1) Bound states and strong interactions

Strong interactions between quarks and gluons are described by Quantum Chromo-Dynamics (QCD). The QCD Lagrangian for the light quarks u, d, s is

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G_{\mu\nu}^{a} G^{a\mu\nu} + \bar{q}(i\gamma^\mu D_\mu - m_q)q; \quad D_\mu = \partial_\mu + g_s G_\mu^a G^a,$$

with

$$q = \begin{pmatrix} u \\ d \end{pmatrix}; \quad m_q = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}.$$

Here, $G_{\mu\nu}^{a}$ is the gluon field strength tensor,

$$G_{\mu\nu}^{a} = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g_s f^{abc} G_\mu^b G_\nu^c; \quad [T^a, T^b] = if^{abc} T^c.$$

The coupling $g_s(\mu)$ is not a constant, but decreases as the energy scale $\mu$ increases; a phenomenon called asymptotic freedom. The "running" of $\alpha_s(\mu) = g_s^2(\mu)/4\pi$ is

$$\frac{d\alpha_s(\mu)}{d\ln\mu} = -2\beta_0 \frac{\alpha_s^2(\mu)}{4\pi} + O(\alpha_s^3(\mu)); \quad \beta_0 = M - \frac{2}{3} N_q,$$

where $N_q$ is the number of quark flavors. Thus,

$$\alpha_s(\mu^2) = \frac{1}{\frac{1}{\alpha_s(\mu_0^2)} + \frac{\beta_0}{4\pi} \ln \left( \frac{\mu^2}{\mu_0^2} \right)} \quad \rightarrow \quad \alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln \left( \frac{\mu^2}{\Lambda_{\text{QCD}}^2} \right)}.$$

For $\mu^2 \to \Lambda_{\text{QCD}}^2$, $\alpha_s(\mu)$ becomes very large and the perturbative expansion breaks down. (def. $\Lambda_{\text{QCD}} = \mu e^{-\frac{2\pi}{\beta_0 \alpha_s(\mu_0)}}$)
Experimentally, one measures
\[ Z^{(q+\bar{q})}_s (\mu = M_Z) = 0.184 \pm 0.0011. \]  
(Particle Data Group, 2016)

From this result, one deduces \( \Lambda_{QCD} \approx 200 \text{MeV} \). Around this scale, quarks are expected (and observed!) to be confined in hadrons, i.e., bound states that are neutral under strong interactions. For example,

**mesons**: \( q \bar{q}' \); **baryons**: \( qq'q'' \).

At \( \mu = \Lambda_{QCD} \gg m_q \), it is useful to consider the
**chiral limit** \( m_q \to 0 \) of QCD, where

\[ Z_q = \overline{q} i \gamma^\mu D_\mu q = \overline{q}_L i \gamma^\mu D_\mu q_L + \overline{q}_R i \gamma^\mu D_\mu q_R \]

has an \( SU(3)_L \times SU(3)_R \) chiral symmetry

\[ q_L \to V_L q_L; \quad q_R \to V_R q_R. \]

Here \( V_L, V_R \) are \( 3 \times 3 \) matrices describing \( SU(3)_L \) and \( SU(3)_R \) transformations, respectively.

The chiral symmetry is spontaneously broken by the QCD vacuum expectation value \( V_s \sim 1/\Lambda_{QCD}^3 \), the so-called quark condensate

\[ \langle \overline{q}_R q_L \rangle = V_s \delta_{ij}. \]

Under \( SU(3)_L \times SU(3)_R \):

\[ \langle \overline{q}_R^i q_L^j \rangle \to V_s (V_R^i V_L)_ij \frac{V_R^1 V_L^1}{V_s} \delta_{ij}. \]
The symmetry of the vacuum state of QCD is thus reduced to the diagonal subgroup $SU(3)_V$,

$$SU(3)_L \times SU(3)_R \xrightarrow{V_5} SU(3)_V = L + R.$$ 

The eight broken generators of the chiral symmetry correspond with eight (massless) Goldstone bosons. These can be written as a $3 \times 3$ special unitary matrix

$$\Sigma = \exp \left( \frac{2iM}{f} \right); \quad M = \begin{pmatrix} \frac{\pi^0}{2} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{\pi^0}{2} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-\eta}{\sqrt{6}} \end{pmatrix}.$$ 

The Goldstone bosons of QCD chiral symmetry breaking are physical! They correspond with the mesons made of light quarks; for instance,

$$\pi^+ : u \bar{d}; \quad K^0 : d \bar{s}; \quad \pi^0 : \frac{1}{\sqrt{2}} (u \bar{u} - d \bar{d}).$$

But these mesons are not massless. Experimentally,

$$M_{\pi^+} \approx 140 \text{ MeV} \approx M_{\pi^0} ; \quad M_{K^0} \approx 498 \text{ MeV} > M_{\pi^+, 0}.$$ 

Since $M_{K^0} > M_{\pi^+, 0}$, meson masses break $SU(3)_V$. Nevertheless, the effective theory based on chiral symmetry is useful to describe hadronic processes with kaons and pions ("chiral perturbation theory").