H LORENTZ Invariance of Electrodynamics

23 LORENTZ Transformation

23.a GALILEI and LORENTZ Transformation

The equations of NEWTON's mechanics are invariant under the GALILEI transformation (GALILEI invariance)

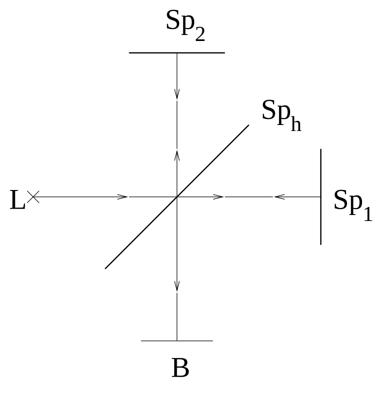
$$x' = x, \quad y' = y, \quad z' = z - vt, \quad t' = t. \tag{23.1}$$

We will see in the following that MAXWELL's equations are invariant under appropriate transformations of fields, currents and charges against linear transformations of the coordinates x, y, z, and t, which leave the velocity of light invariant (LORENTZ invariance). Such a transformation reads

$$x' = x, \quad y' = y, \quad z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{vz}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{23.2}$$

Consider two charges q und $-q$, which are for $t \le 0$ at the same point and which are also for $t \ge \Delta t$ at the same point, which move however in the time interval $0 < t < \Delta t$ against each other. They separate at time 0 at $\mathbf{r}_0$ and they meet again at time Δt at $\mathbf{r}_1$. They generate according to (21.14) and (21.15) a field, which propagates with light-velocity. It is different from zero at point $\mathbf{r}$ at time t only, if $t > |\mathbf{r} - \mathbf{r}_0|/c$ and $t < \Delta t + |\mathbf{r} - \mathbf{r}_1|/c$ holds. This should hold independently of the system of inertia in which we consider the wave. (We need only assume that the charges do not move faster than with light-velocity.) If we choose an infinitesimal Δt then the light flash arrives at time $t = |\mathbf{r} - \mathbf{r}_0|/c$, since it propagates with light-velocity. Since the LORENTZ transformation is not in agreement with the laws of NEWTON's mechanics and the GALILEI transformation not with MAXWELL's equations (in a moving inertial frame light would have a velocity dependent on the direction of light-propagation) the question arises which of the three following possibilities is realized in nature:
(i) there is a preferred system of inertia for electrodynamics, in which MAXWELL's equations hold (ether-hypothesis),
(ii) NEWTON's mechanics has to be modified
(iii) MAXWELL's equations have to be modified.

The decision can only be made experimentally: An essential experiment to refute (i) is the MICHELSON-MORLEY experiment: A light beam hits a half-transparent mirror Sp_h, is split into two beams, which are reflected at mirror Sp_1 and Sp_2, resp. at distance l and combined again at the half-transparent mirror. One observes the interference fringes of both beams at B. If the apparatus moves with velocity v in the direction of the mirror Sp_1, then the time t_1 the light needs to propagate from the half-transparent mirror to Sp_1 and back is



$$t_1 = \frac{l}{c - v} + \frac{l}{c + v} = \frac{2lc}{c^2 - v^2} = \frac{2l}{c}(1 + \frac{v^2}{c^2} + ...). \tag{23.3}$$

The time t_2 the light needs to the mirror Sp_2 is

$$t_2 = \frac{2l}{\sqrt{c^2 - v^2}} = \frac{2l}{c}(1 + \frac{v^2}{2c^2} + ...), \tag{23.4}$$

since the light velocity c has to be separated into the two components v and $\sqrt{c^2 - v^2}$. Thus there remains the time difference

$$t_1 - t_2 = \frac{lv^2}{c^3}, \tag{23.5}$$

which would be measurable by a displacement of the interference fringes, if for example the velocity v is the velocity of the earth against the sun. This displacement has not been observed. One may object that this is due to a drag of the ether by the earth. There are however many other experiments, which are all in agreement with LORENTZ invariance, i.e. the constancy of the velocity of light in vacuum independent of the system of inertia. The consequences in mechanics for particles with velocities comparable to the velocity of light in particular for elementary particles have confirmed LORENTZ invariance very well.

Development of the Theory of Relativity

In order to determine the velocity of the earth against the postulated ether MICHELSON and MORLEY performed their experiment initially in 1887 with the negative result: No motion against the ether was detected. In order to explain this FITZGERALD (1889) and LORENTZ (1892) postulated that all material objects are contracted in their direction of motion against the ether (compare LORENTZ contraction, subsection 23.b.β).

In the following we will develop the idea of a four-dimensional space-time, in which one may perform transformations similar to orthogonal transformations in three-dimensional space, to which we are used. However this space is not a EUCLIDean space, i.e. a space with definite metric. Instead space and time have a metric with different sign (see the metric tensor g, eq. 23.10). This space is also called MINKOWSKI space. We use the modern four-dimensional notation introduced by MINKOWSKI in 1908.

Starting from the basic ideas of special relativity

The laws of nature and the results of experiments in a system of inertia are independent of the motion of such a system as whole.

The velocity of light is the same in each system of inertia and independent of the velocity of the source

we will introduce the LORENTZ-invariant formulation of MAXWELL's equations and of relativistic mechanics.

23.b LORENTZ Transformation

We introduce the notation

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \tag{23.6}$$

or shortly

$$(x^\mu) = (ct, \mathbf{r}) \tag{23.7}$$

and denotes them as the contravariant components of the vector. Further one introduces

$$(x_\mu) = (ct, -\mathbf{r}). \tag{23.8}$$

which are called covariant components of the vector. Then we may write

$$x^\mu = g^{\mu\nu} x_\nu, \quad x_\mu = g_{\mu\nu} x^\nu \tag{23.9}$$

(summation convention)

$$(g^{\cdot\cdot}) = (g_{\cdot\cdot}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{23.10}$$

One calls g the metric tensor. Generally one has the rules for lifting and lowering of indices

$$C \cdot \cdot^{\mu} \cdot \cdot = g^{\mu\nu} C \cdot \cdot_{\nu} \cdot \cdot, \quad C \cdot \cdot_{\mu} \cdot \cdot = g_{\mu\nu} C \cdot \cdot^{\nu} \cdot \cdot \tag{23.11}$$

We introduce the convention: Indices κ, λ, μ, ν run from 0 to 3, indices α, β, γ, ... from 1 to 3. One observes that according to (23.11) $g_\mu{}^\nu = g_{\mu\kappa} g^{\kappa\nu} = \delta_\mu{}^\nu$, $g^\mu{}_\nu = g^{\mu\kappa} g_{\kappa\nu} = \delta^\mu{}_\nu$ with the KRONECKER delta.

If a light-flash is generated at time $t = 0$ at $\mathbf{r} = \mathbf{0}$, then its wave front is described by

$$s^2 = c^2 t^2 - \mathbf{r}^2 = x^\mu x_\mu = 0. \tag{23.12}$$

We denote the system described by the coordinates x^μ by S. Now we postulate with EINSTEIN: Light in vacuum propagates in each inertial system with the same velocity c. (principle of the constance of light velocity) Then the propagation of the light flash in the uniformly moving system S' whose origin agrees at $t = t' = 0$ with that of S is given by

$$s'^2 = x'^\mu x'_\mu = 0. \tag{23.13}$$

Requiring a homogeneous space-time continuum the transformation between x' and x has to be linear

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{23.14}$$

and $s'^2 = f s^2$ with some constant f has to hold. If we require that space is isotropic and no system of inertia is preferred, then $f = 1$ has to hold. The condition $s'^2 = s^2$ implies

$$s'^2 = x'^\mu x'_\mu = \Lambda^\mu{}_\nu x^\nu \Lambda_\mu{}^\kappa x_\kappa = s^2 = x^\nu \delta_\nu{}^\kappa x_\kappa, \tag{23.15}$$

which is fulfilled for arbitrary x, if

$$\Lambda^\mu{}_\nu \Lambda_\mu{}^\kappa = \delta_\nu{}^\kappa \tag{23.16}$$

holds. The inverse transformation of (23.14) follows from

$$x^\kappa = \delta_\nu{}^\kappa x^\nu = \Lambda_\mu{}^\kappa \Lambda^\mu{}_\nu x^\nu = \Lambda_\mu{}^\kappa x'^\mu. \tag{23.17}$$

From (23.16) one obtains in particular for $\nu = \kappa = 0$ the relation $(\Lambda^{00})^2 - \sum_\alpha (\Lambda^{\alpha 0})^2 = 1$. Note that $\Lambda^\alpha{}_0 = +\Lambda^{\alpha 0}$, $\Lambda_\alpha{}^0 = -\Lambda^{\alpha 0}$. Thus one has $|\Lambda^{00}| > 1$. One distinguishes between transformations with positive and negative Λ^{00}, since there is no continuous transition between these two classes. The condition $\Lambda^{00} > 0$ means that $\Lambda^{00} = \frac{dt}{dt'}|_{\mathbf{r}'} > 0$, that is a clock which is at rest in S' changes its time seen from S with the same direction as the clock at rest in S (and not backwards).
Finally we can make a statement on $\det(\Lambda^\mu{}_\nu)$. From (23.16) it follows that

$$\Lambda^\mu{}_\nu g_{\mu\lambda} \Lambda^\lambda{}_\rho g^{\rho\kappa} = \delta_\nu{}^\kappa. \tag{23.18}$$

Using the theorem on the multiplication of determinants we obtain

$$\det(\Lambda^\mu{}_\nu)^2 \det(g_{\mu\lambda}) \det(g^{\rho\kappa}) = 1. \tag{23.19}$$

Since $\det(g_{\mu\lambda}) = \det(g^{\rho\kappa}) = -1$ one obtains

$$\det(\Lambda^\mu{}_\nu) = \pm 1. \tag{23.20}$$

If we consider only a right-basis-system then we have $\det(\Lambda^\mu{}_\nu) = +1$. Transformations which fulfill

$$\Lambda^{00} > 0, \quad \det(\Lambda^\mu{}_\nu) = 1 \tag{23.21}$$

are called proper LORENTZ transformations.
Eq. (23.21) has the consequence that the fourdimensional space time volume is invariant

$$dt' d^3r' = \frac{1}{c} d^4x' = \frac{1}{c} \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} d^4x = \frac{1}{c} \det(\Lambda^\mu{}_\nu) d^4x = \frac{1}{c} d^4x = dt d^3r. \tag{23.22}$$

If the direction of the z- and the z'-axes point into the direction of the relative velocity between both inertial systems and $x' = x$, $y' = y$, then the special transformation (23.2) follows. The corresponding matrix Λ reads

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{23.23}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \beta = \frac{v}{c}. \tag{23.24}$$

23.b.α Time Dilatation

We consider now a clock in the system S which is at rest in S'. From

$$t = \gamma(t' + \frac{vz'}{c^2}) \tag{23.25}$$

we find that

$$\Lambda_0{}^0 = \left.\frac{\partial t}{\partial t'}\right|_{\mathbf{r}'} = \gamma. \tag{23.26}$$

Thus the clock at rest in S' runs slower when seen from S

$$\Delta t' = \left.\frac{\partial t'}{\partial t}\right|_{\mathbf{r}'} \Delta t = \frac{1}{\gamma}\Delta t = \sqrt{1 - \frac{v^2}{c^2}}\Delta t. \tag{23.27}$$

This phenomenon is called time dilatation.

23.b.β LORENTZ Contraction

From

$$z' = \gamma(z - vt) \tag{23.28}$$

one obtains

$$\Lambda^3{}_3 = \left.\frac{\partial z'}{\partial z}\right|_t = \gamma \tag{23.29}$$

and therefore

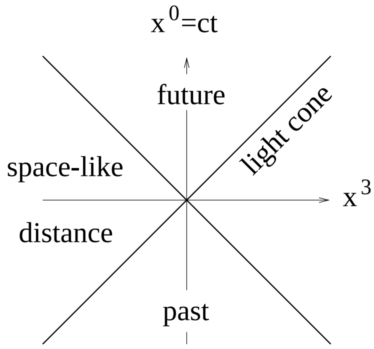
$$\Delta z = \left.\frac{\partial z}{\partial z'}\right|_t \Delta z' = \frac{1}{\gamma}\Delta z' = \sqrt{1 - \frac{v^2}{c^2}}\Delta z'. \tag{23.30}$$

A length-meter which is at rest in S' and is extended in the direction of the relative movement, appears consequently contracted in S. This is called LORENTZ contraction or FITZGERALD-LORENTZ contraction. However, the distances perpendicular to the velocity are unaltered: $\Delta x' = \Delta x$, $\Delta y' = \Delta y$.

This contraction has the effect that in (23.3) the length l has to be replaced by $l\sqrt{1 - \frac{v^2}{c^2}}$. Then the two times the light has to travel agree independent of the velocity v, $t_1 = t_2$.

24 Four-Scalars and Four-Vectors

24.a Distance and Proper Time as Four-Scalars



A quantity which is invariant under LORENTZ transformations is called four-scalar.
Example: Given two points in space-time (events) (x^μ), $(\bar{x}^\mu)$. The quantity

$$s^2 = (x^\mu - \bar{x}^\mu)(x_\mu - \bar{x}_\mu) \tag{24.1}$$

is a four-scalar. It assumes the same number in all systems of inertia.
Especially for $\bar{x}^\mu = 0$ (origin) it is $s^2 = x^\mu x_\mu$.

24.a.α Space-like distance $s^2 < 0$

If $s^2 < 0$, then there are systems of inertia, in which both events occur at the same time $x'^0 = 0$. If for example $(x^\mu) = (ct, 0, 0, z)$. Then one obtains from 23.2)

$$t' = \frac{t - \frac{vz}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{24.2}$$

with $v = tc^2/z$

$$t' = 0, \quad z' = \frac{z(1 - \frac{v^2}{c^2})}{\sqrt{1 - \frac{v^2}{c^2}}} = z\sqrt{1 - \frac{v^2}{c^2}} = \pm\sqrt{z^2 - c^2t^2} = \pm\sqrt{-s^2}. \tag{24.3}$$

Thus one calls such two events space-like separated.

24.a.β Time-like distance $s^2 > 0$

In this case there exists a system of inertia in which both events take place at the same point in space ($\mathbf{x}' = \mathbf{0}$). We choose $v = z/t$ in the transformation (23.2) and obtain

$$t' = \frac{t(1 - \frac{v^2}{c^2})}{\sqrt{1 - \frac{v^2}{c^2}}} = t\sqrt{1 - \frac{v^2}{c^2}} = \operatorname{sign}(t)\sqrt{t^2 - \frac{z^2}{c^2}} = \operatorname{sign}(t)\frac{s}{c}, \quad z' = 0. \tag{24.4}$$

One event takes place before the other that is the sign of t' agrees with that of t.
Proper Time τ
The proper time τ is the time which passes in the rest system under consideration. If a point moves with velocity $\mathbf{v}(t)$ its proper time varies as

$$\mathrm{d}\tau = \frac{\mathrm{d}s}{c} = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\mathrm{d}t, \tag{24.5}$$

that is

$$\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{\mathbf{v}^2(t)}{c^2}}\mathrm{d}t. \tag{24.6}$$

The proper time is independent of the system of inertia, thus it is a four-scalar.

24.a.γ Light-like distance $s^2 = 0$

If a light flash propagates directly from one event to another, then the distance of these two events $s = 0$. The time measured in a system of inertia depends on the system of inertia and may be arbitrarily long or short, however, the sequence of the events (under proper LORENTZ transformation) cannot be reversed.
Another four-scalar is the charge.

24.b World Velocity as Four-Vector

If a four-component quantity (A^μ) transforms by the transition from one system of inertia to another as the space-time coordinates (x^μ), then it is a four-vector

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \tag{24.7}$$

An example is the world velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma v^\mu \text{ with } v^0 = \frac{dx^0}{dt} = c\frac{dt}{dt} = c. \tag{24.8}$$

The world velocity $(u^\mu) = (c\gamma, \mathbf{v}\gamma)$ is a four-vector. Since τ is invariant under LORENTZ transformations, its components transform like (x^μ). However, $(c, \mathbf{v})$ is not a four-vector. One has

$$u^\mu u_\mu = (c^2 - \mathbf{v}^2)\gamma^2 = c^2. \tag{24.9}$$

Quite generally the scalar product of two four-vectors (A^μ) and (B^μ) is a four-scalar

$$A'^\mu B'_\mu = \Lambda^\mu{}_\nu \Lambda_\mu{}^\kappa A^\nu B_\kappa = \delta^\kappa_\nu A^\nu B_\kappa = A^\nu B_\nu. \tag{24.10}$$

We show the following lemma: If (a^μ) is an arbitrary four-vector (or one has a complete set of four-vectors) and $a^\mu b_\mu$ is a four-scalar then (b^μ) is a four-vector too. Proof:

$$a^\mu b_\mu = a'^\kappa b'_\kappa = \Lambda^\kappa{}_\mu a^\mu b'_\kappa. \tag{24.11}$$

Since this holds for all (a^μ) or for a complete set, one has $b_\mu = \Lambda^\kappa{}_\mu b'_\kappa$. This, however, is the transformation formula (23.17) for four-vectors.

Addition theorem for velocities

The system of inertia S' moves with velocity v in z-direction with respect to S. A point in S' moves with velocity w' also in z-direction. With which velocity does the point move in S? We have

$$z = \frac{z' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t = \frac{t' + \frac{vz'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{24.12}$$

With $z' = w't'$ one obtains

$$z = \frac{(v + w')t'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t = \frac{(1 + \frac{vw'}{c^2})t'}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{24.13}$$

From this one obtains the velocity of the point in S

$$w = \frac{z}{t} = \frac{w' + v}{1 + \frac{w'v}{c^2}}. \tag{24.14}$$

We observe

$$1 - \frac{w^2}{c^2} = 1 - \left(\frac{\frac{w'}{c} + \frac{v}{c}}{1 + \frac{w'v}{c^2}}\right)^2 = \frac{(1 - \frac{w'^2}{c^2})(1 - \frac{v^2}{c^2})}{(1 + \frac{w'v}{c^2})^2}. \tag{24.15}$$

If $|w'| < c$ and $|v| < c$, then this expression is positive. Then one obtains also $|w| < c$. Example: $w' = v = 0.5c$, then one obtains $w = 0.8c$.

24.c Current Density Four-Vector

We combine charge- and current-density in the charge-current density

$$(j^\mu) = (c\rho, \mathbf{j}) \tag{24.16}$$

and convince us that j^μ is a four-vector. For charges of velocity $\mathbf{v}$ one has (the contributions of charges of different velocities can be superimposed)

$$j^\mu = \rho v^\mu, \quad (v^0 = c), \quad j^\mu = \rho\sqrt{1-\beta^2}u^\mu \tag{24.17}$$

If $\rho\sqrt{1-\beta^2}$ is a four-scalar then indeed j^μ is a four-vector. Now one has

$$\rho = \frac{q}{V} = \frac{q}{V_0\sqrt{1-\beta^2}} \tag{24.18}$$

with the volume V_0 in the rest system and the LORENTZ contraction $V = V_0\sqrt{1-\beta^2}$. Since the charge q and V_0 are four-scalars this holds also $\rho\sqrt{1-\beta^2}$.
We bring the equation of continuity in LORENTZ-invariant form. From $\dot\rho + \operatorname{div}\mathbf{j} = 0$ one obtains

$$\frac{\partial j^\mu}{\partial x^\mu} = 0, \tag{24.19}$$

since $\partial j^0/\partial x^0 = \partial\rho/\partial t$. We consider now the transformation properties of the derivatives $\partial/\partial x^\mu$

$$\frac{\partial f}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial f}{\partial x^\nu} = \Lambda_\mu{}^\nu \frac{\partial f}{\partial x^\nu}, \tag{24.20}$$

that is the derivatives transform according to

$$\frac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} \tag{24.21}$$

as $x'_\mu = \Lambda_\mu{}^\nu x_\nu$. Thus one writes

$$\frac{\partial}{\partial x^\mu} = \partial_\mu, \quad (\partial_\mu) = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla). \tag{24.22}$$

Watch the positions of the indices. Similarly one has

$$\frac{\partial}{\partial x_\mu} = \partial^\mu, \quad (\partial^\mu) = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla). \tag{24.23}$$

Then the equation of continuity can be written

$$\partial_\mu j^\mu = 0. \tag{24.24}$$

Generally the four-divergency $\partial_\mu P^\mu = \partial^\mu P_\mu$ of a four-vector P is a four-scalar.

24.d Four-Potential

We combine the potentials $\mathbf{A}$ and Φ in the four-potential

$$(A^\mu) = (\Phi, \mathbf{A}), \tag{24.25}$$

then one has

$$\Box A^\mu = -\frac{4\pi}{c} j^\mu \tag{24.26}$$

in the LORENZ gauge with the gauge condition

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\dot\Phi = 0 \to \partial_\mu A^\mu = 0. \tag{24.27}$$

There the D'ALEMBERT operator

$$\Box = \triangle - \frac{1}{c^2}\partial_t^2 = -\partial_\mu\partial^\mu \tag{24.28}$$

is a four-scalar $\Box' = \Box$.

We now show that the retarded solution A_r^μ is manifestly LORENTZ-invariant. We claim

$$A_r^\mu(x) = \frac{1}{c}\int d^4y\, j^\mu(y)\delta(\frac{1}{2}s^2)\theta(x^0-y^0) \quad (24.29)$$

$$s^2 = (x^\mu-y^\mu)(x_\mu-y_\mu) = c^2(t_y-t_x)^2-(\mathbf{x}-\mathbf{y})^2 \quad (24.30)$$

$$\theta(x^0) = \begin{cases} 1 & x^0>0 \\ 0 & x^0<0 \end{cases} \quad (24.31)$$

We consider now generally the integration over a δ-function, which depends on a function f. Apparently only the zeroes t_i of f contribute,

$$\int g(t)\delta(f(t))dt = \sum_i \int_{t_i-\epsilon}^{t_i+\epsilon} g(t)\delta(f(t))dt \quad \text{with} \quad f(t_i)=0. \quad (24.32)$$

With $z=f(t)$, $dz=f'(t)dt$ one obtains

$$\int g(t)\delta(f(t))dt = \sum_i \int_{-\epsilon f'(t_i)}^{\epsilon f'(t_i)} g(t_i)\delta(z)\frac{dz}{f'(t_i)} = \sum_i \frac{g(t_i)}{|f'(t_i)|}. \quad (24.33)$$

Thus the zeroes in the δ-function of (24.29) are $t_y = t_x \pm |\mathbf{x}-\mathbf{y}|/c$ and their derivatives are given by $f'(t_y) = c^2(t_y-t_x) = \pm c|\mathbf{x}-\mathbf{y}|$, which yields

$$A_r^\mu(x) = \frac{1}{c}\int d^4y\, j^\mu\delta(\frac{1}{2}s^2)\theta(t_x-t_y) = \int d^3y \frac{1}{c|\mathbf{x}-\mathbf{y}|} j^\mu(\mathbf{y}, t_x-\frac{|\mathbf{x}-\mathbf{y}|}{c}). \quad (24.34)$$

The factor $\theta(t_x-t_y)$ yields the retarded solution. The solution is in agreement with (21.14) and (21.15). If we replace the θ-function by $\theta(t_y-t_x)$, then we obtain the advanced solution. Remember that the sign of the time difference does not change under proper LORENTZ transformations.

25 Electromagnetic Field Tensor

25.a Field Tensor

We obtain the fields **E** and **B** from the four-potential

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \quad \mathbf{E} = -\operatorname{grad}\Phi - \frac{1}{c}\dot{\mathbf{A}}, \tag{25.1}$$

for example

$$B_1 = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \partial^3 A^2 - \partial^2 A^3, \quad E_1 = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = \partial^1 A^0 - \partial^0 A^1. \tag{25.2}$$

Thus we introduce the electromagnetic field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad F^{\mu\nu} = -F^{\nu\mu}. \tag{25.3}$$

It is an antisymmetric four-tensor. It reads explicitely

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{25.4}$$

25.b Maxwell's Equations

25.b.α The Inhomogeneous Equations

The equation $\operatorname{div}\mathbf{E} = 4\pi\rho$ reads

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{4\pi}{c} j^0. \tag{25.5}$$

From the 1-component of $\operatorname{curl}\mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{j}$ one obtains

$$\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3} - \frac{\partial E_1}{\partial x^0} = \frac{4\pi}{c} j^1 \rightarrow \partial_2 F^{21} + \partial_3 F^{31} + \partial_0 F^{01} = \frac{4\pi}{c} j^1, \tag{25.6}$$

similarly for the other components. These four component-equations can be combined to

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu. \tag{25.7}$$

If we insert the representation of the fields by the potentials, (25.3), we obtain

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{4\pi}{c} j^\nu. \tag{25.8}$$

Together with the condition for the Lorenz gauge $\partial_\mu A^\mu = 0$, (24.27) one obtains

$$\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} j^\nu \tag{25.9}$$

in agreement with (24.26) and (24.28).

25.b.β The Homogeneous Equations

Similarly the homogeneous MAXWELL's equations can be written. From $\operatorname{div}\mathbf{B} = 0$ one obtains

$$\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0 \tag{25.10}$$

and $(\operatorname{curl}\mathbf{E} + \frac{1}{c}\dot{\mathbf{B}})_x = 0$ reads

$$-\partial^2 F^{30} - \partial^3 F^{02} - \partial^0 F^{23} = 0. \tag{25.11}$$

These equations can be combined to

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \tag{25.12}$$

One observes that these equations are only non-trivial for $\lambda \neq \mu \neq \nu \neq \lambda$. If two indices are equal, then the left-hand side vanishes identically. One may represent the equations equally well by the dual field tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}. \tag{25.13}$$

Here $\epsilon^{\kappa\lambda\mu\nu}$ is completely antisymmetric against interchange of the four indices. Thus it changes sign, if two of the indices are exchanged. This implies that it vanishes, if two indices are equal. It is only different from zero, if all four indices are different. It is normalized to $\epsilon^{0123} = 1$. Then one obtains explicitely

$$(\tilde{F}^{\mu\nu}) = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}. \tag{25.14}$$

and (25.12) can be written

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{25.15}$$

One should convince oneself that ϵ is an invariant pseudo-tensor of fourth order, i.e.

$$\epsilon'^{\mu\nu\kappa\lambda} = \det(\Lambda)\epsilon^{\mu\nu\kappa\lambda}, \tag{25.16}$$

where $\det(\Lambda)$ takes only the values ± 1 according to the discussion after (23.19). For proper LORENTZ transformations it equals $+1$ (23.21).

25.c Transformation of the Electric and Magnetic Fields

Since (∂^μ) and (A^ν) transform like four-vectors, one has

$$F'^{\mu\nu} = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda F^{\kappa\lambda} \tag{25.17}$$

for the transformation of the electromagnetic field. If we choose in particular

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{25.18}$$

then one obtains

$$E_1' = F'^{10} = \Lambda^1{}_\kappa \Lambda^0{}_\lambda F^{\kappa\lambda} = \gamma F^{10} - \beta\gamma F^{13} = \gamma(E_1 - \beta B_2), \tag{25.19}$$

thus

$$E_1' = \gamma(E_1 - \frac{v}{c}B_2), \tag{25.20}$$

similarly

$$B_1' = \gamma(B_1 + \frac{v}{c}E_2) \tag{25.21}$$

$$E_2' = \gamma(E_2 + \frac{v}{c}B_1), \qquad B_2' = \gamma(B_2 - \frac{v}{c}E_1) \tag{25.22}$$

$$E_3' = E_3, \qquad B_3' = B_3, \tag{25.23}$$

which can be combined to

$$E'_{\|} = E_{\|}, \qquad B'_{\|} = B_{\|}, \qquad \text{component } \parallel \mathbf{v} \tag{25.24}$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \frac{\mathbf{v}}{c}\times\mathbf{B}), \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c}\times\mathbf{E}), \qquad \text{components } \perp \mathbf{v}. \tag{25.25}$$

25.d Fields of a Point Charge in Uniform Motion

From this we can determine the fields of a charge which moves with constant velocity $\mathbf{v} = v\mathbf{e}_z$. In the rest system S' of the charge, which is supposed to be in the origin of S', one has

$$\mathbf{E}' = q\frac{\mathbf{r}'}{r'^3}, \quad \mathbf{B}' = \mathbf{0}. \tag{25.26}$$

In the system S the coordinates of the charge are $x_q = y_q = 0$, $z_q = vt$. Now we express $\mathbf{r}'$ by $\mathbf{r}$ and t and obtain

$$\mathbf{E}' = \left(\frac{qx}{N}, \frac{qy}{N}, \frac{q\gamma(z-vt)}{N}\right), \tag{25.27}$$

$$\mathbf{B}' = \mathbf{0}, \tag{25.28}$$

$$N = r'^3 = (x^2+y^2+\gamma^2(z-vt)^2)^{3/2}. \tag{25.29}$$

It follows that

$$\left.\begin{array}{r} E_1 = \gamma(E'_1 + \frac{v}{c}B'_2) = \frac{q\gamma x}{N} \\ E_2 = \gamma(E'_2 - \frac{v}{c}B'_1) = \frac{q\gamma y}{N} \\ E_3 = E'_3 = \frac{q\gamma(z-vt)}{N} \end{array}\right\} \mathbf{E} = \frac{q\gamma(\mathbf{r}-\mathbf{v}t)}{N} \tag{25.30}$$

$$\left.\begin{array}{r} B_1 = \gamma(B'_1 - \frac{v}{c}E'_2) = -\frac{q\gamma\beta y}{N} \\ B_2 = \gamma(B'_2 + \frac{v}{c}E'_1) = \frac{q\gamma\beta x}{N} \\ B_3 = B'_3 = 0 \end{array}\right\} \mathbf{B} = \frac{q\gamma(\mathbf{v}\times\mathbf{r})}{cN}. \tag{25.31}$$

Areas of constant N are oblate rotational ellipsoids. The ratio short half-axis / long half-axis is given by $1/\gamma = \sqrt{1-\frac{v^2}{c^2}}$, thus the same contraction as for the Lorentz contraction.

25.e Doppler Effect

We consider a monochromatic plane wave

$$\mathbf{E} = \mathbf{E}_0 e^{i\phi}, \quad \mathbf{B} = \mathbf{B}_0 e^{i\phi} \quad \text{with } \phi = \mathbf{k}\cdot\mathbf{r} - \omega t. \tag{25.32}$$

We know, how $\mathbf{E}$ and $\mathbf{B}$ and thus $\mathbf{E}_0$ and $\mathbf{B}_0$ transform. Thus we are left with considering the phase ϕ which is a four-scalar. If we write

$$(k^\mu) = (\frac{\omega}{c}, \mathbf{k}), \tag{25.33}$$

then

$$\phi = -k_\mu x^\mu \tag{25.34}$$

follows. Since (x^μ) is an arbitrary four-vector and ϕ is a four-scalar, it follows that (k^μ) has to be a four-vector. Thus one obtains for the special Lorentz transformation (25.18)

$$\omega' = ck'^0 = c\gamma(k^0 - \beta k^3) = \gamma(\omega - \beta c k^3), \quad k'^1 = k^1, \quad k'^2 = k^2, \quad k'^3 = \gamma(k^3 - \beta\frac{\omega}{c}). \tag{25.35}$$

If the angle between z-axis and direction of propagation is θ, then $k^3 = \frac{\omega}{c}\cos\theta$ holds, and one obtains

$$\omega' = \omega\gamma(1 - \beta\cos\theta). \tag{25.36}$$

Thus if **v** is parallel and antiparallel to the direction of propagation, resp., then one deals with the longitudinal DOPPLER shift

$$\theta = 0: \quad \omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}} \tag{25.37}$$

$$\theta = \pi: \quad \omega' = \omega\sqrt{\frac{1+\beta}{1-\beta}}. \tag{25.38}$$

If however $\theta = \pi/2$ and $\theta' = \pi/2$, resp., then one deals with the transverse DOPPLER shift.

$$\theta = \frac{\pi}{2}: \quad \omega' = \frac{\omega}{\sqrt{1-\beta^2}} \tag{25.39}$$

$$\theta' = \frac{\pi}{2}: \quad \omega' = \omega\sqrt{1-\beta^2}. \tag{25.40}$$

Here θ' is the angle between the z'-axis and the direction of propagation in S'.

26 Relativistic Mechanics

Einstein realized that the constance of light velocity in vacuum and the resulting Lorentz transformation is not restricted to electrodynamics, but is generally valid in physics. Here we consider its application to mechanics starting from the force on charges.

26.a Lorentz Force Density

The force density on moving charges reads

$$\mathbf{k} = \rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}, \tag{26.1}$$

that is e.g. for the first component

$$k^1 = \rho E_1 + \frac{1}{c}(j^2B_3 - j^3B_2) = \frac{1}{c}(j^0F^{10} - j^2F^{12} - j^3F^{13}) = \frac{1}{c}j_\nu F^{1\nu}. \tag{26.2}$$

Thus one introduces the four-vector of the Lorentz force density

$$k^\mu = \frac{1}{c}j_\nu F^{\mu\nu}. \tag{26.3}$$

We consider the time-like component

$$k^0 = \frac{1}{c}j_\nu F^{0\nu} = \frac{1}{c}\mathbf{j}\cdot\mathbf{E}. \tag{26.4}$$

The time-like component gives the mechanical energy acquired per time and volume, whereas the space-like components give the rate of change of mechanic momentum per time and volume

$$(k^\mu) = (\frac{1}{c}\mathbf{j}\cdot\mathbf{E}, \mathbf{k}). \tag{26.5}$$

26.b Lorentz Force Acting on a Point Charge

The four-current-density at $\mathbf{x}$ of a point charge q at $\mathbf{x}_\mathrm{q}$ reads

$$j^\nu(\mathbf{x}, t) = q\delta^3(\mathbf{x} - \mathbf{x}_\mathrm{q}(t))v^\nu. \tag{26.6}$$

Thus the force acting on the point charge is given by

$$K^\mu = \frac{q}{c}v_\nu F^{\mu\nu}. \tag{26.7}$$

This is not a four-vector, since (v^μ) is not a four-vector. If we multiply it by γ then we obtain a four-vector, the Minkowski force

$$\gamma K^\mu = \frac{q}{c}u_\nu F^{\mu\nu}. \tag{26.8}$$

K is the momentum which is fed into the point charge per time unit, cK^0 is the power fed into it. The Minkowski force is the momentum and the energy divided by c, resp., fed into it per proper time.

26.c Energy and Momentum of a Mass Point

We assume that also mechanical momentum and energy/c combine to a four-vector, since the change of momentum and energy divided by c are components of a four-vector

$$(G^\mu) = (\frac{1}{c}E, \mathbf{G}). \tag{26.9}$$

In the rest system S' we expect $\mathbf{G}' = \mathbf{0}$ to hold, i.e.

$$(G'^{\mu}) = (\frac{1}{c}E_0, \mathbf{0}). \tag{26.10}$$

In the system S the special transformation (23.23) yields for $\mathbf{v} = v\mathbf{e}_z$

$$\mathbf{G} = \gamma \tfrac{v}{c^2} E_0 \mathbf{e}_z = \gamma \mathbf{v} \frac{E_0}{c^2}, \tag{26.11}$$

$$E = cG^0 = c\gamma G'^0 = \gamma E_0. \tag{26.12}$$

For velocities small in comparison to light-velocity one obtains

$$\mathbf{G} = \frac{E_0}{c^2}\mathbf{v}\Big(1 + \frac{v^2}{2c^2} + ...\Big). \tag{26.13}$$

In NEWTON's mechanics we have

$$\mathbf{G}_{\text{Newton}} = m\mathbf{v} \tag{26.14}$$

for a mass point of mass m. For velocities $v \ll c$ the momentum of NEWTON's and of the relativistic mechanics should agree. From this one obtains

$$m = \frac{E_0}{c^2} \to E_0 = mc^2, \quad \mathbf{G} = m\gamma\mathbf{v}. \tag{26.15}$$

Then one obtains for the energy E

$$E = mc^2\gamma = mc^2 + \frac{m}{2}v^2 + O(v^4/c^2). \tag{26.16}$$

One associates a rest energy $E_0 = mc^2$ with the masses. At small velocities the contribution $\frac{m}{2}v^2$ known from NEWTONian mechanics has to be added

$$G^{\mu} = mu^{\mu}. \tag{26.17}$$

This G is called four-momentum. We finally observe

$$G^{\mu}G_{\mu} = m^2u^{\mu}u_{\mu} = m^2c^2, \tag{26.18}$$

from which one obtains

$$-\mathbf{G}^2 + \frac{1}{c^2}E^2 = m^2c^2, \quad E^2 = m^2c^4 + \mathbf{G}^2c^2. \tag{26.19}$$

One does not observe the rest energy $E_0 = mc^2$ as long as the particles are conserved. However they are observed when the particles are converted, for example, when a particle decays into two other ones

$$\Lambda^0 \to \pi^- + p^+. \tag{26.20}$$

With the masses

$$m_\Lambda = 2182m_e, \quad m_\pi = 273m_e, \quad m_p = 1836m_e \tag{26.21}$$

one obtains the momentum and energy bilance for the Λ which is at rest before the decay

$$m_\Lambda c^2 = \sqrt{m_\pi^2c^4 + \mathbf{G}_\pi^2c^2} + \sqrt{m_p^2c^4 + \mathbf{G}_p^2c^2} \tag{26.22}$$

$$\mathbf{0} = \mathbf{G}_\pi + \mathbf{G}_p. \tag{26.23}$$

The solution of the system of equations yields

$$|\mathbf{G}| = 4c\sqrt{M(m_\Lambda - M)(M - m_\pi)(M - m_p)}/m_\Lambda, \quad 2M = m_\Lambda + m_\pi + m_p. \tag{26.24}$$

By means of the four-vectors one may solve

$$G_\Lambda{}^{\mu} = G_\pi{}^{\mu} + G_p{}^{\mu} \tag{26.25}$$

with respect to G_p and take the square

$$G_p{}^{\mu} G_{p\mu} = (G_{\Lambda}^{\mu} - G_{\pi}^{\mu})(G_{\Lambda\mu} - G_{\pi\mu}) = G_{\Lambda}^{\mu} G_{\Lambda\mu} + G_{\pi}^{\mu} G_{\pi\mu} - 2G_{\Lambda}^{\mu} G_{\pi\mu}. \tag{26.26}$$

This yields

$$m_p^2 c^2 = m_\Lambda^2 c^2 + m_\pi^2 c^2 - 2m_\Lambda E_\pi \tag{26.27}$$

and therefore

$$E_\pi = \frac{c^2}{2m_\Lambda}\left(m_\Lambda^2 + m_\pi^2 - m_p^2\right) \tag{26.28}$$

and analogously

$$E_p = \frac{c^2}{2m_\Lambda}\left(m_\Lambda^2 - m_\pi^2 + m_p^2\right). \tag{26.29}$$

26.d Equation of Motion

Finally we write down explicitely the equations of motion for point masses

$$\frac{\mathrm{d}G^\mu}{\mathrm{d}t} = K^\mu. \tag{26.30}$$

As mentioned before these equations are not manifestly Lorentz-invariant. We have, however,

$$\frac{\mathrm{d}G^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}G^\mu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}G^\mu}{\mathrm{d}t} = \gamma K^\mu, \tag{26.31}$$

where the right-hand side is the Minkowski force. In this form the equations of motion are manifestly Lorentz invariant.
If the force does not change the rest energy of a particle, one obtains from

$$G^\mu G_\mu = m^2 c^2 \to \frac{\mathrm{d}}{\mathrm{d}\tau}(G^\mu G_\mu) = 0 \to G^\mu \gamma K_\mu = 0 \to u^\mu K_\mu = 0. \tag{26.32}$$

The force is orthogonal on the world velocity. An example is the Lorentz force

$$u_\mu K^\mu = \frac{q}{c}\gamma v_\mu v_\nu F^{\mu\nu} = 0, \tag{26.33}$$

since $F^{\mu\nu}$ is antisymmetric. We observe

$$v^\mu K_\mu = -\mathbf{v}\cdot\mathbf{K} + \frac{c}{c}\frac{\mathrm{d}E}{\mathrm{d}t} = 0. \tag{26.34}$$

Thus equation (26.32) is equivalent to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{v}\cdot\mathbf{K}, \tag{26.35}$$

which yields the power fed into the kinetic energy of the mass.

27 Lagrangian Formulation

27.a Lagrangian of a Massive Charge in the Electromagnetic Field

We claim that the Lagrangian $\mathcal{L}$ of a point charge q of mass m in an electromagnetic field can be written

$$\begin{aligned}\mathcal{L} &= -mc^2\sqrt{1-\frac{\dot{\mathbf{r}}^2}{c^2}} - q\Phi(\mathbf{r},t) + \frac{q}{c}\mathbf{A}(\mathbf{r},t)\cdot\dot{\mathbf{r}} \\ &= -mc^2\sqrt{1+\frac{\dot{x}^\alpha\dot{x}_\alpha}{c^2}} - \frac{q}{c}A^\mu(x)\dot{x}_\mu. \end{aligned} \tag{27.1}$$

Then the action I can be written

$$I = \int \mathrm{d}t\mathcal{L} = -mc^2\int \mathrm{d}\tau - \frac{q}{c}\int \mathrm{d}tA^\mu\frac{\mathrm{d}x_\mu}{\mathrm{d}t} = \int \mathrm{d}\tau(-mc^2 - \frac{q}{c}A^\mu u_\mu), \tag{27.2}$$

that is as a four-scalar.
Now we show that this Lagrangian yields the correct equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}_\alpha} - \frac{\partial\mathcal{L}}{\partial x_\alpha} = 0, \tag{27.3}$$

from which by use of

$$-\frac{\partial\mathcal{L}}{\partial\dot{x}_\alpha} = \frac{m\dot{x}^\alpha}{\sqrt{1-\frac{\dot{\mathbf{r}}^2}{c^2}}} + \frac{q}{c}A^\alpha(\mathbf{r}(t),t) = G^\alpha + \frac{q}{c}A^\alpha \tag{27.4}$$

one finally obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} + \frac{q}{c}\dot{\mathbf{A}} + \frac{q}{c}(\mathbf{v}\cdot\nabla)\mathbf{A} + q\nabla\Phi - \frac{q}{c}\nabla(\mathbf{v}\cdot\mathbf{A}) = 0. \tag{27.5}$$

Note that $\dot{\mathbf{A}}$ contains only the partial time-derivative of $\mathbf{A}$, thus we have $\mathrm{d}\mathbf{A}/\mathrm{d}t = \dot{\mathbf{A}} + (\mathbf{v}\cdot\nabla)\mathbf{A}$. By suitable combination of the contributions one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} + q(\nabla\Phi + \frac{1}{c}\dot{\mathbf{A}}) - \frac{q}{c}\mathbf{v}\times(\nabla\times\mathbf{A}) = 0 \tag{27.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} - q\mathbf{E} - \frac{q}{c}\mathbf{v}\times\mathbf{B} = 0. \tag{27.7}$$

Thus the Lagrangian given above yields the correct equation of motion.

27.b Lagrangian Density of the Electromagnetic Field

The Lagrangian density L of the electromagnetic field of a system of charges consists of three contributions

$$L = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}A^\mu j_\mu + L_{\mathrm{mech}}. \tag{27.8}$$

The mechanical part for point charges of mass m_i reads

$$L_{\mathrm{mech}} = -\sum_i m_ic^3\int d\tau\delta^4(x - x_i(\tau)), \tag{27.9}$$

which yields after integration over d^4x the corresponding contribution to the action I given in (27.1). The second contribution in (27.8) describes the interaction between field and charge. Integration of this contribution for point charges using

$$j_\mu(\mathbf{r},t) = \sum_i q_i\frac{dx_{i,\mu}}{dt}\delta^3(\mathbf{r}-\mathbf{r}_i) \tag{27.10}$$

yields the corresponding contribution in (27.1). The first contribution is that of the free field. Below we will see that it yields MAXWELL's equations correctly. The action itself reads

$$I = \frac{1}{c}\int d^4x L(x) = \int dt \int d^3x L(\mathbf{x},t) = \int dt \mathcal{L}(t), \quad \mathcal{L}(t) = \int d^3x L(\mathbf{x},t). \tag{27.11}$$

The action has to be extremal if the fields A are varied. There we have to consider F as function of A (25.3), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then the variation with respect to A yields

$$\delta L = -\frac{1}{8\pi}F_{\mu\nu}\delta F^{\mu\nu} - \frac{1}{c}j_\nu \delta A^\nu \tag{27.12}$$

$$\delta F^{\mu\nu} = \delta(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu \tag{27.13}$$

$$F_{\mu\nu}\delta F^{\mu\nu} = F_{\mu\nu}\partial^\mu \delta A^\nu - F_{\mu\nu}\partial^\nu \delta A^\mu = 2F_{\mu\nu}\partial^\mu \delta A^\nu \tag{27.14}$$

$$\delta L = -\frac{1}{4\pi}F_{\mu\nu}\partial^\mu \delta A^\nu - \frac{1}{c}j_\nu \delta A^\nu. \tag{27.15}$$

Thus the variation of the action with respect to A is

$$\begin{aligned}\delta I &= \int d^4x \Big(-\frac{1}{4\pi c}F_{\mu\nu}\partial^\mu \delta A^\nu - \frac{1}{c^2}j_\nu \delta A^\nu \Big) \\ &= -\int d^4x \frac{1}{4\pi c}\partial^\mu (F_{\mu\nu}\delta A^\nu) + \int d^4x \Big(\frac{1}{4\pi c}\partial^\mu F_{\mu\nu} - \frac{1}{c^2}j_\nu\Big)\delta A^\nu.\end{aligned} \tag{27.16}$$

The first term of the second line is a surface-term (in four dimensions). From the second term one concludes MAXWELL's inhomogeneous equations (25.7)

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\nu. \tag{27.17}$$

MAXWELL's homogeneous equations are already fulfilled due to the representation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$.
Generally one obtains for a Lagrangian density, which depends on a field (A^μ) and its derivatives by variation

$$\begin{aligned} c\delta I &= \int d^4x \delta L(x) \\ &= \int d^4x \left(\frac{\delta L}{\delta A^\nu(x)}\delta A^\nu(x) + \frac{\delta L}{\delta \partial^\mu A^\nu(x)}\partial^\mu \delta A^\nu(x) \right) \\ &= \int d^4x \partial^\mu \left(\frac{\delta L}{\delta \partial^\mu A^\nu(x)}\delta A^\nu(x) \right) + \int d^4x \left(\frac{\delta L}{\delta A^\nu(x)} - \partial^\mu \left(\frac{\delta L}{\delta \partial^\mu A^\nu(x)} \right) \right) \delta A^\nu(x). \end{aligned} \tag{27.18}$$

Usually one denotes the partial derivatives of L with respect to A and ∂A by $\delta L/\delta ...$. Since the variation has to vanish, one obtains in general the equations of motion

$$\partial^\mu \left(\frac{\delta L}{\delta \partial^\mu A^\nu(x)} \right) - \frac{\delta L}{\delta A^\nu(x)} = 0. \tag{27.19}$$

This is the generalization of LAGRANGE's equations of motion (27.3) for fields. There appear derivatives of $\delta L/\delta \nabla A^\nu$ with respect to the space variables besides the time-derivatives of $\delta L/\delta \dot{A}^\nu$.

28 Energy Momentum Tensor and Conserved Quantities

28.a The Tensor

In section 15.b we have calculated the conservation law for momentum from the density of the LORENTZ force in vacuo that is without considering additional contributions due to matter

$$-\mathbf{k} = \frac{\partial}{\partial t}\mathbf{g}_\mathrm{s} - \frac{\partial}{\partial x^\beta}T^{\alpha\beta}\mathbf{e}^\alpha, \tag{28.1}$$

$$\mathbf{g}_\mathrm{s} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}, \tag{28.2}$$

$$T^{\alpha\beta} = \frac{1}{4\pi}\Big(E_\alpha E_\beta + B_\alpha B_\beta\Big) - \frac{\delta_{\alpha\beta}}{8\pi}\Big(E^2 + B^2\Big). \tag{28.3}$$

The zeroth component is the energy-density. For this density we have obtainded in section 15.a

$$-k^0 = -\frac{1}{c}\mathbf{j}\cdot\mathbf{E} = \frac{1}{c}\,\mathrm{div}\,\mathbf{S} + \frac{1}{c}\dot{u} \tag{28.4}$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \tag{28.5}$$

$$u = \frac{1}{8\pi}\Big(E^2 + B^2\Big). \tag{28.6}$$

We summarize

$$-k^\mu = -\partial_\nu T^{\mu\nu} \tag{28.7}$$

with the electromagnetic energy-momentum tensor

$$(T^{\mu\nu}) = \begin{pmatrix} -u & -\frac{1}{c}S_1 & -\frac{1}{c}S_2 & -\frac{1}{c}S_3 \\ -cg_{\mathrm{s}1} & T_{11} & T_{12} & T_{13} \\ -cg_{\mathrm{s}2} & T_{21} & T_{22} & T_{23} \\ -cg_{\mathrm{s}3} & T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{28.8}$$

This energy-momentum tensor is built up from the energy density u, the POYNTING vector (density of energy current) $\mathbf{S}$, the momentum density $\mathbf{g}$, and the stress tensor T. One observes that $T^{\mu\nu}$ is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, since $T_{\alpha\beta}$ is symmetric and $c\mathbf{g}_\mathrm{s} = \frac{1}{c}\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{B}$ holds. One easily checks that

$$T^{\mu\nu} = \frac{1}{4\pi}\Big(-F^\mu{}_\lambda F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^\kappa{}_\lambda F^\lambda{}_\kappa\Big) \tag{28.9}$$

holds either by explicit calculation and comparison or from

$$k^\mu = \frac{1}{c}j_\lambda F^{\mu\lambda} = \frac{1}{4\pi}(\partial^\nu F_{\nu\lambda})F^{\mu\lambda} = \frac{1}{4\pi}\partial^\nu(F_{\nu\lambda}F^{\mu\lambda}) - \frac{1}{4\pi}F_{\nu\lambda}\partial^\nu F^{\mu\lambda}. \tag{28.10}$$

From

$$F_{\nu\lambda}\Big(\partial^\nu F^{\mu\lambda} + \partial^\mu F^{\lambda\nu} + \partial^\lambda F^{\nu\mu}\Big) = 0 \tag{28.11}$$

one obtains the relation

$$\frac{1}{2}\partial^\mu\Big(F_{\nu\lambda}F^{\lambda\nu}\Big) + 2F_{\nu\lambda}\partial^\nu F^{\mu\lambda} = 0, \tag{28.12}$$

so that finally we obtain

$$\begin{aligned} k^\mu &= \frac{1}{4\pi}\partial^\nu\Big(F_{\nu\lambda}F^{\mu\lambda}\Big) + \frac{1}{16\pi}\partial^\mu\Big(F_{\nu\lambda}F^{\lambda\nu}\Big) \\ &= \frac{1}{4\pi}\partial_\nu\Big(-F^\mu{}_\lambda F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^\kappa{}_\lambda F^\lambda{}_\kappa\Big). \end{aligned} \tag{28.13}$$

$T^{\mu\nu}$ is a symmetric four-tensor, i.e. it transforms according to

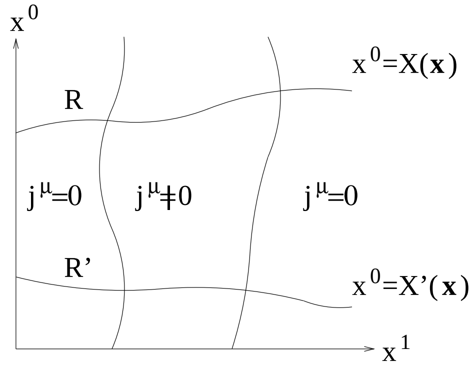
$$T'^{\mu\nu} = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda T^{\kappa\lambda}. \tag{28.14}$$

28.b Conservation Laws

We start out from a four-vector field $(j^\mu(x)$. In any three-dimensional space-like subspace R of the four-dimensional space be (j^μ) different from zero only in a finite region. We call a space space-like if any two points in this space have a space-like distance. A world-line, i.e. a line which everywhere has a velocity below light-speed hits a space-like subspace in exactly one point. If one plots the subspace as a function $x^0(\mathbf{r})$ then its slope is everywhere less then 1. The slope of the world-line is everywhere larger than 1. For example, the points of constant time in an inertial frame constitute such a space-like space. We now integrate the divergence $\partial_\mu j^\mu$ over the four-dimensional volume Ω, which is bounded by two space-like spaces R and R' and obtain

$$\int_\Omega \mathrm{d}^4x \frac{\partial j^\mu}{\partial x^\mu} = \int_R \mathrm{d}^3x\Big(j^0 - \frac{\partial X}{\partial x^\alpha} j^\alpha\Big) - \int_{R'} \mathrm{d}^3x\Big(j^0 - \frac{\partial X'}{\partial x^\alpha} j^\alpha\Big). \tag{28.15}$$

The contribution $\partial_\mu j^\mu$ is integrated in x^μ-direction until the boundary R or R' or until j^μ vanishes. This yields immediately the contribution given for the 0-component. For the 1-component one obtains initially the integral $\pm \int \mathrm{d}x^0 \mathrm{d}x^2 \mathrm{d}x^3 j^1$ at the boundary.The $\mathrm{d}x^0$-integration may be transformed into an $\mathrm{d}x^1 \frac{\partial X}{\partial x^1}$-integration. If $X = x^0$ increases (decreases) at the boundary with x^1, then this is the lower (upper) limit of the integration. Thus we have a minus-sign in front of $\frac{\partial X}{\partial x^1}$, similarly for the other space-components.We may convince ourselves that



$$\int_R \mathrm{d}^3x\Big(j^0 - \frac{\partial X}{\partial x^\alpha} j^\alpha\Big) = \int_R \mathrm{d}V_\mu j^\mu \tag{28.16}$$

with $(\mathrm{d}V_\mu) = (1, -\nabla X)\mathrm{d}^3x$ is a four-scalar. If we introduce a four-vector $(\overline{j}^\mu)$, so that

$$\overline{j}^\mu = \begin{cases} j^\mu & \text{in } R \\ 0 & \text{in } R' \end{cases}, \tag{28.17}$$

then it follows that

$$\int_R \mathrm{d}V_\mu j^\mu = \int_R \mathrm{d}V_\mu \overline{j}^\mu = \int_\Omega \mathrm{d}^4x \frac{\partial \overline{j}^\mu}{\partial x^\mu}, \tag{28.18}$$

where the last integral is obviously a four-scalar, since both d^4x and the four-divergence of $\overline{j}$ is a four-scalar. Since the field (j^μ) is arbitrary, we find that $\mathrm{d}V_\mu j^\mu$ has to be a four-scalar for each infinitesimal $(\mathrm{d}V_\mu)$ in R. Since (j^μ) is a four-vector, $(\mathrm{d}V^\mu)$ must be a four-vector, too. Then (28.16) reads

$$\int_\Omega \mathrm{d}^4x \partial_\mu j^\mu = \int_R \mathrm{d}V_\mu j^\mu - \int_{R'} \mathrm{d}V_\mu j^\mu. \tag{28.19}$$

This is the divergence theorem in four dimensions.
From this we conclude:

28.b.α Charge

(j^μ) be the four-vector of the current density. One obtains from the equation of continuity $\partial_\mu j^\mu = 0$ for each space-like R the same result

$$q = \frac{1}{c} \int_R \mathrm{d}V_\mu j^\mu \tag{28.20}$$

for the charge, since the integral of the divergence in Ω in (28.19) vanishes, (since the integrand vanishes) and since one may always choose the same R'. Thus the charge is a conserved quantity, more precisely we have found a consistent behaviour, since we already have assumed in subsection 24.c that charge is conserved. New is that it can be determined in an arbitrary space-like three-dimensional space.

28.b.β Energy and Momentum

From
$$k^\mu = \partial_\nu T^{\mu\nu} \tag{28.21}$$
one obtains
$$\int_\Omega d^4x k^\mu = \int_R dV_\nu T^{\mu\nu} - \int_{R'} dV_\nu T^{\mu\nu}. \tag{28.22}$$
In a charge-free space ($k^\mu = 0$), i.e. for free electromagnetic waves one finds that the components of the momentum of radiation
$$G_s^\mu = -\frac{1}{c}\int_R dV_\nu T^{\mu\nu} \tag{28.23}$$
are independent of R. Thus they are conserved. Now be (b_μ) an arbitrary but constant four-vector. Then $b_\mu T^{\mu\nu}$ is a four-vector and $\partial_\nu(b_\mu T^{\mu\nu}) = 0$. Then $b_\mu G_s^\mu$ is a four-scalar and G_s^μ is a four-vector.
If there are charges in the four-volume Ω, then one obtains.
$$G_s^\mu(R) = -\frac{1}{c}\int_\Omega d^4x k^\mu + G_s^\mu(R'). \tag{28.24}$$
For point-charges q_i one has (26.7, 26.30)
$$\frac{1}{c}\int d^4x k^\mu = \sum_i \int dt K_i^\mu = \sum_i \int dt \dot{G}_i^\mu = \sum_i (G_i^\mu(R) - G_i^\mu(R')). \tag{28.25}$$
Here $G_i^\mu(R) = m_i u_i^\mu(R)$ is the four-momentum of the charge #i at the point where its worldline hits the three-dimensional space R. Then
$$G^\mu = G_s^\mu(R) + \sum_i G_i^\mu(R) \tag{28.26}$$
is the conserved four-momentum.

28.b.γ Angular Momentum and Movement of Center of Mass

Eq. (28.7) yields
$$\partial_\nu\left(x^\lambda T^{\mu\nu} - x^\mu T^{\lambda\nu}\right) = x^\lambda k^\mu - x^\mu k^\lambda + T^{\mu\lambda} - T^{\lambda\mu}. \tag{28.27}$$
Since the tensor T is symmetric, the last two terms cancel. We introduce the tensor
$$M_s^{\lambda\mu}(R) = -\frac{1}{c}\int_R dV_\nu (x^\lambda T^{\mu\nu} - x^\mu T^{\lambda\nu}\big). \tag{28.28}$$
It is antisymmetric $M_s^{\lambda\mu} = -M_s^{\mu\lambda}$. Due to (28.19) one has
$$M_s^{\lambda\mu}(R) = -\frac{1}{c}\int_\Omega d^4x\left(x^\lambda k^\mu - x^\mu k^\lambda\right) + M_s^{\lambda\mu}(R'). \tag{28.29}$$
For point-charges one obtains
$$\frac{1}{c}\int_\Omega d^4x\left(x^\lambda k^\mu - x^\mu k^\lambda\right) = \sum_i \int dt\left(x_i^\lambda K_i^\mu - x_i^\mu K_i^\lambda\right) = \sum_i \int dt \frac{d}{dt}\left(x_i^\lambda G_i^\mu - x_i^\mu G_i^\lambda\right), \tag{28.30}$$
since $\dot{x}^\lambda G^\mu = \dot{x}^\mu G^\lambda$. Therefore
$$M^{\lambda\mu}(R) = M_s^{\lambda\mu}(R) + M_m^{\lambda\mu}(R) \tag{28.31}$$
including the mechanical contribution
$$M_m^{\lambda\mu}(R) = \sum_i \left(x_i^\lambda G_i^\mu - x_i^\mu G_i^\lambda\right)\Big|_R \tag{28.32}$$
is a conserved quantity, i.e. $M^{\lambda\mu}(R)$ is independent of the choice of R. Simultaneously $(M^{\lambda\mu})$ is a four-tensor.

Finally we have to determine the meaning of M. For this purpose we consider M in the three-dimensional space R given by constant time t for a system of inertia S. Then we have

$$M^{\lambda\mu} = -\frac{1}{c}\int d^3x\Big(x^\lambda T^{\mu 0} - x^\mu T^{\lambda 0}\Big) + \sum_i \Big(x_i^\lambda G_i^\mu - x_i^\mu G_i^\lambda\Big) \tag{28.33}$$

First we consider the space-like components

$$M^{\alpha\beta} = \int d^3x\Big(x^\alpha g_s^\beta - x^\beta g_s^\alpha\Big) + \sum_i \Big(x_i^\alpha G_i^\beta - x_i^\beta G_i^\alpha\Big). \tag{28.34}$$

This is for $\alpha \neq \beta$ a component of the angular momentum $\mathbf{L}$, namely $\epsilon_{\alpha\beta\gamma}L_\gamma$. Thus we have found the conservation of angular momentum.
If one component is time-like then one finds

$$M^{0\alpha} = ct\Big(\int d^3x g_s^\alpha + \sum_i G_i^\alpha\Big) - \frac{1}{c}\Big(\int d^3x x^\alpha u + \sum_i x_i^\alpha E_i\Big). \tag{28.35}$$

The first contribution is ct multiplied by the total momentum. The second contribution is the sum of all energies times their space-coordinates x^α divided by c. This second contribution can be considered as the center of energy (actually its α-component) multiplied by the total energy divided by c. Since total momentum and total energy are constant, one concludes that the center of energy moves with the constant velocity $c^2 \frac{\text{total momentum}}{\text{total energy}}$.
For non-relativistic velocities the mechanical part of the energy reduces to

$$M_m^{0\alpha} = c\left(t\sum_i G_i^\alpha - \sum_i m_i x_i^\alpha\right). \tag{28.36}$$

Then the conservation of this quantity comprises the uniform movement of the center of mass with the velocity total momentum divided by total mass. In the theory of relativity this transforms into a uniform moving center of energy. Lorentz invariance combines this conservation with the conservation of angular momentum to the antisymmetric tensor M.

29 Field of an arbitrarily Moving Point-Charge

29.a LIÉNARD-WIECHERT Potential

First we determine the potential at a point (x^μ) of a point-charge q which moves along a world-line $\mathbf{r}_q(t)$. Its four-current density reads

$$j^\mu(x') = qv^\mu\delta^3(\mathbf{x}' - \mathbf{r}_q(t)), \quad v^\mu = (c, \dot{\mathbf{r}}_q(t)). \tag{29.1}$$

According to (24.29) the four-potential reads

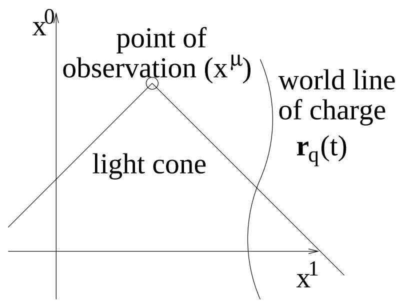
$$A^\mu(x) = \frac{1}{c}\int d^4x'\, j^\mu(x')\delta(\frac{1}{2}s^2)\theta(t-t') = q\int dt'\, v^\mu(t')\delta(\frac{1}{2}s^2)\theta(t-t') \tag{29.2}$$

with

$$s^2 = a^\nu a_\nu, \quad a^\nu = x^\nu - x^\nu_q(t'). \tag{29.3}$$

(a^ν) is a function of (x^ν) and t'. The differential of $\frac{1}{2}s^2$ is given by

$$d(\frac{1}{2}s^2) = a_\nu da^\nu = a_\nu dx^\nu - a_\nu v^\nu dt'. \tag{29.4}$$



Thus one obtains the LIÉNARD-WIECHERT potential

$$A^\mu(x) = qv^\mu(t')\frac{1}{|\frac{\partial\frac{1}{2}s^2}{\partial t'}|} = \left.\frac{qv^\mu}{a_\nu v^\nu}\right|_r = \left.\frac{qu^\mu}{a_\nu u^\nu}\right|_r. \tag{29.5}$$

Here the two expressions with the index $_r$ are to be evaluated at the time t' at which $s^2 = 0$ and $t > t'$.
We note that $a_\nu v^\nu = ac - \mathbf{a}\cdot\mathbf{v} > 0$, since $a = c(t-t') = |\mathbf{a}|$. $a_\nu u^\nu/c$ is the distance between point of observation and charge in the momentary rest system of the charge.

29.b The Fields

Starting from the potentials we calculate the fields

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{29.6}$$

In order to do this we have to determine the derivatives of v, a and t'

$$\partial^\mu v^\nu = \frac{\partial v^\nu}{\partial t'}\frac{\partial t'}{\partial x_\mu} \tag{29.7}$$

$$\partial^\mu a^\nu = \partial^\mu(x^\nu - x^\nu_q(t')) = g^{\mu\nu} - v^\nu\frac{\partial t'}{\partial x_\mu} \tag{29.8}$$

$$\frac{\partial t'}{\partial x_\mu} = \frac{a^\mu}{(a\cdot v)}, \tag{29.9}$$

where the last expression has been obtained from $s^2 = 0$ by means of (29.4). Here and in the following we use

$$(a\cdot v) = a^\nu v_\nu = ac - \mathbf{a}\cdot\mathbf{v} = c(a - \mathbf{a}\cdot\boldsymbol{\beta}) \tag{29.10}$$

$$(v\cdot v) = v^\nu v_\nu = c^2 - v^2 = c^2(1-\beta^2) \tag{29.11}$$

$$(a\cdot\dot{v}) = a^\nu\dot{v}_\nu = -\mathbf{a}\cdot\dot{\mathbf{v}}. \tag{29.12}$$

One evaluates

$$\partial^\mu v^\nu = \frac{\dot{v}^\nu a^\mu}{(a\cdot v)} \tag{29.13}$$

$$
\begin{aligned}
\partial^\mu a^\nu &= g^{\mu\nu} - \frac{v^\nu a^\mu}{(a\cdot v)} & (29.14)\\
\partial^\mu (a\cdot v) &= (\partial^\mu a^\kappa) v_\kappa + a_\kappa (\partial^\mu v^\kappa)\\
&= g^{\mu\kappa} v_\kappa - \frac{v^\kappa a^\mu}{(a\cdot v)} v_\kappa + a_\kappa \frac{\dot v^\kappa a^\mu}{(a\cdot v)}\\
&= v^\mu - a^\mu \frac{(v\cdot v)}{(a\cdot v)} + a^\mu \frac{(a\cdot \dot v)}{(a\cdot v)}. & (29.15)
\end{aligned}
$$

Then one obtains

$$
\begin{aligned}
\partial^\mu A^\nu &= \partial^\mu \left(q \frac{v^\nu}{(a\cdot v)} \right) = q \frac{\partial^\mu v^\nu}{(a\cdot v)} - q \frac{v^\nu \partial^\mu (a\cdot v)}{(a\cdot v)^2}\\
&= a^\mu b^\nu - q \frac{v^\mu v^\nu}{(a\cdot v)^2}, & (29.16)\\
b^\nu &= q \frac{v^\nu (v\cdot v) - v^\nu (a\cdot \dot v) + \dot v^\nu (a\cdot v)}{(a\cdot v)^3}. & (29.17)
\end{aligned}
$$

Therefore

$$
(b^\nu) = \frac{q}{(a - \mathbf{a}\cdot\boldsymbol\beta)^3} \left(1 - \beta^2 + \frac{\mathbf{a}\cdot\dot{\boldsymbol\beta}}{c}, \boldsymbol\beta (1-\beta^2) + \frac{1}{c}\boldsymbol\beta (\mathbf{a}\cdot\dot{\boldsymbol\beta}) + \frac{1}{c}(a - \mathbf{a}\cdot\boldsymbol\beta)\dot{\boldsymbol\beta} \right) \quad (29.18)
$$

and the fields read

$$
\begin{aligned}
F^{\mu\nu} &= a^\mu b^\nu - a^\nu b^\mu & (29.19)\\
\mathbf{E} = \mathbf{a} b^0 - a\mathbf{b} &= \frac{q(1-\beta^2)(\mathbf{a} - \boldsymbol\beta a)}{(a - \mathbf{a}\cdot\boldsymbol\beta)^3} + \frac{q\mathbf{a}\times((\mathbf{a}-\boldsymbol\beta a)\times\dot{\boldsymbol\beta})}{c(a - \mathbf{a}\cdot\boldsymbol\beta)^3} & (29.20)\\
\mathbf{B} &= -\mathbf{a}\times\mathbf{b} = \frac{\mathbf{a}\times\mathbf{E}}{a} & (29.21)
\end{aligned}
$$

The contribution proportional to the acceleration $\dot{\boldsymbol\beta}$ decreases like $1/a$; $\mathbf{a}$, $\mathbf{E}$, and $\mathbf{B}$ constiute an orthogonal system for this contribution. The contribution independent of $\dot{\boldsymbol\beta}$ falls off like $1/a^2$.

29.c Uniform Motion

(compare section 25.d). The scalar $\gamma a^\lambda v_\lambda / c$ is the distance between the point of observation and the point of the charge in the rest-system of the charge. Thus one has

$$
a - \mathbf{a}\cdot\boldsymbol\beta = \frac{1}{\gamma}|\mathbf{r}'|, \quad (a - \mathbf{a}\cdot\boldsymbol\beta)^3 = N/\gamma^3. \quad (29.22)
$$

Considering that $\mathbf{a} = \mathbf{r} - \mathbf{v}t'$, $a = c(t - t')$, one obtains

$$
\mathbf{a} - \boldsymbol\beta a = \mathbf{r} - \mathbf{v}t' - \mathbf{v}t + \mathbf{v}t' = \mathbf{r} - \mathbf{v}t \quad (29.23)
$$

and thus

$$
\mathbf{E} = \frac{q\gamma(\mathbf{r} - \mathbf{v}t)}{N}, \quad \mathbf{B} = \frac{(\mathbf{r} - \mathbf{v}t')\times(\mathbf{r} - \mathbf{v}t)q\gamma}{c(t-t')N} = \frac{q\gamma\mathbf{v}\times\mathbf{r}}{cN} \quad (29.24)
$$

in accordance with (25.30) and (25.31).

29.d Accelerated Charge Momentarily at Rest

The equations (29.20) and (29.21) simplify for $\beta = 0$ to

$$
\begin{aligned}
\mathbf{E} &= \frac{q\mathbf{a}}{a^3} + \frac{q}{ca^3}\mathbf{a}\times(\mathbf{a}\times\dot{\boldsymbol\beta}) & (29.25)\\
\mathbf{B} &= -\frac{q}{ca^2}(\mathbf{a}\times\dot{\boldsymbol\beta}), & (29.26)
\end{aligned}
$$

from which the power radiated into the solid angle $d\Omega$ can be determined with the energy-current density $\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}$

$$\frac{\mathrm{d}\dot{U}_\mathrm{s}}{\mathrm{d}\Omega} = a^2\mathbf{S}\cdot\mathbf{n} = \frac{ca}{4\pi}[\mathbf{a},\mathbf{E},\mathbf{B}] = \frac{q^2}{4\pi ca^2}(\mathbf{a}\times\dot{\boldsymbol{\beta}})^2 = \frac{q^2}{4\pi c^3}(\mathbf{n}\times\dot{\mathbf{v}})^2 \tag{29.27}$$

and the total radiated power

$$\dot{U}_\mathrm{s} = \frac{2}{3}\frac{q^2}{c^3}\dot{\mathbf{v}}^2 \tag{29.28}$$

(LARMOR-formula) follows.
For a harmonic motion $\mathbf{r}_\mathrm{q} = \mathbf{r}_{0\mathrm{q}}\cos(\omega t)$ and $\dot{\mathbf{v}} = -\mathbf{r}_{0\mathrm{q}}\omega^2\cos(\omega t)$ one obtains

$$\dot{U}_\mathrm{s} = \frac{2}{3}\frac{q^2\mathbf{r}_{0\mathrm{q}}^2}{c^3}\omega^4(\cos(\omega t))^2, \quad \overline{\dot{U}_\mathrm{s}} = \frac{1}{3}\frac{p_0^2}{c^3}\omega^4 \tag{29.29}$$

in agreement with section 22.b. This applies for $\beta \ll 1$. Otherwise one has to take into account quadrupole and higher multipole contributions in 22.b, and here that β cannot be neglected anymore, which yields additional contributions in order ω^6 and higher orders.

29.e Emitted Radiation $\beta \neq 0$

We had seen that in the system momentarily at rest the charge emits the power $\dot{U}_\mathrm{s} = \frac{2}{3}\frac{q^2}{c^3}\dot{\mathbf{v}}^2$. The emitted momentum vanishes because of the symmetry of the radiation (without consideration of the static contribution of $\mathbf{E}$, which, however, decays that fast that it does not contribute for sufficiently large a)

$$\mathbf{E}(-\mathbf{a}) = \mathbf{E}(\mathbf{a}), \quad \mathbf{B}(-\mathbf{a}) = -\mathbf{B}(\mathbf{a}), \quad T_{\alpha\beta}(-\mathbf{a}) = T_{\alpha\beta}(\mathbf{a}). \tag{29.30}$$

Thus we may write the energy-momentum-vector emitted per proper time

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\frac{1}{c}U_\mathrm{s},\mathbf{G}_\mathrm{s}\Big) = \frac{u^\mu}{c}\frac{2q^2}{3c^3}\left(-\frac{\mathrm{d}u^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}u_\lambda}{\mathrm{d}\tau}\right), \tag{29.31}$$

since $\dot{u}^0 = c\dot{\gamma} \propto \mathbf{v}\cdot\dot{\mathbf{v}} = 0$. Since the formula is written in a lorentz-invariant way, it holds in each inertial frame, i.e.

$$\begin{aligned}\frac{\mathrm{d}U_\mathrm{s}}{\mathrm{d}t} &= \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{u^0}{c}\frac{2q^2}{3c^3}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\left(-\frac{\mathrm{d}(\gamma v^\lambda)}{\mathrm{d}t}\frac{\mathrm{d}(\gamma v_\lambda)}{\mathrm{d}t}\right)\\ &= \frac{2q^2}{3c^3}\gamma^2\left((\gamma\mathbf{v})\dot{}(\gamma\mathbf{v})\dot{} - c^2\dot{\gamma}^2\right)\\ &= \frac{2q^2}{3c^3}\gamma^2\left(\gamma^2\dot{\mathbf{v}}^2 + 2\gamma\dot{\gamma}(\mathbf{v}\cdot\dot{\mathbf{v}}) + \dot{\gamma}^2(\mathbf{v}^2 - c^2)\right).\end{aligned} \tag{29.32}$$

With $\mathrm{d}\tau/\mathrm{d}t\cdot u^0/c = 1$ and

$$\dot{\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \gamma^3\frac{\mathbf{v}\cdot\dot{\mathbf{v}}}{c^2} \tag{29.33}$$

one obtains finally

$$\dot{U}_\mathrm{s} = \frac{2}{3}\frac{q^2}{c^3}\left(\gamma^4\dot{\mathbf{v}}^2 + \gamma^6\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right). \tag{29.34}$$

Orbiting in a synchrotron of radius r a charge undergoes the acceleration $\dot{v} = v^2/r$ perpendicular to its velocity. Thus one has

$$\dot{U}_\mathrm{s} = \frac{2}{3}q^2c\beta^4\gamma^4/r^2 = \frac{2}{3}q^2c(\gamma^2-1)^2/r^2. \tag{29.35}$$

The radiated energy per circulation is

$$\Delta U_\mathrm{s} = \frac{2\pi r}{v}\dot{U}_\mathrm{s} = \frac{4\pi}{3}q^2\beta^3\gamma^4/r. \tag{29.36}$$

At Desy one obtains for an orbiting electron of energy E =7.5 GeV and mass m_0c^2 =0.5 MeV a value $\gamma = E/(m_0c^2) = 15000$. For $r = 32$m one obtains $\Delta U = 9.5$MeV. Petra yields with $E = 19$GeV a $\gamma = 38000$ and with $r = 367$m a radiation $\Delta U = 34$MeV per circulation.
Exercise Hera at Desy has $r = 1008$m and uses electrons of $E_\mathrm{e} = 30$GeV and protons of Ep = 820GeV. Calculate the energy radiated per circulation.