A Appendix: Formulae

A.a Γ-function

The Γ-function is defined by

$$\Gamma(n+1) = n! = \int_0^\infty \mathrm{d}x\, x^n \mathrm{e}^{-x} \tag{A.1}$$

One shows by partial integration

$$\Gamma(n+1) = n\Gamma(n), \quad n! = n(n-1)! \tag{A.2}$$

Special values are

$$\Gamma(1) = 0! = 1, \qquad \Gamma(n) = (n-1)! = 1 \cdot 2 \cdot 3 ... (n-1) \tag{A.3}$$

$$(-1/2)! = \Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(n+\frac{1}{2}) = \sqrt{\pi}\frac{1 \cdot 3 \cdot 5 ... (2n-1)}{2^n}. \tag{A.4}$$

For large argument one obtains the Stirling-formula

$$n! = \Gamma(n+1) = \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n. \tag{A.5}$$

A.b Bose- and fermi-integrals with ζ-function

A number of integrals for bosons and fermions can be expressed in terms of the Riemannian ζ-function. It is defined by

$$\zeta(p) = \sum_{n=1}^\infty \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + ... \tag{A.6}$$

In calculations for bosons one finds

$$\int_0^\infty \mathrm{d}x \frac{x^p}{\mathrm{e}^x - 1} = \int_0^\infty \mathrm{d}x \frac{x^p \mathrm{e}^{-x}}{1 - \mathrm{e}^{-x}} = \sum_{n=1}^\infty \int_0^\infty \mathrm{d}x x^p \mathrm{e}^{-nx} = \sum_{n=1}^\infty \frac{1}{n^{p+1}} \int_0^\infty \mathrm{d}y y^p \mathrm{e}^{-y} = \Gamma(p+1)\zeta(p+1), \tag{A.7}$$

where $y = nx$ has been substituted. Similarly one finds

$$-\int_0^\infty \mathrm{d}x x^p \ln(1 - \mathrm{e}^{-x}) = \Gamma(p+1)\zeta(p+2). \tag{A.8}$$

For fermions one has similar integrals. By means of

$$\frac{1}{\mathrm{e}^x + 1} = \frac{1}{\mathrm{e}^x - 1} - \frac{2}{\mathrm{e}^{2x} - 1} \tag{A.9}$$

one obtains

$$\int_0^\infty \mathrm{d}x \frac{x^p}{\mathrm{e}^x + 1} = (1 - 2^{-p}) \int_0^\infty \mathrm{d}x \frac{x^p}{\mathrm{e}^x - 1} = (1 - 2^{-p})\Gamma(p+1)\zeta(p+1) \tag{A.10}$$

and from

$$\ln(1 + \mathrm{e}^{-x}) = \ln(1 - \mathrm{e}^{-2x}) - \ln(1 - \mathrm{e}^{-x}) \tag{A.11}$$

one obtains

$$\int_0^\infty \mathrm{d}x x^p \ln(1 + \mathrm{e}^{-x}) = (2^{-p-1} - 1) \int_0^\infty \mathrm{d}x x^p \ln(1 - \mathrm{e}^{-x}) = (1 - 2^{-p-1})\Gamma(p+1)\zeta(p+2). \tag{A.12}$$

Special values are

$$\begin{aligned} &\zeta(3/2) = 2.612, && \zeta(2) = \pi^2/6 = 1.645, \\ &\zeta(5/2) = 1.341, && \zeta(3) = 1.202, \\ &\zeta(7/2) = 1.127, && \zeta(4) = \pi^4/90 = 1.0823. \end{aligned} \tag{A.13}$$

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