Exercise 1: Dirac mass term

In quantum field theory, a Dirac fermion is described by a spinor field \( \psi = (\psi_L, \psi_R)^T \), where \( \psi_L \) and \( \psi_R \) denote left- and right-chiral components.

Show that the mass term for a Dirac particle with mass \( m \) can be written as

\[
m \bar{\psi} \psi = m \left( \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \right),
\]

i.e. that the terms \( \bar{\psi}_L \psi_L \) and \( \bar{\psi}_R \psi_R \) vanish.

Hint: Express \( \psi_L \) and \( \psi_R \) in terms of \( \psi \) and the projection operators \( P_L = (1 - \gamma_5)/2 \) and \( P_R = (1 + \gamma_5)/2 \).

Exercise 2: Charge conjugation and Majorana spinor

The charge conjugation transformation of a Dirac spinor is defined by \( \psi^C \equiv C \bar{\psi}^T \), with \( C = i \gamma^0 \gamma^2 \).

a) Show that under charge conjugation \( (\psi_L)^C = (\psi^C)_R \) and \( (\psi_R)^C = (\psi^C)_L \).

A Majorana particle is its own anti-particle, defined by the condition \( \psi^C = \psi \).

b) By imposing this condition on a general spinor, derive a relation between \( \psi_L \) and \( \psi_R \). How does the four-component Majorana spinor \( \psi_M \) then read?

Exercise 3: Neutrinoless double beta decay

A signature for neutrinos being Majorana particles is the neutrinoless double beta \( (0\nu2\beta) \) decay.

Draw a Feynman diagram for quark interactions contributing to the \( 0\nu2\beta \) decay of a nucleus \( N = (A, Z) \) with mass number \( A \) and atomic number \( Z \) at the quark level,

\[
(A, Z) \rightarrow (A, Z + 2) + 2e^-.
\]