introduced magnetization quantitatively. In 1828 this theory was extended by Green.
William Thomson (Lord Kelvin) introduced the equations $\operatorname{div}\mathbf{B} = 0$ and $\operatorname{curl}\mathbf{H} = \mathbf{0}$ for current-free space, introduced the relation $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, obtained the expression for the magnetic energy density and concluded that in the relation from the expression $\mathbf{B} = \mu\mathbf{H}$, which had already been given in 1824 by Poisson with a tensor μ for anisotropic crystals the tensor μ has to be symmetric. He coined the notions susceptibility and permeability.

30.d Set out for Electrodynamics

For a long time electricity and magnetism were two separate phenomena. A first hint on a connection was the observation that lightnings made compass needles deflected. Occasionally it happened that during lightning the magnetization of magnets was reversed or that iron became magnetic: It is reported that in 1731 a flash hit a box filled with knives and forks, which melted. When they were taken up some nails which laid around were attracted. In 1681 a ship bound for Boston was hit by a flash. After this stroke the compasses showed into the opposite direction.
About 1800 the experimental situation improved when Volta invented what is called the Volta pile, a prototype of battery. Now it was possible to generate continuous electric currents with a power improved by a factor 1000 over the electrostatic one.
In 1820 Ørsted observed that a magnetic needle was deflected by a parallel flowing current. This discovery spread like wildfire in Europe. Biot and Savart determined in the same year quantitatively the force of a straight current on a magnet. On the basis of a calculation of Laplace for the straight wire and another experiment with a V-shaped wire, Biot abstracted in 1824 the force between a magnetic pole and a current element, which is basically what we now call the law of Biot and Savart.
In 1820 Ampère assumed a law of force of the form

$$\mathbf{K} = I_1 I_2 \oint\oint \hat{\mathbf{r}}_{12} \left(f_1(r_{12})(\mathrm{d}\mathbf{r}_1 \cdot \mathrm{d}\mathbf{r}_2) + f_2(r_{12})(\hat{\mathbf{r}}_{12} \cdot \mathrm{d}\mathbf{r}_1)(\hat{\mathbf{r}}_{12} \cdot \mathrm{d}\mathbf{r}_2) \right),$$
$$\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2, \quad \hat{\mathbf{r}}_{12} = \frac{\mathbf{r}_{12}}{r_{12}}. \qquad (30.1)$$

between two circuits with currents I_1 and I_2. In comparison with his measurements he obtained $f_1 = A/r_{12}^2$, $f_2 = B/r_{12}^2$. Each one of these contributions yields separately by an appropriate choice of A and B, resp., the force between two closed circuits, compare (9.21). Ampère had already observed that the force on a line element of the conductor is perpendicular to it, which is fulfilled by $B = -3A/2$. Thus his force law includes already the Lorentz force, although he did not use the notion of a magnetic field.
After some preliminary work by Ampère and Arago, William Sturgeon constructed in 1825 an electro-magnet which could hold twenty times its own weight.
In 1821 Humphrey Davy found that the conductance ('conducting power') of a metal is proportional to its cross-section and inverse proportional to its length. Georg Wilhelm Ohm found in his *Die Galvanische Kette* (1826-1827) the linearity between the current through a conductor and the voltage applied to the conductor. In 1845 Kirchhoff formulated the current and the voltage laws (13.10, 13.11) named after him.
In 1812 Michael Faraday, a bookbinder journeyman interested in science applied for a position at the Royal Institution in London. Its director, Humphrey Davy accepted the application, hardly anticipating that he had accepted one of the greatest future experimentalists to his institute. (After Davy's death Faraday became director of the institute.) Shortly after Ørsted's discovery Faraday investigated the known experiments in electricity and magnetism, which he reviewed in his *Historical Sketch of Electro-Magnetism* (1821). Inspired by the influence i.e. the effect of a charge on charges on a conductor, he investigated, whether a current may excite a current on another circuit. He found that this happened when the current in the first circuit changed. This was the starting point for the law of induction. (1831)

When a politician asked Faraday, what his discoveries are worth, he answered 'Presently I do not know, but may be they can be taxed one day.' Well-known are also Faraday's investigations on electrolysis. Since he himself did not enjoy a classical education he asked William Wheeler, a philosopher and mathematician from Cambridge to help him choose appropriate termini. They introduced the names electrode, anode, cathode, ion, electrolysis which are still in use. Faraday discovered diamagnetism, too.
Faraday often used the concept of electric and magnetic field lines. He made them visible by plaster shavings and iron filings. These procedures were not new, but they were not popular with mathematical physicists in the succession of Newton who preferred the concept of long-distance action. Already Wilcke made electric field lines visible. Many experiments of Faraday on electrostatics were already performed by Wilcke. A survey of experiments of both physicists on the same topic is given in the *History of Physics* by Hoppe. The lines of magnetic force were already made visible by Niccolo Cabeo (1629) and by Petrus Peregrinus (1269). The reader should consider why electric and magnetic lines of forces can be made visible by prolate bodies of large dielectric constant and susceptibility, resp.
Faraday had a rather precise imagination of the magnetic field. He considered it as tubes of lines with the property that the product of magnitude and cross-section is constant, which corresponds to a divergency free field. He stated that the induced current is proportional to the number of field lines crossed by the circuit; we say today proportional to the change of the magnetic flux.
In 1890 the name electron was coined by Johnstone Stoney. Before also (today's) electrons were called ions.

30.e Electrodynamics and Waves

In 1845 Faraday observed that polarized light transversing glass changes its plain of polarization if a magnetic field is applied parallel to the ray. From this he conjectured that light is an electromagnetic phenomenon.
In order to obtain a unified theory of electromagnetism there were mainly two directions of effort. One started out from the law of induction and introduced the vector-potential **A**, the other stayed mainly with the theory of action on distance following Ampère's investigations and introduced velocity dependent forces.
The vector-potential was introduced on the basis of various considerations. In 1845/48 Franz Neumann found in that the voltage of induction could be expressed as the time-derivative of the integral $\oint d\mathbf{r}\cdot\mathbf{A}(\mathbf{r})$. In 1846 the vector potential was also introduced by Wilhelm Weber and William Thomson (Lord Kelvin) on the basis of other considerations which today are no longer that convincing. In 1857 Kirchhoff used it.
In 1848 Kirchhoff and in 1858 Riemann realized that the equations of forces for charges and currents differed by a factor which is the square of a velocity c. Two charges q_1 and q_2 at distance r exert the Coulomb force q_1q_2/r^2 on each other, two wires of length l at distance r ($r \ll l$) carrying currents I_1 and I_2 exert the force $kI_1I_2l/(c^2r)$ on each other with a number k, which may be determined by the reader. The determination of c showed that this velocity agreed well with that of light. In 1834 first measurements of the propagation of electricity were performed by Wheatstone, in 1849 by Fizeau and Gounelle, and in 1850 by Foucault. They yielded velocities which were larger or smaller by factors of two or three-half from the velocity of light. (That some velocities were larger then light velocity was only possible because some arrangements were not linear).
In 1851 the construction of cables under water for the transmission of electric signals began (Dover-Calais). In 1854 William Thomson (Kelvin) found that at sufficiently high frequencies a damped wave propagates with approximately constant velocity. Kirchhoff showed by calculation that the velocity for a circular cross-section agrees with the velocity c, which appears in the ratio of the forces between charges and currents. This value had been measured shortly before by Weber and Kohlrausch to 3.1×10^{10}cm/sec.
Finally it was Maxwell who succeeded due to his imagination and analytic facilities to present the equations of electrodynamics in closed form. Studying Faraday's *Experimental Researches* he had learned a lot and still maintained the necessary abstraction. In 1857 he wrote to Forbes that he was 'by no means as yet a convert to the views which Faraday maintained', but in 1858 he wrote about Faraday as 'the nucleus of everything electric since 1830.'
Maxwell still worked a lot using mechanical analogies when he considered the fields **B** and **D** as velocities of an incompressible fluid. In 1861 he realized that in the equation $\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j}$ the displacement current $\dot{D}/(4\pi)$ had to be added to **j**, so that conservation of charge was guaranteed. From these equations he found that the velocity of light in vacuum was given by the factor c appearing in the ratio between forces between charges and currents, which agreed very well with the measured ones. He concluded: 'We can scarcely avoid the interference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic

phenomena.' Maxwell's equations contained the potentials Φ and $\mathbf{A}$, where he used the gauge we call Coulomb gauge. In 1864 he presented the complete set of electrodynamic equations in his paper *On a Dynamical Theory of the Electromagnetic Field*. In 1871 his monograph *Treatise on Electricity and Magnetism* was published.
In 1867 Ludvig Valentin Lorenz published his theory of electromagnetism, which contained the displacement current and the expressions (21.14) and (21.15) for the retarded potentials in which he had used the gauge named after him. The paper was based on the potential theory of Franz Neumann. In 1858 Riemann had found the retarded potentials, too. However, his paper was published only in 1867 together with that of Lorenz. Much of what Ludvig Lorenz found, was later attributed to the Dutch Hendrick Lorentz who wrote comprehensive papers on electrodynamics. This might be also due to their nearly equal names as well as Maxwell's inadequate criticism (1868) 'From the assumptions of both these papers we may draw the conclusions, first, that action and reaction are not always equal and opposite, and second, that apparatus may be constructed to generate any amount of work from its resources.' Ironically Maxwell did not realize that the fields contained energy and momentum. The Lorentz-Lorenz relation (1880) which is equivalent to the Clausius-Mossotti relation (6.34) when one replaces ϵ by the square n^2 of the index of refraction goes back to both of them.
In his *Treatise on Electricity and Magnetism* Maxwell derived the stress tensor of an electromagnetic field. The Poynting vector as the current density of electromagnetic energy was found by Poynting (1884) and by Heaviside (1885). In 1893 J.J. Thomson finally found that electromagnetic momentum can be expressed by the Poynting vector.
In 1889 Heaviside gave the expression (1.17) for the force on a charge moving in a magnetic field. J.J. Thomson who investigated cathode rays, had given it as half this amount in 1881. In 1895 Lorentz gives the correct result in his treatise. Today it is called Lorentz force. Already in 1864 Maxwell gave the contribution $\mathbf{v} \times \mathbf{B}$ to the electromotive force in a moving body.
Already Wilcke (1758) and Faraday (1837) introduced the notion of polarization of an insulator. The idea that magnetization is related to atomic currents was already found in the work of Coulomb, Ampère and Thomson (Kelvin). This connection is not clearly stated in Maxwell's formulation. It is the merit of Lorentz that in 1895 he introduced in his *Elektronentheorie* the fields $\mathbf{E}$ and $\mathbf{B}$ as fundamental fields and clarified that $\mathbf{D}$ and $\mathbf{H}$ are due to polarization and magnetization. 'Seat of the electromagnetic field is the empty space. In this space there is only one electric and one magnetic field-vector. This field is generated by atomistic electronic charges, onto which the fields in turn act ponderomotorically. A connection of the electromagnetic field with the ponderable matter exists only since the electric elementary charges are rigidly tied to the atomistic building blocks of matter.' Lorentz was able to provide a clear cut between electrodynamics and the properties of condensed matter.
Alfred Liénard (1898) and Emil Wiechert (1900) determined the potentials of an arbitrarily moving point charge.
In 1873 Maxwell already realized that the magnetic field is invariant under a gauge transformation $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$. However, he did not consider the consequences for the scalar potential. In 1904 Lorentz gave the general gauge transformation.
Besides to Lorentz we are indebted to Henri Poincaré, Oliver Heaviside and Heinrich Hertz for working out Maxwell's theory more clearly so that it found a broad distribution.
In 1900 and 1903, resp., Larmor and Schwarzschild introduced the principle of least action for the combined system of the electromagnetic field and charged particles.
Since 1878 Michelson and collaborators determined the velocity of light with high precision. Finally Heinrich Hertz succeeded in 1886 to produce electromagnetic waves (Hertzscher Dipol) and to detect them, initially in the range of meters, later also shorter ones. In 1890 Wiener proved the wave-nature of light by reflecting it on a mirror and obtaining a periodic blackening of the photographic emulsion by the standing waves.

30.f Theory of Relativity

In order to determine the velocity of the earth against the postulated ether Michelson and Morley performed their experiment initially in 1887 with the negative result: No motion against the ether was detected. In 1889 Fitzgerald postulated that all material objects are contracted in their direction of motion against the ether. Lorentz gave an expression for this contraction in 1892 up to order v^2/c^2 (Lorentz contraction, subsection 23.b.β). Essential was Lorentz's observation that the assumption of an aether carried along with matter was wrong.
In 1887 Voigt realized that the homogeneous equation $\Box\Phi = 0$ with the d'Alembert operator $\Box$ (20.13) is form

invariant under a class of linear transformations of $\mathbf{x}$ and t. Larmor gives in his paper *Ether and Matter* written in 1898 and published in 1900 already the transformation (23.2). It is unknown whether this had an influence on Lorentz. Already in 1898 Poincaré expressed doubts on the concept of simultaneity. In 1899 Lorentz stated the transformation called after him, but with an undetermined scale factor, which corresponds to the factor f after eq. (23.14).

In 1904 Lorentz found that Maxwell's equations without charges and currents are invariant under the transformations (23.2) provided that the fields are transformed in an appropriate way (see section 25). In 1905 Poincaré realized that the charge and current densities could be transformed so that the full set of Maxwell's equations are invariant in form under Lorentz transformations (compare sections 24 and 25).

In 1905 Einstein without the knowledge of Lorentz's paper and simultaneously with Poincaré's work mentioned above formulated the theory of special relativity in a general and complete way. He realized that the idea of a constant velocity of light in all systems of inertia constitutes a reality which governs all physics including mechanics and not only electrodynamics and which has to replace Galileian invariance. The reason that it took so long to develop the theory of (special) relativity and to convince scientists that it describes the reality, is the role of time in this theory.

It was (and still is for some persons) difficult to accept that the idea of absolute (that is independent of the system of inertia) simultaneity has to be abandoned. More on the history can be found in *A. Pais, "Subtle is the Lord ..." Albert Einstein, Oxford University Press*. Another problem is that now the ether as a system of reference disappeared.

An elegant formulation of the four-dimensional space was introduced by Minkowski in 1908, which was considered by Einstein initially as superfluous, but later as very useful. Starting from the special theory of relativity, which acts in a planar space, Einstein developped the general theory of relativity assuming that gravitation yields a curved space.

30.g From Classical to Quantum Electrodynamics

In 1900 Max Planck derived an interpolation formula between the two limit cases of the energy distribution of a black body radiator as a function of the frequency radiated, namely the Rayleigh-Jeans law (1900-1905) for low frequencies and the Wien law (1896) for high frequencies, the Planck radiation law. It agreed excellently with the observations. A few months later he postulated that this can be explained by the fact that electromagnetic radiation of frequency $\nu = \omega/(2\pi)$ cannot have arbitary energies but only integer multiples of $h\nu$, where h is a new fundamental constant now called Planck constant. This quantization of energy was soon confirmed by the photoelectric effect: The kinetic energy of electrons emitted from the surface of a metal by means of light is independent of the intensity of light but depends on its frequency (Lenard 1902).

It took a quarter of a century from this observation to the quantum theory of electrodynamics. First the quantum theory for the particles which hitherto had been considered point masses had to be developed until it was possible to quantize the electromagnetic field (P.A.M. Dirac 1927, P. Jordan and W. Pauli, 1928; W. Heisenberg and W. Pauli, 1929; see e.g. W. Heitler, The Quantum Theory of Radiation).

31 Gravitational Time Dilatation

31.a Light Quantum in the Gravitational Field

Finally we will consider an effect of the general theory of relativity, which can be derived in an elementary way, namely, the different behaviour of clocks in a gravitational potential. The statement is that clocks at different distances from a massive body run differently fast, those at further distance faster, the closer ones slower. This is an effect which had been observed in the Hafele-Keating experiment. In this experiment cesium atomic beam clocks were carried in an airplane around the earth (J.C. Hafele and R. E. Keating, Science 177, 166 (1972)). In this experiment one can observe time dilatation due to different velocities of the airplanes with respect to the center of the earth; but the effect that clocks run differently in different gravitational potentials is of the same order of magnitude. We will now explain this second effect.
We give two explanations: The first one uses the conservation of energy. (Actually, the theorem of conservation of energy does not hold in generally in the general theory of relativity. If however, the space becomes sufficiently plane at large distances then it is still valid. Therefore we need not consider this objection.) If a body of mass m falls the height h in a field of gravitation of acceleration g, then it gains $\delta E = mgh$ of kinetic energy. This holds at least for masses of velocity $v \ll c$.
As a consequence, light quanta will gain energy in falling in the field of gravitation and they loose energy while climbing against the field. If this were not true, then one could construct a perpetuum mobile by letting particles and anti-particles falling in the gravitational field, and having them irradiated into light quanta. These could now move up and recombine to a particle anti-particle pair, where one could extract the gained potential energy from the system. Since the energies of all masses are changed by $\delta E = mgh = \frac{gh}{c^2}E$, the same has to hold for light quanta, that is we find for light quanta of energy $E = \hbar\omega$

$$\delta\omega = \frac{\delta E}{\hbar} = \frac{gh}{c^2}\frac{E}{\hbar} = \frac{gh}{c^2}\omega. \tag{31.1}$$

This loss of frequency while leaving a gravitational field is known as the red shift in a gravitational field. It can be measured by means of the Mössbauer effect. A loss of frequency at a height of about 20 m is already sufficient. Thus if we compare the course of two atomic clocks down and up at a difference of height h, then one observes that the frequency of the lower clock is smaller by $\delta\omega$. The upper clock is thus faster by a factor of

$$1 + \frac{\delta\omega}{\omega} = 1 + \frac{gh}{c^2}. \tag{31.2}$$

31.b Principle of Equivalence

The general theory of relativity does not make use of quantum theory, i.e. it does not use the relation $E = \hbar\omega$. Instead it uses the principle of equivalence. This principle says that a system of reference which moves freely in the gravitational field behaves like a system of inertia. Let us assume we consider a system which moves like a freely falling elevator. Let us assume the lower clock is at a certain time relative to the elevator at rest and radiates upwards with frequency ω. It takes the time $t = h/c$ until the light has arrived the upper clock. During that time the earth and the upper clock have gained the velocity $v = gt$ upwards as seen from the elevator. Thus an observer at the upper clock will observe a Doppler shift by the frequency $\delta\omega = \omega v/c$ (for the weak gravitational field we consider here it is sufficient to consider in subsection 25.e only the contribution linear in β). Thus we obtain the Doppler shift

$$\delta\omega = \frac{gh}{c^2}\omega, \tag{31.3}$$

which agrees with the result obtained above.
Now you may ask, how can one apply the principle of equivalence, if the gravitational field does not point everywhere in the same direction and is of the same strength. Then, indeed, the description becomes more complicated. Then the description can no longer be founded on a flat space, and one has to dig seriously into the general theory of relativity.

Appendices

A Connection between different Systems of Units

Besides the GAUSSian system of units a number of other cgs-systems is used as well as the SI-system (international system of units, GIORGI-system).
Whereas all electromagnetic quantities in the GAUSSIAN system are expressed in cm, g und s, the GIORGI-system uses besides the mechanical units m, kg und s two other units, A (ampere) und V (volt). They are not independent, but related by the unit of energy

$$1 \text{ kg m}^2 \text{ s}^{-2} = 1 \text{ J} = 1 \text{ W s} = 1 \text{ A V s}. \tag{A.1}$$

The conversion of some conventional systems of units can be described by three conversion factors ϵ_0, μ_0 and ψ. The factors ϵ_0 and μ_0 (known as the dielectric constant and permeability constant of the vacuum in the SI-system) and the interlinking factor

$$\gamma = c\sqrt{\epsilon_0\mu_0} \tag{A.2}$$

can carry dimensions whereas ψ is a dimensionless number. One distinguishes between rational systems $\psi = 4\pi$) and non-rational systems ($\psi = 1$). The conversion factors of some conventional systems of units are

System of Units	ϵ_0	μ_0	γ	ψ
GAUSSIAN	1	1	c	1
Electrostatic (esu)	1	c^{-2}	1	1
Electromagnetic (emu)	c^{-2}	1	1	1
HEAVISIDE-LORENTZ	1	1	c	4π
GIORGI (SI)	$(c^2\mu_0)^{-1}$	$\frac{4\pi}{10^7}\frac{\text{Vs}}{\text{Am}}$	1	4π

The field intensities are expressed in GAUSSian units by those of other systems (indicated by an asterisk) in the following way

$$\begin{aligned}
\mathbf{E} &= \sqrt{\psi\epsilon_0}\mathbf{E}^* && \text{analogously electric potential} \\
\mathbf{D} &= \sqrt{\psi/\epsilon_0}\mathbf{D}^* && \\
\mathbf{P} &= 1/\sqrt{\psi\epsilon_0}\mathbf{P}^* && \text{analogously charge, current and their densities, electric moments} \\
\mathbf{B} &= \sqrt{\psi/\mu_0}\mathbf{B}^* && \text{analogously vector potential, magnetic flux} \\
\mathbf{H} &= \sqrt{\psi\mu_0}\mathbf{H}^* && \\
\mathbf{M} &= \sqrt{\mu_0/\psi}\mathbf{M}^* && \text{analogously magnetic moments}
\end{aligned} \tag{A.3}$$

One has for the quantities connected with conductivity and resistance

$$\begin{aligned}
\sigma &= 1/(\psi\epsilon_0)\sigma^* && \text{analogously capacity} \\
R &= \psi\epsilon_0 R^* && \text{analogously inductance}
\end{aligned} \tag{A.4}$$

For the electric and magnetic susceptibilities one obtains

$$\chi = \chi^*/\psi. \tag{A.5}$$

We obtain the following equations for arbitrary systems of units (the * has now been removed): MAXWELL's equations in matter read now

$$\operatorname{curl}\mathbf{H} = \frac{1}{\gamma}(\dot{\mathbf{D}} + \frac{4\pi}{\psi}\mathbf{j}_{\mathrm{f}}), \tag{A.6}$$

$$\operatorname{div}\mathbf{D} = \frac{4\pi}{\psi}\rho_{\mathrm{f}}, \tag{A.7}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{\gamma}\dot{\mathbf{B}}, \tag{A.8}$$

$$\operatorname{div}\mathbf{B} = 0. \tag{A.9}$$

The material equations read

$$\mathbf{D} = \epsilon_0\mathbf{E} + \frac{4\pi}{\psi}\mathbf{P}, \tag{A.10}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \frac{4\pi}{\psi}\mathbf{M}. \tag{A.11}$$

For the LORENTZ force one obtains

$$\mathbf{K} = q(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{\gamma}) \tag{A.12}$$

For the energy density u and the POYNTING vector $\mathbf{S}$ on obtains

$$u = \frac{\psi}{4\pi}\int(\mathbf{E}\cdot d\mathbf{D} + \mathbf{H}\cdot d\mathbf{B}), \tag{A.13}$$

$$\mathbf{S} = \frac{\psi\gamma}{4\pi}\mathbf{E}\times\mathbf{H}. \tag{A.14}$$

Whereas in GAUSSian units all the fields **E**, **D**, **P**, **B**, **H** und **M** are measured in units

$$\sqrt{\mathrm{dyn}}/\mathrm{cm} = \sqrt{\mathrm{erg/cm^3}} \tag{A.15}$$

the GIORGI system measures **E** in V/m, **D** and **P** in $\mathrm{As/m^2}$, **B** in $\mathrm{Vs/m^2}$, **H** and **M** in A/m. Depending on the quantity 1 $\mathrm{dyn^{1/2}\ cm^{-1}}$ in units of the GAUSSIAN system corresponds to (analogously for the quantities listed in (A.3) and (A.4))

$$\mathbf{E} = 3\cdot 10^4\ \mathrm{V/m} \tag{A.16}$$

$$\mathbf{D} = 10^{-5}/(12\pi)\ \mathrm{As/m^2} \tag{A.17}$$

$$\mathbf{P} = 10^{-5}/3\ \mathrm{As/m^2} \tag{A.18}$$

$$\mathbf{B} = 10^{-4}\ \mathrm{Vs/m^2} \tag{A.19}$$

$$\mathbf{H} = 10^3/(4\pi)\ \mathrm{A/m} \tag{A.20}$$

$$\mathbf{M} = 10^3\ \mathrm{A/m}. \tag{A.21}$$

For resistors one has $c^{-1} \widehat{=} 30\Omega$. For precise calculations the factors 3 (including the 3 in $12 = 4\cdot 3$) are to be replaced by the factor 2.99792458. This number multiplied by 10^8 m/s is the speed of light.
There are special names for the following often used units in the GAUSSian and electromagnetic system

magnetic induction	1 $\mathrm{dyn^{1/2}\ cm^{-1}}$=1 G (Gauß)
magnetic field intensity	1 $\mathrm{dyn^{1/2}\ cm^{-1}}$=1 Oe (Oerstedt)
magnetic flux	1 $\mathrm{dyn^{1/2}\ cm}$ =1 Mx (Maxwell)

The following quantities besides Ampere and Volt have their own names in the SI-system:

charge	1 As =1 C (Coulomb)
resistance	1 V/A =1 Ω (Ohm)
conductance	1 A/V =1 S (Siemens)
capacitance	1 As/V =1 F (Farad)
inductivity	1 Vs/A =1 H (Henry)
magnetic flux	1 Vs =1 Wb (Weber)
magnetic induction	1 $\mathrm{Vs/m^2}$=1 T (Tesla).

Historically the international or SI system was derived from the electromagnetic system. Since the units of this system where inconveniently large or small one introduced as unit for the current 1 A = 10^{-1} $\text{dyn}^{1/2}$ and for the voltage 1 V = 10^{8} $\text{dyn}^{1/2}$ cm s^{-1}. Giorgi realized that changing to mks-units one obtains the relation (A.1). However, one changed also from non-rational to rational units.

B Formulae for Vector Calculus

The reader is asked to solve the exercises B.11, B.15, B.34-B.50 and the exercise after B.71 by her- or himself or to take the results from the script where they are used.

B.a Vector Algebra

B.a.α Summation Convention and Orthonormal Basis

We use the summation convention which says that summation is performed over all indices, which appear twice in a product. Therefore

$$\mathbf{a} = a_\alpha \mathbf{e}_\alpha \tag{B.1}$$

stands for

$$\mathbf{a} = \sum_{\alpha=1}^{3} a_\alpha \mathbf{e}_\alpha = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

In the following we assume that the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in (B.1) represent an orthonormal and space independent right-handed basis. Then a_1, a_2, a_3 are the components of the vector $\mathbf{a}$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

B.a.β Scalar Product

The scalar product is defined by

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = a_\alpha b_\alpha, \tag{B.2}$$

in particular we have

$$\mathbf{e}_\alpha\cdot\mathbf{e}_\beta = \delta_{\alpha,\beta} = \begin{cases} 1 \text{ for } \alpha=\beta \\ 0 \text{ for } \alpha\neq\beta \end{cases} \tag{B.3}$$

with the KRONECKER symbol $\delta_{\alpha,\beta}$ which is symmetric in its indices, and

$$\mathbf{a}\cdot\mathbf{e}_\alpha = a_\alpha. \tag{B.4}$$

B.a.γ Vector Product

The vector product is given by

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = \epsilon_{\alpha,\beta,\gamma} a_\alpha b_\beta \mathbf{e}_\gamma = (a_2b_3-a_3b_2)\mathbf{e}_1 + (a_3b_1-a_1b_3)\mathbf{e}_2 + (a_1b_2-a_2b_1)\mathbf{e}_3 \tag{B.5}$$

with the total antisymmetric LEVI-CIVITA symbol

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} +1 & \text{for } (\alpha,\beta,\gamma) = (1,2,3),(2,3,1),(3,1,2) \\ -1 & \text{for } (\alpha,\beta,\gamma) = (1,3,2),(2,1,3),(3,2,1) \\ 0 & \text{otherwise} \end{cases} \tag{B.6}$$

Using determinants it can be written

$$\epsilon_{\alpha,\beta,\gamma} = \begin{vmatrix} \delta_{\alpha,1} & \delta_{\beta,1} & \delta_{\gamma,1} \\ \delta_{\alpha,2} & \delta_{\beta,2} & \delta_{\gamma,2} \\ \delta_{\alpha,3} & \delta_{\beta,3} & \delta_{\gamma,3} \end{vmatrix}. \tag{B.7}$$

From (B.5) one obtains by multiplication with a_α, b_β and $\mathbf{e}_\gamma$ and summation

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & b_1 & \mathbf{e}_1 \\ a_2 & b_2 & \mathbf{e}_2 \\ a_3 & b_3 & \mathbf{e}_3 \end{vmatrix}. \tag{B.8}$$

In particular one obtains

$$\mathbf{a}\times\mathbf{a} = \mathbf{0} \tag{B.9}$$

and

$$\mathbf{e}_\alpha \times \mathbf{e}_\beta = \epsilon_{\alpha,\beta,\gamma}\mathbf{e}_\gamma. \tag{B.10}$$

Express the sum

$$\epsilon_{\alpha,\beta,\gamma}\epsilon_{\zeta,\eta,\gamma} = \tag{B.11}$$

by means of Kronecker deltas.

B.a.δ Multiple Products

For the scalar triple product one has

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \epsilon_{\alpha,\beta,\gamma}a_\alpha b_\beta c_\gamma = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{B.12}$$

One has

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{b},\mathbf{a},\mathbf{c}] = -[\mathbf{c},\mathbf{b},\mathbf{a}]. \tag{B.13}$$

For the vector triple product one has

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{ac})\mathbf{b} - (\mathbf{ab})\mathbf{c}. \tag{B.14}$$

Express the quadruple product

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \tag{B.15}$$

by means of (B.11) or (B.14) in terms of scalar products.

B.b Vector Analysis

B.b.α Differentiation in Space, Del Operator

Differentiation in space is performed by means of the del-Operator ∇. It is a differential operator with vector properties. In cartesian coordinates it is written

$$\nabla = \mathbf{e}_\alpha\partial_\alpha, \tag{B.16}$$

where ∂_α stands for $\partial/\partial x_\alpha$. One calls

$$\nabla\Phi(\mathbf{r}) = \mathbf{e}_\alpha\partial_\alpha\Phi(\mathbf{r}) = \operatorname{grad}\Phi(\mathbf{r}) \tag{B.17}$$

the gradient,

$$(\mathbf{b}(\mathbf{r})\nabla)\mathbf{a}(\mathbf{r}) = b_\alpha(\mathbf{r})\partial_\alpha\mathbf{a}(\mathbf{r}) = (\mathbf{b}(\mathbf{r})\operatorname{grad})\mathbf{a}(\mathbf{r}) \tag{B.18}$$

the vector gradient

$$\nabla\mathbf{a}(\mathbf{r}) = \partial_\alpha a_\alpha(\mathbf{r}) = \operatorname{div}\mathbf{a}(\mathbf{r}) \tag{B.19}$$

the divergence and

$$\nabla\times\mathbf{a}(\mathbf{r}) = (\mathbf{e}_\alpha\times\mathbf{e}_\beta)\partial_\alpha a_\beta(\mathbf{r}) = \epsilon_{\alpha,\beta,\gamma}\partial_\alpha a_\beta(\mathbf{r})\mathbf{e}_\gamma = \operatorname{curl}\mathbf{a}(\mathbf{r}) \tag{B.20}$$

the curl.

B.b.β Second Derivatives, Laplacian

As far as differentiations do commute one has

$$\nabla \times \nabla = \mathbf{0}, \tag{B.21}$$

from which

$$\begin{aligned} \text{curl grad}\,\Phi(\mathbf{r}) &= \mathbf{0}, && \text{(B.22)} \\ \text{div curl}\,\mathbf{a}(\mathbf{r}) &= 0 && \text{(B.23)} \end{aligned}$$

follows. The scalar product

$$\nabla \cdot \nabla = \partial_\alpha \partial_\alpha = \triangle \tag{B.24}$$

is called the Laplacian. Therefore one has

$$\text{div grad}\,\Phi(\mathbf{r}) = \triangle\Phi(\mathbf{r}). \tag{B.25}$$

One obtains

$$\triangle\mathbf{a}(\mathbf{r}) = \text{grad div}\,\mathbf{a}(\mathbf{r}) - \text{curl curl}\,\mathbf{a}(\mathbf{r}), \tag{B.26}$$

by replacing **a** and **b** by ∇ in (B.14) and bringing the vector **c** always to the right.

B.b.γ Derivatives of Products

Application of the del operator onto a product of two factors yields according to the product rule two contributions. In one contribution one differentiates the first factor and keeps the second one constant, in the other contribution one differentiates the second factor and keeps the first constant. Then the expressions have to be rearranged, so that the constant factors are to the left, those to be differentiated to the right of the del operator. In doing this one has to keep the vector character of the del in mind. Then one obtains

$$\begin{aligned} \text{grad}\,(\Phi\Psi) &= \Phi\,\text{grad}\,\Psi + \Psi\,\text{grad}\,\Phi && \text{(B.27)} \\ \text{div}\,(\Phi\mathbf{a}) &= \Phi\,\text{div}\,\mathbf{a} + \mathbf{a}\cdot\,\text{grad}\,\Phi && \text{(B.28)} \\ \text{curl}\,(\Phi\mathbf{a}) &= \Phi\,\text{curl}\,\mathbf{a} + (\text{grad}\,\Phi)\times\mathbf{a} && \text{(B.29)} \\ \text{div}\,(\mathbf{a}\times\mathbf{b}) &= \mathbf{b}\cdot\,\text{curl}\,\mathbf{a} - \mathbf{a}\cdot\,\text{curl}\,\mathbf{b} && \text{(B.30)} \\ \text{curl}\,(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}\,\text{div}\,\mathbf{b} - \mathbf{b}\,\text{div}\,\mathbf{a} + (\mathbf{b}\,\text{grad}\,)\mathbf{a} - (\mathbf{a}\,\text{grad}\,)\mathbf{b} && \text{(B.31)} \\ \text{grad}\,(\mathbf{a}\cdot\mathbf{b}) &= \mathbf{a}\times\,\text{curl}\,\mathbf{b} + \mathbf{b}\times\,\text{curl}\,\mathbf{a} + (\mathbf{b}\,\text{grad}\,)\mathbf{a} + (\mathbf{a}\,\text{grad}\,)\mathbf{b} && \text{(B.32)} \\ \triangle(\Phi\Psi) &= \Phi\triangle\Psi + \Psi\triangle\Phi + 2(\,\text{grad}\,\Phi)\cdot(\,\text{grad}\,\Psi). && \text{(B.33)} \end{aligned}$$

B.c Special Expressions

Calculate for $r = |\mathbf{r}|$ and constant vector **c**

$$\begin{aligned} \text{grad}\,r^2 &= && \text{(B.34)} \\ \text{div}\,\mathbf{r} &= && \text{(B.35)} \\ \text{curl}\,\mathbf{r} &= && \text{(B.36)} \\ \text{grad}\,(\mathbf{c}\cdot\mathbf{r}) &= && \text{(B.37)} \\ (\mathbf{c}\,\text{grad}\,)\mathbf{r} &= && \text{(B.38)} \\ \text{grad}\,f(r) &= && \text{(B.39)} \\ \text{div}\,(\mathbf{c}\times\mathbf{r}) &= && \text{(B.40)} \\ \text{curl}\,(\mathbf{c}\times\mathbf{r}) &= && \text{(B.41)} \\ \text{grad}\,\frac{1}{r} &= && \text{(B.42)} \end{aligned}$$

$$\operatorname{div} \frac{\mathbf{c}}{r} = \tag{B.43}$$

$$\operatorname{curl} \frac{\mathbf{c}}{r} = \tag{B.44}$$

$$\operatorname{div} \frac{\mathbf{r}}{r^3} = \tag{B.45}$$

$$\operatorname{curl} \frac{\mathbf{r}}{r^3} = \tag{B.46}$$

$$\operatorname{grad} \frac{\mathbf{c} \cdot \mathbf{r}}{r^3} = \tag{B.47}$$

$$\operatorname{div} \frac{\mathbf{c} \times \mathbf{r}}{r^3} = \tag{B.48}$$

$$\operatorname{curl} \frac{\mathbf{c} \times \mathbf{r}}{r^3} = \tag{B.49}$$

$$\operatorname{grad} \frac{1}{|\mathbf{r} - \mathbf{c}|} = \quad , \tag{B.50}$$

with the exception of singular points.

B.d Integral Theorems

B.d.α Line Integrals

For a scalar or a vector field $A(\mathbf{r})$ one has

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} (\mathrm{d}\mathbf{r}\nabla) A(\mathbf{r}) = A(\mathbf{r}_2) - A(\mathbf{r}_1), \tag{B.51}$$

that is

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{d}\mathbf{r} \operatorname{grad} \Phi(\mathbf{r}) = \Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1), \tag{B.52}$$

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} (\mathrm{d}\mathbf{r} \operatorname{grad}) \mathbf{a}(\mathbf{r}) = \mathbf{a}(\mathbf{r}_2) - \mathbf{a}(\mathbf{r}_1). \tag{B.53}$$

B.d.β Surface Integrals

According to Stokes a surface integral over F of the form

$$\int_F (\mathrm{d}\mathbf{f} \times \nabla) A(\mathbf{r}) = \oint_{\partial F} \mathrm{d}\mathbf{r} A(\mathbf{r}) \tag{B.54}$$

can be rewritten as a line integral over the curve ∂F bounding the surface. The direction is given by the right-hand rule; that is, if the thumb of your right hand points in the direction of d**f**, your fingers curve in the direction d**r** of the line integral,

$$\int_F \mathrm{d}\mathbf{f} \times \operatorname{grad} \Phi(\mathbf{r}) = \oint_{\partial F} \mathrm{d}\mathbf{r} \Phi(\mathbf{r}), \tag{B.55}$$

$$\int_F \mathrm{d}\mathbf{f} \cdot \operatorname{curl} \mathbf{a}(\mathbf{r}) = \oint_{\partial F} \mathrm{d}\mathbf{r} \cdot \mathbf{a}(\mathbf{r}). \tag{B.56}$$

B.d.γ Volume Integrals

According to Gauss a volume integral of the form

$$\int_V \mathrm{d}^3r \nabla A(\mathbf{r}) = \int_{\partial V} \mathrm{d}\mathbf{f} A(\mathbf{r}) \tag{B.57}$$

can be converted into an integral over the surface ∂V of the volume. The vector $\mathrm{d}\mathbf{f}$ points out of the volume. In particular one has

$$\int_V \mathrm{d}^3r \,\mathrm{grad}\,\Phi(\mathbf{r}) = \int_{\partial V} \mathrm{d}\mathbf{f}\Phi(\mathbf{r}), \tag{B.58}$$

$$\int_V \mathrm{d}^3r \,\mathrm{div}\,\mathbf{a}(\mathbf{r}) = \int_{\partial V} \mathrm{d}\mathbf{f}\cdot\mathbf{a}(\mathbf{r}), \tag{B.59}$$

$$\int_V \mathrm{d}^3r \,\mathrm{curl}\,\mathbf{a}(\mathbf{r}) = \int_{\partial V} \mathrm{d}\mathbf{f}\times\mathbf{a}(\mathbf{r}). \tag{B.60}$$

B.d.δ Volume Integrals of Products

If one substitutes products for $\Phi(\mathbf{r})$ or $\mathbf{a}(\mathbf{r})$ in equations (B.58-B.60) and applies equations (B.27-B.30), then one obtains

$$\int_V \mathrm{d}^3r\Phi(\mathbf{r})\,\mathrm{grad}\,\Psi(\mathbf{r}) + \int_V \mathrm{d}^3r\Psi(\mathbf{r})\,\mathrm{grad}\,\Phi(\mathbf{r}) = \int_{\partial V}\mathrm{d}\mathbf{f}\Phi(\mathbf{r})\Psi(\mathbf{r}), \tag{B.61}$$

$$\int_V \mathrm{d}^3r\Phi(\mathbf{r})\,\mathrm{div}\,\mathbf{a}(\mathbf{r}) + \int_V \mathrm{d}^3r\mathbf{a}(\mathbf{r})\cdot\,\mathrm{grad}\,\Phi(\mathbf{r}) = \int_{\partial V}\mathrm{d}\mathbf{f}\cdot\mathbf{a}(\mathbf{r})\Phi(\mathbf{r}), \tag{B.62}$$

$$\int_V \mathrm{d}^3r\Phi(\mathbf{r})\,\mathrm{curl}\,\mathbf{a}(\mathbf{r}) + \int_V \mathrm{d}^3r(\,\mathrm{grad}\,\Phi(\mathbf{r}))\times\mathbf{a}(\mathbf{r}) = \int_{\partial V}\mathrm{d}\mathbf{f}\times\mathbf{a}(\mathbf{r})\Phi(\mathbf{r}), \tag{B.63}$$

$$\int_V \mathrm{d}^3r\mathbf{b}(\mathbf{r})\cdot\,\mathrm{curl}\,\mathbf{a}(\mathbf{r}) - \int_V \mathrm{d}^3r\mathbf{a}(\mathbf{r})\cdot\,\mathrm{curl}\,\mathbf{b}(\mathbf{r}) = \int_{\partial V}\mathrm{d}\mathbf{f}\cdot(\mathbf{a}(\mathbf{r})\times\mathbf{b}(\mathbf{r})). \tag{B.64}$$

These equations allow the transformation of a volume integral into another one and a surface integral. This is the generalization of integration by parts from one dimension to three. In many cases the surface integral vanishes in the limit of infinite volume, so that the equations (B.61-B.64) allow the conversion from one volume integral into another one.
If one replaces $\mathbf{a}(\mathbf{r})$ in (B.62) by $\mathrm{curl}\,\mathbf{a}(\mathbf{r})$ or $\mathbf{b}(\mathbf{r})$ in (B.64) by $\mathrm{grad}\,\Phi(\mathbf{r})$, then one obtains with (B.22) and (B.23)

$$\int_V \mathrm{d}^3r\,\mathrm{grad}\,\Phi(\mathbf{r})\cdot\,\mathrm{curl}\,\mathbf{a}(\mathbf{r}) = \int_{\partial V}\mathrm{d}\mathbf{f}\cdot(\mathbf{a}(\mathbf{r})\times\,\mathrm{grad}\,\Phi(\mathbf{r})) = \int_{\partial V}\mathrm{d}\mathbf{f}\cdot(\Phi(\mathbf{r})\,\mathrm{curl}\,\mathbf{a}(\mathbf{r})). \tag{B.65}$$

Similarly one obtains from (B.63)

$$\int_V \mathrm{d}^3r\,\mathrm{grad}\,\Phi(\mathbf{r})\times\,\mathrm{grad}\,\Psi(\mathbf{r}) = \int_{\partial V}\mathrm{d}\mathbf{f}\times(\,\mathrm{grad}\,\Psi(\mathbf{r}))\Phi(\mathbf{r}) = -\int_{\partial V}\mathrm{d}\mathbf{f}\times(\,\mathrm{grad}\,\Phi(\mathbf{r}))\Psi(\mathbf{r}). \tag{B.66}$$

If one replaces $\mathbf{a}(\mathbf{r})$ in (B.59) by $\Phi\,\mathrm{grad}\,\Psi - \Psi\,\mathrm{grad}\,\Phi$, then one obtains Green's theorem

$$\int_V \mathrm{d}^3r(\Phi(\mathbf{r})\triangle\Psi(\mathbf{r}) - \Psi(\mathbf{r})\triangle\Phi(\mathbf{r})) = \int_{\partial V}\mathrm{d}\mathbf{f}\cdot(\Phi(\mathbf{r})\,\mathrm{grad}\,\Psi(\mathbf{r}) - \Psi(\mathbf{r})\,\mathrm{grad}\,\Phi(\mathbf{r})). \tag{B.67}$$

B.e The Laplacian of $1/r$ and Related Expressions

B.e.α The Laplacian of $1/r$

For $r \neq 0$ one finds $\triangle(1/r) = 0$. If one evaluates the integral over a sphere of radius R by use of (B.59),

$$\int \triangle(\frac{1}{r})\mathrm{d}^3r = \int \mathrm{d}\mathbf{f} \cdot \operatorname{grad}(\frac{1}{r}) = -\int \mathbf{r} r \mathrm{d}\Omega \cdot \frac{\mathbf{r}}{r^3} = -4\pi \tag{B.68}$$

with the solid-angle element $\mathrm{d}\Omega$, then one obtains -4π. Therefore one writes

$$\triangle(\frac{1}{r}) = -4\pi\delta^3(\mathbf{r}), \tag{B.69}$$

where DIRAC's delta "function" $\delta^3(\mathbf{r})$ (actually a distribution) has the property

$$\int_V \mathrm{d}^3r f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{r}_0) = \begin{cases} f(\mathbf{r}_0) & \text{if } \mathbf{r}_0 \in V \\ 0 & \text{otherwise.} \end{cases} \tag{B.70}$$

From

$$\triangle \frac{\mathbf{c}}{|\mathbf{r}-\mathbf{r}'|} = \mathbf{c}\triangle \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\mathbf{c}\delta^3(\mathbf{r}-\mathbf{r}')$$

one obtains with (B.26,B.43,B.44)

$$4\pi\mathbf{c}\delta^3(\mathbf{r}-\mathbf{r}') = -\operatorname{grad}\operatorname{div}\frac{\mathbf{c}}{|\mathbf{r}-\mathbf{r}'|} + \operatorname{curl}\operatorname{curl}\frac{\mathbf{c}}{|\mathbf{r}-\mathbf{r}'|} = \operatorname{grad}\frac{\mathbf{c}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \operatorname{curl}\frac{\mathbf{c}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{B.71}$$

Determine the δ-function contributions in (B.45) to (B.49). What is the dimension of $\delta^3(\mathbf{r})$?

B.e.β Representation of a Vector Field as a Sum of an Irrotational and a Divergence-free Field

We rewrite the vector field $\mathbf{a}(\mathbf{r})$ in the form

$$\mathbf{a}(\mathbf{r}) = \int \mathrm{d}^3r' \mathbf{a}(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}') \tag{B.72}$$

and obtain from (B.71), since $\mathbf{a}(\mathbf{r}')$ does not depend on $\mathbf{r}$

$$\mathbf{a}(\mathbf{r}) = \frac{1}{4\pi}\int \mathrm{d}^3r' \operatorname{grad}\frac{\mathbf{a}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{1}{4\pi}\int \mathrm{d}^3r' \operatorname{curl}\frac{\mathbf{a}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{B.73}$$

which may be written as

$$\mathbf{a}(\mathbf{r}) = -\operatorname{grad}\Phi(\mathbf{r}) + \operatorname{curl}\mathbf{A}(\mathbf{r}) \tag{B.74}$$

with

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\int \mathrm{d}^3r' \frac{\mathbf{a}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{B.75}$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int \mathrm{d}^3r' \frac{\mathbf{a}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{B.76}$$

If the integrals (B.75) and (B.76) exist, then one obtains in this way a representation of $\mathbf{a}(\mathbf{r})$ as sum of the irrotational field $-\operatorname{grad}\Phi(\mathbf{r})$ and the divergence free field $\operatorname{curl}\mathbf{A}(\mathbf{r})$. With (B.48) one finds

$$\operatorname{div}\mathbf{A}(\mathbf{r}) = 0. \tag{B.77}$$

C Spherical Harmonics

C.a Eigenvalue Problem and Separation of Variables

We are looking for the eigen functions Y of

$$\Delta_\Omega Y(\theta,\phi) = \lambda Y(\theta,\phi) \tag{C.1}$$

with

$$\Delta_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{C.2}$$

where the operators (multiplication by functions and differentiation) apply from right to left (compare 5.16). One introduces the ansatz

$$Y = g(\cos\theta)h(\phi). \tag{C.3}$$

With

$$\xi = \cos\theta, \quad \frac{dg}{d\theta} = -\sin\theta\frac{dg}{d\cos\theta} = -\sqrt{1-\xi^2}\frac{dg}{d\xi} \tag{C.4}$$

one obtains by insertion into the eigenvalue equation and division by $h(\phi)$

$$\frac{d}{d\xi}((1-\xi^2)\frac{dg}{d\xi}) + \frac{g(\xi)}{1-\xi^2}(\frac{d^2h(\phi)}{d\phi^2}/h(\phi)) = \lambda g(\xi). \tag{C.5}$$

This equation can only be fulfilled, if $d^2h(\phi)/d\phi^2/h(\phi)$ is constant. Since moreover one requires $h(\phi+2\pi) = h(\phi)$, it follows that

$$h(\phi) = e^{im\phi} \text{ with integer } m. \tag{C.6}$$

This reduces the differential equation for g to

$$\frac{d}{d\xi}((1-\xi^2)\frac{dg}{d\xi}) - \frac{m^2 g(\xi)}{1-\xi^2} = \lambda g(\xi). \tag{C.7}$$

C.b Associated Legendre Functions

Considering that (at least for positive m) the factor $e^{im\phi}$ comes from the analytic function $(x+iy)^m = r^m(\sin\theta)^m e^{im\phi}$, it seems appropriate to extract a factor $(\sin\theta)^m$ out of g

$$g(\xi) = (\sin\theta)^m G(\xi) = (1-\xi^2)^{m/2}G(\xi), \tag{C.8}$$

so that one obtains the equation

$$-m(m+1)G(\xi) - 2(m+1)\xi G'(\xi) + (1-\xi^2)G'' = \lambda G(\xi) \tag{C.9}$$

for G.
For G we may assume a Taylor expansion

$$G(\xi) = \sum_k a_k\xi^k, \quad G'(\xi) = \sum_k k a_k \xi^{k-1}, \quad G''(\xi) = \sum_k k(k-1)a_k\xi^{k-2} \tag{C.10}$$

and find by comparison of the coefficients

$$[m(m+1) + 2(m+1)k + k(k-1) + \lambda]a_k = (k+2)(k+1)a_{k+2}. \tag{C.11}$$

If we put

$$\lambda = -l(l+1), \tag{C.12}$$

then the recurrence formula reads

$$\frac{a_{k+2}}{a_k} = \frac{(m+k+l+1)(m+k-l)}{(k+1)(k+2)}. \tag{C.13}$$

The series expansion comes to an end at finite k, if the numerator vanishes, in particular for integer not negative $k = l - m$. We continue to investigate this case. Without closer consideration we mention that in the other cases the function Y develops a nonanalyticity at $\cos\theta = \pm 1$.
The leading term has then the coefficient a_{l-m}. Application of the recurrence formula yields

$$\begin{aligned}
a_{l-m-2} &= -\frac{(l-m)(l-m-1)}{(2l-1)2} a_{l-m} \\
&= -\frac{(l-m)(l-m-1)l}{(2l-1)2l} a_{l-m}, && \text{(C.14)} \\
a_{l-m-4} &= \frac{(l-m)(l-m-1)(l-m-2)(l-m-3)}{(2l-1)(2l-3)2\cdot 4} a_{l-m} \\
&= \frac{(l-m)(l-m-1)(l-m-2)(l-m-3)l(l-1)}{(2l-1)(2l-3)2l(2l-2)2} a_{l-m}, && \text{(C.15)} \\
a_{l-m-2k} &= (-)^k \frac{(l-m)!l!(2l-2k)!}{(l-m-2k)!(l-k)!(2l)!k!} a_{l-m}. && \text{(C.16)}
\end{aligned}$$

Conventionally one chooses

$$a_{l-m} = \frac{(-)^m (2l)!}{(l-m)!2^l l!}. \tag{C.17}$$

Then it follows that

$$\begin{aligned}
G(\xi) &= \frac{(-)^m}{2^l l!} \sum_k \frac{(2l-2k)!}{(l-m-2k)!} \frac{l!}{k!(l-k)!} (-)^k \xi^{l-m-2k} && \text{(C.18)} \\
&= \frac{(-)^m}{2^l l!} \sum_k \binom{l}{k} (-)^k \frac{d^{l+m}\xi^{2l-2k}}{d\xi^{l+m}} = \frac{(-)^m}{2^l l!} \frac{d^{l+m}(\xi^2-1)^l}{d\xi^{l+m}}. && \text{(C.19)}
\end{aligned}$$

The solutions $g(\xi)$ in the form

$$P_l^m(\xi) = (1-\xi^2)^{m/2} \frac{(-)^m}{2^l l!} \frac{d^{l+m}}{d\xi^{l+m}} (\xi^2-1)^l \tag{C.20}$$

are called associated LEGENDRE functions. $Y_{lm}(\theta, \phi)$ is given by $P_l^m(\cos\theta)e^{im\phi}$ apart from the normalization.
The differential equation for g depends only on m^2, but not on the sign of m. Therefore we compare P_l^m and P_l^{-m}. Be $m \geq 0$, then it follows that

$$\begin{aligned}
\frac{d^{l-m}}{d\xi^{l-m}}(\xi^2-1)^l &= \sum_{k=0}^{l-m} \binom{l-m}{k} \frac{d^k(\xi-1)^l}{d\xi^k} \frac{d^{l-m-k}(\xi+1)^l}{d\xi^{l-m-k}} \\
&= \sum_{k=0}^{l-m} \frac{(l-m)!l!l!}{k!(l-m-k)!(l-k)!(m+k)!} (\xi-1)^{l-k}(\xi+1)^{m+k}, && \text{(C.21)} \\
\frac{d^{l+m}}{d\xi^{l+m}}(\xi^2-1)^l &= \sum_{k=0}^{l-m} \binom{l+m}{k+m} \frac{d^{m+k}(\xi-1)^l}{d\xi^{m+k}} \frac{d^{l-k}(\xi+1)^l}{d\xi^{l-k}} \\
&= \sum_{k=0}^{l-m} \frac{(l+m)!l!l!}{(m+k)!(l-k)!(l-m-k)!k!} (\xi-1)^{l-k-m}(\xi+1)^k. && \text{(C.22)}
\end{aligned}$$

Comparison shows

$$P_l^{-m}(\xi) = \frac{(l-m)!}{(l+m)!} (-)^m P_l^m(\xi), \tag{C.23}$$

that is, apart from the normalization both solutions agree.

C.c Orthogonality and Normalization

We consider the normalization integral

$$N_{lml'm'} = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta P_l^m(\cos\theta)e^{-im\phi} P_{l'}^{m'}(\cos\theta)e^{im'\phi}. \tag{C.24}$$

The integration over ϕ yields

$$\begin{aligned} N_{lml'm'} &= 2\pi\delta_{mm'} \int_{-1}^{+1} P_l^m(\xi)P_{l'}^m(\xi)d\xi \\ &= 2\pi\delta_{mm'}(-)^m \frac{(l'+m)!}{(l'-m)!} \int_{-1}^{+1} P_l^m(\xi)P_{l'}^{-m}(\xi)d\xi \\ &= \frac{2\pi(l'+m)!}{(l'-m)!} \frac{\delta_{mm'}}{2^{2l}l!^2} I_m^{ll'} \end{aligned} \tag{C.25}$$

with

$$I_m^{ll'} = (-)^m \int_{-1}^{+1} \frac{d^{l+m}(\xi^2-1)^l}{d\xi^{l+m}} \frac{d^{l'-m}(\xi^2-1)^{l'}}{d\xi^{l'-m}} d\xi. \tag{C.26}$$

Partial integration yields

$$I_m^{ll'} = (-)^m \left[\frac{d^{l+m}(\xi^2-1)^l}{d\xi^{l+m}} \frac{d^{l'-m-1}(\xi^2-1)^{l'}}{d\xi^{l'-m-1}} \right]_{-1}^{+1} + I_{m+1}^{ll'}. \tag{C.27}$$

The first factor in square brackets contains at least $-m$, the second $m+1$ zeroes at $\xi = \pm 1$. The contents in square brackets vanishes therefore. Thus $I_m^{ll'}$ is independent of m for $-l \le m \le l'$. For $l' > l$ it follows that $I_m^{ll'} = I_{l'}^{ll'} = 0$, since the first factor of the integrand of $I_{l'}^{ll'}$ vanishes. For $l' < l$ one obtains $I_m^{ll'} = I_{-l}^{ll'} = 0$, since the second factor of the integrand of $I_{-l}^{ll'}$ vanishes. For $l = l'$ we evaluate

$$I_m^{ll} = I_l^{ll} = (-)^l \int_{-1}^{+1} \frac{d^{2l}(\xi^2-1)^l}{d\xi^{2l}} (\xi^2-1)^l d\xi. \tag{C.28}$$

The first factor in the integrand is the constant $(2l)!$

$$I_m^{ll} = (2l)! \int_{-1}^{+1} (1-\xi^2)^l d\xi. \tag{C.29}$$

The last integral yields $2^{2l+1}l!^2/(2l+1)!$ (one obtains this by writing the integrand $(1+\xi)^l(1-\xi)^l$ and performing partial integration l times, by always differentiating the power of $1-\xi$ and integrating that of $1+\xi$. This yields the norm

$$N_{lml'm'} = 2\pi \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{ll'}\delta_{m,m'}. \tag{C.30}$$

Thus the normalized spherical harmonics read

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{im\phi}. \tag{C.31}$$

C.d Remark on Completeness

If we expand a function f which is analytic in the Cartesian coordinates x, y, z in the vicinity of the origin in a Taylor expansion

$$f(\mathbf{r}) = \sum_{ijk} a_{ijk} x^i y^j z^k = \sum_n r^n f_n(\theta,\phi), \tag{C.32}$$

then the contributions proportional to r^n are contained in those with $i + j + k = n$. These are in total $(n + 1) + n + (n - 1) + ... = (n + 2)(n + 1)/2$ terms

$$f_n(\theta, \phi) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} a_{n-j-k,j,k} (\frac{x}{r})^{n-j-k} (\frac{y}{r})^j (\frac{z}{r})^k. \tag{C.33}$$

On the other hand we may represent the function f_n equally well by the functions $Y_{lm}(\theta, \phi) = \sqrt{...} P_l^{|m|}(\cos\theta) e^{im\phi}$, since they can be written $(\sin\theta)^{|m|} e^{im\phi} = ((x \pm iy)/r)^{|m|}$ multiplied by a polynomial in $\cos\theta$ of order $l - |m|$. The appearing powers of the $\cos\theta$ can be written $(\cos\theta)^{l-|m|-2k} = (z/r)^{l-|m|-2k}((x^2 + y^2 + z^2)/r^2)^k$. In addition we introduce a factor $((x^2 + y^2 + z^2)/r^2)^{(n-l)/2}$. Then we obtain contributions for $l = n, n - 2, n - 4, ...$ Since m runs from $-l$ to l, one obtains in total $(2n + 1) + (2n - 3) + (2n - 7) + ... = (n + 2)(n + 1)/2$ linearly independent (since orthogonal) contributions. Therefore the space of these functions has the same dimension as that of the f_n's. Thus we may express each f_n as a linear combination of the spherical harmonics.

Name Index

Subject Index

Contents Electrodynamics