For a single Wilson coefficient, the solution in QCD at leading order is

\[
C(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{-\frac{\beta_0}{2\beta_0}} (1 + O(\alpha_s))
\]

\[\approx (1 + \beta_0 \frac{\alpha_s(M_W)}{4\pi} \ln \frac{M_W^2}{\mu^2})^{-\frac{\beta_0}{2\beta_0}} = 1 - \frac{\beta_0}{2} \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2} + O(\alpha_s^2 \ln^2).
\]

Back to our example, \(b \to u \bar{c}s\):

Since the one-loop diagrams \(2\) and \(3\) contribute to both \(C_1\) and \(C_2\), the operators \(O_1\) and \(O_2\) mix under the renormalization group. At LO in QCD, we obtain

\[
\frac{d}{d\alpha_s(\mu)} \left( \frac{C_1(\mu)}{C_2(\mu)} \right) = \frac{\beta_0}{-2\alpha_s} \left( \frac{C_1(\mu)}{C_2(\mu)} \right) = \frac{4}{-2\alpha_s} \frac{(6 - 2) (C_1(\mu))}{(C_2(\mu))}.
\]

It is convenient to choose a basis in which the RGE is diagonal, i.e., \(O_+ = \frac{O_1 + O_2}{2}\), \(C_\pm = C_1 \pm C_2\).

The solutions for \(C_\pm(\mu = \mu_b)\) at some scale \(\mu_b\) around the \(b\)-quark mass are then obtained as \((C_1(M_b) = 1, C_2(M_b) = 0)\)

\[
C_+(\mu_b) = \left( \frac{\alpha_s(\mu_b)}{\alpha_s(M_W)} \right)^{-\frac{6}{23}} \approx 0.8; \quad C_-(\mu_b) = \left( \frac{\alpha_s(\mu_b)}{\alpha_s(M_W)} \right)^{+\frac{12}{23}} \approx 1.4.
\]

These are sizeable corrections to the tree-level result \(C_+ = 1 = C_-\).

The amplitude for \(b \to u \bar{c}s\) transitions in the RG-improved effective theory is finally given by

\[
A = \frac{4}{\sqrt{2}} \left\{ C_+ (\mu_b) \langle \sigma_+ (\mu_b) \rangle + C_- (\mu_b) \langle \sigma_- (\mu_b) \rangle \right\}.
\]
Effective flavor-changing neutral currents (FCNCs)

In the SM, FCNC-induced processes occur first at the 1-loop level through weak interactions. This loop suppression could be lifted by a new heavy particle that generates FCNC at tree level. FCNCs are therefore very sensitive probes of new physics.

In the EFT, FCNCs are described by operators of mass dimension \( \geq 6 \). We identify 4 classes of FCNC operators.

1) Current-current operators

\[
0_{4f} = (\bar{s}_L \gamma_{\mu} b_L)(\bar{s}_L \gamma^\mu b_L) \\
(\bar{B}_s - \bar{B}_s \text{ meson mixing})
\]

2) QCD penguin operators

\[
0_{QCD} = (\bar{s}_L \gamma_{\mu} b_L) \sum_{q=u,d,c,s,b} (\bar{q}_L \gamma^\mu q_L) \\
(\bar{B}^0 \rightarrow K^-\pi^0)
\]

3) Electro-weak penguin operators

\[
0_{EW} = \left( \bar{s}_L \gamma_{\mu} s_L \right) \sum_{q=u,d,c,s,b} \frac{3}{2} \epsilon_{q} (\bar{q}_R \gamma^\mu q_R) \\
(\text{using } D_{q} \sigma_{\alpha}^{a} = -\frac{\epsilon}{2} g_{\alpha \beta} \bar{q}_{R} \gamma_{\beta} T_{a} q)
\]
4) Electromagnetic and chromomagnetic dipole operators

\[ O_{q} = - \frac{G_{F}}{\sqrt{2}} s_{L} \delta_{\mu\nu} F^{\mu\nu} b_{R} \]

\[ O_{q} = - \frac{G_{F}}{\sqrt{2}} s_{L} \delta_{\mu\nu} G^{\mu\nu}_{a} T_{a} b_{R} \]

\((B \to X_{s} \gamma)\)

Using the unitarity of the CKM matrix, we can simplify the flavor structure of the relevant terms in the Lagrangian. For penguin operators, we can substitute \( \lambda_{t} = - (\lambda_{u} + \lambda_{c}) \), where \( \lambda_{q} = V_{qL} V_{qS}^{*} \) for \( b \to s \); assuming \( m_{u} = 0, m_{c} = 0 \). Contributions of penguin operators can then be written as

\[ L_{eff} = - \frac{G_{F}}{\sqrt{2}} \sum_{i=a_{0}, a_{1}, E, W, Q} (\lambda_{u} + \lambda_{c}) C_{i} O_{i}. \]

The full operator basis for FCNC transitions can be found in W. Weibert, hep-ph/0512222.

**EFT application: \( B_{d} \to \bar{B}_{d} \) meson mixing**

The dominant contribution to \( B_{d} \to \bar{B}_{d} \) mixing stems from top-quark box diagrams,

\[ O_{3B-2} = (b_{L}^{d} \bar{d}_{L}^{d})(b_{L}^{d} \bar{d}_{L}^{d}) \]

(same as \( O_{1} \) in \( B_{d} \to \tau^{+} \tau^{-} \) example)
\[ L_{\text{eff}} = - \frac{G_F^2}{16 \pi^2} M_W^2 (V_{tb}^* V_{td})^2 C(\mu) \bar{O}_{AB=2} + \text{h.c.} \]

The Wilson coefficient is given by the loop function

\[ C(M_W) = S_0(x_t) = x_t \left( 4 - M_W^2 + x_t^2 \right) \frac{3 x_t^2 \ln x_t}{2(1-x_t)^3} \]

where \( x_t = \frac{m_t^2}{M_W^2} \).

The anomalous dimension of \( O_{AB=2} \) is the same as for \( O_7 \). At LO QCD, we have \( \gamma_+^0 \gamma_+^0 = 4 \).

The renormalization group evolution of \( C(\mu) \) down to the energy scale of \( B_d - \bar{B_d} \) mixing, \( \mu_B = \mu_b \), is thus described by

\[ C(\mu_b) = \left( \frac{\mathcal{L}_s(\mu_b)}{\mathcal{L}_s(M_W)} \right)^{\frac{2}{25}} C(M_W) = \left( \frac{\mathcal{L}_s(\mu_b)}{\mathcal{L}_s(M_b)} \right)^{\frac{6}{25}} S_0(x_t). \]

The matrix element describing \( B_d - \bar{B_d} \) mixing in RG-improved perturbation theory at LO is then obtained as

\[ 2 M_{\bar{B}_d} M_{B_d} = \langle \bar{B}_d | L_{\text{eff}}^{AB=2} | B_d \rangle \]

\[ = \frac{G_F^2}{16 \pi^2} M_W^2 (V_{tb}^* V_{td})^2 C(\mu_b) \langle \bar{B}_d | \bar{O}_{AB=2}(\mu_b) | B_d \rangle, \]

where \( \langle \bar{B}_d | \bar{O}_{AB=2}(\mu_b) | B_d \rangle \equiv \frac{8}{3} B_{\bar{d}d}(\mu_b) f_{\bar{d}}^2 M_{\bar{d}}^2 \).

and \( f_{\bar{d}} \) is the \( \bar{d} \) meson decay constant.

The parameter \( B_{\bar{d}d}(\mu_b) \) comprises non-perturbative QCD effects.
Using calculations of the combination $f_{8d}(\bar{B}_{8d})$, where $\bar{B}_{8d} = B_{8d}(\mu_b) L_8(\mu_b)^{-\frac{\epsilon}{2\pi}}$, from lattice gauge theory, we can eventually calculate the observable $\Delta M$ in $B_d - \bar{B}_d$ mixing,

$$\Delta M_{ld} = 2|M_{12}| = \frac{G_F^2}{6\pi^2} M_W^2 |V_{td}|^2 f_{ld}^2 B_{8d}^2 m_b M_{ld} \text{SU}(x),$$

with $m_b = L_8(M_W)^{\frac{\epsilon}{2\pi}}$ at LO.