The solutions for Assignment 1 are studied in detail in pages 38 – 47 of Matteo Bertolini’s Lectures on Supersymmetry. The complete notes can be found at [http://people.sissa.it/~bertmat/susycourse.pdf](http://people.sissa.it/~bertmat/susycourse.pdf), however the relevant pages are also extracted and included here.
3 Representations of the supersymmetry algebra

In this lecture we will discuss representations of the supersymmetry algebra. Let us first briefly recall how things go for the Poincaré algebra. The Poincaré algebra (2.25) has two Casimir (i.e. two operators which commute with all generators)

\[ P^2 = P_\mu P^\mu \quad \text{and} \quad W^2 = W_\mu W^\mu, \]

where \( W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \) is the so-called Pauli-Lubanski vector. Casimir operators are useful to classify irreducible representations of a group. In the case of the Poincaré group such representations are nothing but what we usually call particles. Let us see how this is realized for massive and massless particles, respectively.

Let us first consider a massive particle with mass \( m \) and go to the rest frame, \( P_\mu = (m, 0, 0, 0) \). In this frame it is easy to see that the two Casimir reduce to \( P^2 = m^2 \) and \( W^2 = -m^2 s(s + 1) \) where \( s \) is the spin. The second equality can be proven by noticing that \( W_\mu P^\mu = 0 \) which implies that in the rest frame \( W_0 = 0 \). Therefore in the rest frame \( W_\mu = (0, \frac{1}{2} \epsilon_{ijk} m M^{jk}) \) from which one immediately gets \( W^2 = -m^2 \vec{J}^2 \). So we see that massive particles are distinguished by their mass and their spin.

Let us now consider massless particles. In this case we have that \( P^2 = 0 \) and \( W^2 = 0 \). In the rest frame \( P_\mu = (E, 0, 0, E) \). This implies that \( W^\mu = M_{12} P^\mu \). In other words, the two operators are proportional for a massless particle, the constant of proportionality being \( M_{12} = \pm s \), the helicity. For these representations the spin is then fixed and the different states are distinguished by their energy and by the sign of the helicity (e.g. the photon is a massless particle with two helicity states, \( \pm 1 \)).

Now, as a particle is an irreducible representation of the Poincaré algebra, we call superparticle an irreducible representation of the supersymmetry algebra. Since the Poincaré algebra is a subalgebra of the supersymmetry algebra, it follows that any irreducible representation of the supersymmetry algebra is a representation of the Poincaré algebra, which in general will be reducible. This means that a superparticle corresponds to a collection of particles, the latter being related by the action of the supersymmetry generators \( Q_\alpha \) and \( \bar{Q}_\dot{\alpha} \) and having spins differing by units of half. Being a multiplet of different particles, a superparticle is often called supermultiplet.

Before discussing in detail specific representations of the supersymmetry algebra, let us list three generic properties any such representation enjoys, all of them having very important physical implications.
1. As compared to the Poincaré algebra, in the supersymmetry algebra $P^2$ is still a Casimir, but $W^2$ is not anymore (this follows from the fact that $M_{\mu\nu}$ does not commute with the supersymmetry generators). Therefore, particles belonging to the same supermultiplet have the same mass and different spin, since the latter is not a conserved quantum number of the representation. The mass degeneracy between bosons and fermions is something we do not observe in known particle spectra; this implies that supersymmetry, if at all realized, must be broken in Nature, at energies of the order the EW scale or higher.

2. In a supersymmetric theory the energy of any state is always $\geq 0$. Consider an arbitrary state $|\phi\rangle$. Using the supersymmetry algebra, we easily get

\[
\langle \phi | \{ Q_I^\dagger, \bar{Q}_{\dot{I}}^\dagger \} | \phi \rangle = 2 \sigma_\alpha \delta^{IJ} \delta_{IJI} (\bar{Q}_{\dot{\alpha}}^I = (Q^I_{\alpha})^\dagger)
\]

\[
= \langle \phi | (Q^I_\alpha (Q^I_{\dot{\alpha}})^\dagger + (Q^I_{\dot{\alpha}})^\dagger Q^I_\alpha) | \phi \rangle
\]

\[
= ||(Q^I_{\dot{\alpha}})^\dagger|\phi\rangle|^2 + ||Q^I_{\alpha}|\phi\rangle|^2 \geq 0 .
\]

The last inequality follows from positivity of the Hilbert space. Summing over $\alpha = \dot{\alpha} = 1, 2$ and recalling that $\text{Tr} \sigma^\mu = 2 \delta^{\mu 0}$ we get

\[
4 \langle \phi | P_0 | \phi \rangle \geq 0 ,
\]

as anticipated.

3. A supermultiplet contains an equal number of bosonic and fermionic d.o.f., $n_B = n_F$. Define a fermion number operator

\[
(-1)^{N_F} = \begin{cases} 
-1 & \text{fermionic state} \\
+1 & \text{bosonic state}
\end{cases}
\]

$N_F$ can be taken to be twice the spin, $N_F = 2s$. Such an operator, when acting on a bosonic, respectively a fermionic state, gives indeed

\[
(-1)^{N_F} |B\rangle = |B\rangle , \quad (-1)^{N_F} |F\rangle = -|F\rangle .
\]

We want to show that $\text{Tr} (-1)^{N_F} = 0$ if the trace is taken over a finite dimensional representation of the supersymmetry algebra. First notice that

\[
\{ Q^I_\alpha , (-1)^{N_F} \} = 0 \rightarrow Q^I_{\alpha} (-1)^{N_F} = -(-1)^{N_F} Q^I_{\alpha} .
\]
Using this property and the ciclicity of the trace one easily sees that
\[ 0 = \text{Tr} \left( -Q^I_{\alpha}(-1)^{N_F} \bar{Q}^I_{\dot{\beta}} + (-1)^{N_F} \bar{Q}^I_{\dot{\beta}}Q^I_{\alpha} \right) \]
\[ = \text{Tr} \left( (-1)^{N_F} \left\{ Q^I_{\alpha}, \bar{Q}^I_{\dot{\beta}} \right\} \right) = 2\sigma^{\mu}_{\alpha\dot{\beta}} \text{Tr} \left[ (-1)^{N_F} \right] P_\mu \delta^{IJ} \]

Summing on $I, J$ and choosing any $P_\mu \neq 0$ it then follows that $\text{Tr} (-1)^{N_F} = 0$, which implies that $n_B = n_F$.

In the following, we discuss (some) representations in detail. Since the mass is a conserved quantity in a supermultiplet, it is meaningful distinguishing between massless and massive representations. Let us start from the former.

### 3.1 Massless supermultiplets

Let us first assume that all central charges vanish, i.e. $Z^{IJ} = 0$ (we will see later that this is the only relevant case, for massless representations). Notice that in this case it follows from eqs. (2.56) and (2.57) that all $Q$’s and all $\bar{Q}$’s commute among themselves. The steps to construct the irreps are as follows:

1. Go to the rest frame where $P_\mu = (E, 0, 0, E).$ In such frame we get
   \[
   \sigma^\mu P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}
   \]
   Plug in this into eq. (2.55) we get
   \[
   \left\{ Q^I_{\alpha}, \bar{Q}^I_{\dot{\beta}} \right\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix} \delta^{IJ} \to \left\{ Q^I_{1}, \bar{Q}^I_{1} \right\} = 0.
   \]
   Due to the positiveness of the Hilbert space, this implies that both $Q^I_{1}$ and $\bar{Q}^I_{1}$ are trivially realized. Indeed
   \[
   0 = \langle \phi | \left\{ Q^I_{1}, \bar{Q}^I_{1} \right\} | \phi \rangle = ||Q^I_{1}|\phi||^2 + ||\bar{Q}^I_{1}|\phi||^2,
   \]
   whose only solution is $Q^I_{1} = \bar{Q}^I_{1} = 0$. We are then left with just $Q^I_{2}$ and $\bar{Q}^I_{2}$, hence only $N$ out of the original $2N$ chiral (respectively antichiral) generators.

2. From the non-trivial generators we can define
   \[
   a_I \equiv \frac{1}{\sqrt{4E}} Q^I_{2} \quad , \quad a^I \equiv \frac{1}{\sqrt{4E}} \bar{Q}^I_{2}.
   \]
These operators satisfy the anticommutation relations of a set of creation and annihilation operators
\[ \{a_I, a_J^\dagger\} = \delta^{IJ}, \quad \{a_I, a_J\} = 0, \quad \{a_I^\dagger, a_J^\dagger\} = 0. \] (3.10)
These are the basic tools we need in order to construct irreps of the supersymmetry algebra. Notice that when acting on some state, the operators \( Q_I^2 \) and \( \bar{Q}_I^2 \) (and hence \( a_I \) and \( a_I^\dagger \)) lower respectively rise the elicity of half unit, since
\[ [M_{12}, Q_I^2] = i(\sigma_{12})_2 Q_I^2 = -\frac{1}{2} Q_I^2, \quad [M_{12}, \bar{Q}_I^2] = \frac{1}{2} \bar{Q}_I^2, \] (3.11)
and \( J_3 = M_{12} \).

3. To construct a representation, one can start by choosing a state annihilated by all \( a_I \)'s (known as the Clifford vacuum): such state will carry some irrep of the Poincaré algebra. Besides having \( m = 0 \), it will carry some helicity \( \lambda_0 \), and we call it \( |E, \lambda_0\rangle \) (\( |\lambda_0\rangle \) for short). For this state
\[ a_I |\lambda_0\rangle = 0. \] (3.12)
Note that this state can be either bosonic or fermionic, and should not be confused with the actual vacuum of the theory, which is the state of minimal energy: the Clifford vacuum is a state with quantum numbers \((E, \lambda_0)\) and which satisfies eq. (3.12).

4. The full representation (aka supermultiplet) is obtained acting on \( |\lambda_0\rangle \) with the creation operators \( a_I^\dagger \) as follows
\[ |\lambda_0\rangle, \quad a_I^\dagger |\lambda_0\rangle \equiv |\lambda_0 + \frac{1}{2}\rangle_I, \quad a_I^\dagger a_J^\dagger |\lambda_0\rangle \equiv |\lambda_0 + 1\rangle_{IJ}, \]
\[ \ldots, \quad a_I^\dagger a_2^\dagger \ldots a_N^\dagger |\lambda_0\rangle \equiv |\lambda_0 + \frac{N}{2}\rangle. \]
Hence, starting from a Clifford vacuum with helicity \( \lambda_0 \), the state with higher helicity in the representation has helicity \( \lambda = \lambda_0 + \frac{k}{2} \). Due to the antisymmetry in \( I \leftrightarrow J \), at elicity level \( \lambda = \lambda_0 + \frac{k}{2} \) we have
\[ \# \text{ of states with helicity } \lambda_0 + \frac{k}{2} = \binom{N}{k}, \] (3.13)
where \( k = 0, 1, \ldots, N \). The total number of states in the irrep will then be
\[ \sum_{k=0}^{N} \binom{N}{k} = 2^N = (2^{N-1})_B + (2^{N-1})_F, \] (3.14)
half of them having integer helicity (bosons), half of them half-integer helicity (fermions).

5. CPT flips the sign of the helicity. Therefore, unless the helicity is distributed symmetrically around 0, which is not the case in general, a supermultiplet is not CPT-invariant. This means that in order to have a CPT-invariant theory one should in general double the supermultiplet we have just constructed adding its CPT conjugate. This is not needed if the supermultiplet is self-CPT conjugate, which can happen only if \( \lambda_0 = -\frac{N}{4} \) (in this case the helicity is indeed distributed symmetrically around 0).

Let us now apply the above procedure and construct several (physically interesting) irreps of the supersymmetry algebra.

**N = 1 supersymmetry**

- **Matter multiplet (aka chiral multiplet):**

  \[
  \lambda_0 = 0 \rightarrow \left( 0, \frac{1}{2} \right) \oplus_{CPT} \left( -\frac{1}{2}, 0 \right). \tag{3.15}
  \]

  The degrees of freedom of this representation are those of one Weyl fermion and one complex scalar (on shell; recall we are constructing supersymmetry representations on states!). In a \( N = 1 \) supersymmetric theory this is the representation where matter sits; this is why such multiplets are called matter multiplets. For historical reasons, these are also known as Wess-Zumino multiplets.

- **Gauge (or vector) multiplet:**

  \[
  \lambda_0 = \frac{1}{2} \rightarrow \left( \frac{1}{2}, 1 \right) \oplus_{CPT} \left( -1, -\frac{1}{2} \right). \tag{3.16}
  \]

  The degrees of freedom are those of one vector and one Weyl fermion. This is the representation one needs to describe gauge fields in a supersymmetric theory. Hence the name. Notice that since internal symmetries (but the \( R \)-symmetry) commute with the supersymmetry algebra, the representation the Weyl fermion should transform under gauge transformations should be the same as the vector field, i.e. the adjoint. Hence, usual SM matter (quarks and leptons) cannot be accommodated in these multiplets.
Although in this course we will focus on rigid supersymmetry and hence not consider supersymmetric theories with gravity, let us list for completeness (and future reference) also representations containing states with higher helicity.

- **Gravitino multiplet:**
  \[ \lambda_0 = 1 \rightarrow \left( 1, \frac{3}{2} \right) \oplus_{CPT} \left( -\frac{3}{2}, -1 \right). \tag{3.17} \]
  The degrees of freedom are those of a spin 3/2 particle and one vector. Notice that in a theory without gravity one cannot accept particles with helicity greater than one. Therefore, this multiplet cannot appear in a \( N = 1 \) supersymmetric theory if also a graviton, with helicity 2, does not appear.

- **Graviton multiplet:**
  \[ \lambda_0 = \frac{3}{2} \rightarrow \left( +\frac{3}{2}, +2 \right) \oplus_{CPT} \left( -2, -\frac{3}{2} \right). \tag{3.18} \]
  The degrees of freedom are those of a graviton, which has helicity 2, and a particle of helicity 3/2, known as the gravitino (which is indeed the supersymmetric partner of the graviton).

Representations constructed from a Clifford vacuum with higher helicity, will inevitably include states with helicity higher than 2. Hence, if one is interested in interacting local field theories, the story stops here. Recall that massless particles with helicity higher than \( \frac{1}{2} \) should couple to conserved quantities at low momentum. The latter are: conserved internal symmetry generators for (soft) massless vectors, supersymmetry generators for (soft) gravitinos and four-vector \( P_\mu \) for (soft) gravitons. The supersymmetry algebra does not allow for generators other than these ones; hence, supermultiplets with helicity \( \lambda \geq \frac{5}{2} \) are ruled out: they may exist, but they cannot have couplings that survive in the low energy limit.

**N = 2 supersymmetry**

- **Matter multiplet (aka hypermultiplet):**
  \[ \lambda_0 = -\frac{1}{2} \rightarrow \left( -\frac{1}{2}, 0, 0, +\frac{1}{2} \right) \oplus_{CPT} \left( -\frac{1}{2}, 0, 0, +\frac{1}{2} \right). \tag{3.19} \]
  The degrees of freedom are those of two Weyl fermions and two complex scalars. This is where matter sits in a \( N = 2 \) supersymmetric theory. In \( N = 1 \)
language this representation corresponds to two Wess-Zumino multiplets with opposite chirality (CPT flips the chirality). Notice that in principle this representation enjoys the CPT self-conjugate condition $\lambda_0 = -\frac{N}{4}$. However, a closer look shows that an hypermultiplet cannot be self-conjugate (that’s why we added the CPT conjugate representation). The way the various states are constructed out of the Clifford vacuum, shows that under $SU(2)$ $R$-symmetry the helicity 0 states behave as a doublet while the fermionic states are singlets. If the representation were CPT self-conjugate the two scalar degrees of freedom would have been both real. Such states cannot form a $SU(2)$ doublet since a two-dimensional representation of $SU(2)$ is pseudoreal, and hence the doublet should be complex.

- **Gauge (or vector) multiplet:**
  \[
  \lambda_0 = 0 \rightarrow \left(0, +\frac{1}{2}, +\frac{1}{2}, +1\right) \oplus_{CPT} \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right). \quad (3.20)
  \]
  The degrees of freedom are those of one vector, two Weyl fermions and one complex scalar. In $N = 1$ language this is just a vector and a matter multiplet (beware: both transforming in the same, i.e. adjoint representation of the gauge group).

- **Gravitino multiplet:**
  \[
  \lambda_0 = -\frac{3}{2} \rightarrow \left(-\frac{3}{2}, -1, -1, -\frac{1}{2}\right) \oplus_{CPT} \left(\frac{1}{2}, +1, +1, +\frac{3}{2}\right). \quad (3.21)
  \]
  The degrees of freedom are those of a spin 3/2 particle, two vectors and one Weyl fermion.

- **Graviton multiplet:**
  \[
  \lambda_0 = -2 \rightarrow \left(-2, -\frac{3}{2}, -\frac{3}{2}, -1\right) \oplus_{CPT} \left(+1, +\frac{3}{2}, +\frac{3}{2}, +2\right). \quad (3.22)
  \]
  The degrees of freedom are those of a graviton, two gravitinos and a vector, which is usually called graviphoton in the supergravity literature.

- **$N = 4$ supersymmetry**

- **Gauge (or vector) multiplet:**
  \[
  \lambda_0 = -1 \rightarrow \left(-1, 4 \times -\frac{1}{2}, 6 \times 0, 4 \times +\frac{1}{2}, +1\right). \quad (3.23)
  \]
The degrees of freedom are those of one vector, four Weyl fermions and three complex scalars. In \( N = 1 \) language this corresponds to one vector multiplet and three matter multiplets (all transforming in the adjoint). Notice that this multiplet is CPT self-conjugate. This time there are no problems with \( R \)-symmetry transformations. The vector is a singlet under \( SU(4) \), fermions transform under the fundamental representation, and scalars under the two times anti-symmetric representation, which is the fundamental of \( SO(6) \), and is real. The fact that the representations under which the scalars transform is real also explain why for \( \mathcal{N} = 4 \) supersymmetry, the \( R \)-symmetry group is not \( U(4) \) but actually \( SU(4) \).

For \( N = 4 \) it is not possible to have matter in the usual sense, since the number of supersymmetry generators is to high to avoid helicity one states. Therefore, \( N = 4 \) supersymmetry cannot accommodate fermions transforming in fundamental representations. Besides the vector multiplet there are of course also representations with higher helicity, but we refrain to report them here.

One might wonder why we did not discuss \( N = 3 \) representations. This is just because as far as non-gravitational theories are concerned, \( N = 3 \) and \( N = 4 \) are equivalent: when constructing \( N = 3 \) representations, once the CPT conjugate representation is added (in this case we cannot satisfy the condition \( \lambda_0 = -\frac{N}{4} \)) one ends up with a multiplet which is the same as the \( N = 4 \) vector multiplet.

**\( N > 4 \) supersymmetry**

In this case one can easily get convinced that it is not possible to avoid gravity since there do not exist representations with helicity smaller than \( \frac{3}{2} \) when \( N > 4 \). Hence, theories with \( N > 4 \) are all supergravity theories. It is interesting to note that \( N = 8 \) supergravity allows only one possible representation with highest helicity smaller than \( \frac{5}{2} \) and that for higher \( N \) one cannot avoid states with helicity \( \frac{5}{2} \) or higher. Therefore, \( N = 8 \) is an upper bound on the number of supersymmetry generators, as far as interacting local field theories are concerned. Beware: as stated, all these statements are valid in four space-time dimensions. The way to count supersymmetries depends on the dimension of space-time, since spinorial representations get bigger, the more the dimensions. Obviously, completely analogous statements can be made in higher dimensions. For instance, in ten space-time dimensions the maximum allowed supersymmetry to avoid states with helicity \( \frac{5}{2} \) or higher is \( N = 2 \). A dimension-independent statement can be made counting the number of single
component supersymmetry generators. Using this language, the maximum allowed number of supersymmetry generators for non-gravitational theories is 16 (which is indeed $N = 4$ in four dimensions) and 32 for theories with gravity (which is $N = 8$ in four dimensions).

One final, very important comment regards chirality. The SM is a chiral theory, in the sense that there exist particles in the spectrum whose chiral and anti-chiral components transform differently under the gauge group (weak interactions are chiral). When it comes to supersymmetric extensions, it is easy to see that only $N = 1$ theories allow for chiral matter. That $N = 1$ irreps can be chiral is obvious: Wess-Zumino multiplets contain one single Weyl fermion. Therefore, in $N = 1$ supersymmetric extensions of the SM one can accommodate left and right components of leptons and quarks in different multiplets, which can then transform differently under the $SU(2)$ gauge group. Let us now consider extended supersymmetry. First notice that all helicity $\frac{1}{2}$ states belonging to multiplets containing vector fields should transform in the adjoint representation of the gauge group, which is real. Therefore, the only other representation which might allow for helicity $\frac{1}{2}$ states transforming in fundamental representations is the $N = 2$ hypermultiplet. However, as already noticed, a hypermultiplet contains two Wess-Zumino multiplets with opposite chirality. Since for any internal symmetry group $G$, we have that $[G, \text{SuperPoincaré}] = 0$, these two Wess-Zumino multiplets transform in the same representation under $G$. Therefore, $N = 2$ is non-chiral: left and right components of leptons and quarks belong to the same matter multiplet and could not transform differently under the $SU(2)$ SM gauge group. Summarizing, if extended supersymmetry is realized in Nature, it should be broken at some high enough energy scale to an effective $N = 1$ model. This is why at low energy people typically focus just on $N = 1$ extensions of the SM.

The table below summarizes all results we have discussed.
$$\lambda_{\text{max}} = \frac{1}{2}$$

<table>
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<th>$\lambda_{\text{max}} = \frac{1}{2}$</th>
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</tr>
</tbody>
</table>

The numbers in paranthesis represent the helicity, while other numbers represent the multiplicity of states with given helicity. Notice that, as anticipated, any supermultiplet contains particles with spin at least as large as $\frac{1}{4}N$.

### 3.2 Massive supermultiplets

The logical steps one should follow for massive representations are similar to previous ones. There is however one important difference. Let us consider a state with mass $m$ in its rest frame $P_{\mu} = (m, 0, 0, 0)$. One can easily see that, differently from the massless case, the number of non-trivial generators gets not diminished: there remain the full set of $2N$ creation and $2N$ annihilation operators. Indeed, eq. (2.55) is now

$$\{Q^I_{\alpha}, \bar{Q}^J_{\dot{\beta}}\} = 2m \delta_{\alpha\dot{\beta}} \delta^{IJ}$$

and no supersymmetric generators are trivially realized. This means that, generically, massive representations are longer than massless ones. Another important difference is that we better speak of spin rather than helicity, now. A given Clifford vacuum will be defined by mass $m$ and spin $j$, with $j(j+1)$ being the eigenvalue of $J^2$. Hence, the Clifford vacuum will have degeneracy $2j + 1$ since $j_3$ takes value from $-j$ to $+j$.

**N = 1 supersymmetry**

The annihilation and creation operators, satisfying the usual oscillator algebra, now read

$$a_{1,2} \equiv \frac{1}{\sqrt{2m}}Q_{1,2} \quad , \quad a_{1,2}^\dagger \equiv \frac{1}{\sqrt{2m}}\bar{Q}_{1,2}$$

As anticipated these are twice those for the massless case. Notice that $a^\dagger_1$ lowers the