$V = \begin{pmatrix} C_{12} C_{13} & S_{12} C_{13} & S_{13} e^{i\delta} \\ -S_{12} C_{23} - C_{12} S_{23} S_{13} e^{i\delta} & C_{12} C_{23} - S_{12} S_{23} S_{13} e^{i\delta} & S_{23} C_{13} \\ S_{12} S_{23} - C_{12} C_{23} S_{13} e^{i\delta} & -C_{12} S_{23} - S_{12} C_{23} S_{13} e^{i\delta} & C_{23} C_{13} \end{pmatrix}$

where $S_{ij} \equiv \sin \Theta_{ij}$, $C_{ij} \equiv \cos \Theta_{ij}$, $S_{23} \equiv \sin \Theta_{23}$, etc.

From measurements of charged currents, we obtain a hierarchical structure of the CKM elements,

$V \sim \begin{pmatrix} \epsilon^0 & \epsilon^1 & \epsilon^2 \\ \epsilon^1 & \epsilon^0 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 & \epsilon^0 \end{pmatrix}$, with $\epsilon \approx 0 \left(10^{-4}\right)$.

The unitarity condition of the CKM matrix, $VV^T = 1$, implies relations between its rows or columns,

$\sum_k V_{ki} V_{kj}^* = 0 ; \; j \neq i$.

For instance, for $i = d$, $j = b$, we obtain

$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0$.

Normalized to $V_{ud} V_{ub}^*$, this condition can be illustrated in terms of the so-called unitarity triangle,

with the angles

$\delta = \arg \left(-\frac{V_{td} V_{tb}^*}{V_{ud} V_{ub}^*}\right)$,

$\beta = \arg \left(-\frac{V_{cd} V_{cb}^*}{V_{td} V_{tb}^*}\right)$,

$\gamma = \arg \left(-\frac{V_{cd} V_{cb}^*}{V_{ud} V_{ub}^*}\right)$. 

(0,0) $R_c = 1$ (0,1)
The area of the unitarity triangle is given by
\[-\frac{1}{2} \text{Im} \frac{V_{ud} V_{ub}^*}{V_{cd} V_{cb}^*} = -\frac{1}{2} \frac{f}{|V_{cd} V_{cb}^*|}, \text{ with the}\]

Faraday invariant \( f = \text{Im} (V_{ud} V_{ub}^* V_{cb} V_{cd}^*) \).

Notice that \( \lambda, \beta, \delta \) and \( f \) are invariant under phase transformations of the quark fields and thus physical quantities.

\[
\text{Discrete symmetries} \\
| \text{parity } P: x^\mu \rightarrow -x^\mu; \text{ flips sign of spatial coordinates} \ |
| \text{time reversal } T: x^\mu \rightarrow -x^\mu \ |
| \text{charge conjugation } C: q_L \rightarrow q_R^c \ |
\]

Strong interactions conserve \( C \) and \( P \) separately. Weak interactions violate \( C \) and \( P \). To see this, we consider the charged current

\[
\mathcal{L} \sim \overline{u}_L V_{ud} d_L W^{+\mu} + \overline{d}_L V_{ub}^* d_R W^{-\mu}.
\]

\[
P: \overline{u}_L \gamma_{\mu} d_L \rightarrow \overline{u}_R \gamma_{\mu} d_R; \ W^{\pm\mu} \rightarrow W^{\mp\mu},
\]

\[
C: \overline{u}_L \gamma_{\mu} d_L \rightarrow -\overline{d}_R \gamma_{\mu} u_R; \ W^{\pm\mu} \rightarrow -W^{\mp\mu},
\]

\[
\rightarrow C \cdot C^{-1} \sim \overline{u}_R V_{ub} \gamma_{\mu} u_R W^{+\mu} + \overline{d}_R V_{ub}^* \gamma_{\mu} d_R W^{-\mu} \neq L;
\]

\[
P \cdot P^{-1} \sim \overline{u}_R V_{ud} \gamma_{\mu} u_R W^{+\mu} + \overline{d}_R V_{ub}^* \gamma_{\mu} d_R W^{-\mu} \neq L.
\]
How about CP?

$$CP \neq (CP)^\dagger \sim \bar{d}_L \nu_d \gamma^\mu m_L W^\mu + \bar{u}_L V_{ud}^* \gamma^\mu d_L W^\mu$$

$$= \times \text{ only if } V_{ud} = V_{ud}^*.$$  

In the standard model, the CKM phase $\delta$ is the only source of CP violation. In our parametrization of $V$, we can write the CP-violating invariant

$$f = -C_{12} C_{23} C_{13} S_{12} S_{23} S_{13} \sin \delta = O(10^{-5})$$

CP violation in weak interactions thus not only requires (at least) three generations of quarks, but also mixing among different quark flavors. Moreover, CP violation requires quarks with different flavors but equal gauge quantum numbers to have non-degenerate masses, i.e.

$$\Delta m^2_{ij} = m^2_i - m^2_j \neq 0 \ \forall \ i \neq j.$$  

3) Flavor-changing neutral currents (FCNC)

We have seen that the neutral currents due to Z boson interactions are flavor-universal in the SM. This is also true for QED and QCD interactions with a photon or gluons, since these are described by exact gauge symmetries in the Lagrangian.
Can there still be flavor-changing neutral interactions in the SM?

Let's consider two different $B$ meson decays:

$$\text{Br} (B^+ \to D^0 \pi^+ \nu_e) \approx 2 \times 10^{-4};$$

$$\text{Br} (B \to X_s \gamma) \underset{E_\gamma > 1.6 \text{GeV}}{\approx} 4 \times 10^{-4} < \text{Br} (B^+ \to D^0 l^+ \nu_e).$$

$$b \to W^- c \gamma \quad \mu \sim V_{cb} \times q^2 \sim q^2 \varepsilon^2$$

$$b \to W^- c \tau \nu \quad \mu \sim e \frac{q^2}{16 \pi^2} \sum_{i=m, c, t} V_{ib} V_{is}^{*} F \left( \frac{M_i^2}{M_W^2} \right)$$

For $M_i < M_W$, expand $F(x) \approx F(0) + x F'(0) + \ldots$:

$$\mu \sim \sum_{i=m, c, t} V_{ib} V_{is}^{*} F(0) + \sum_{i=m, c, t} V_{ib} V_{is}^{*} \frac{M_i^2}{M_W^2} F'(0).$$

With $V_{cb} V_{ts}^{*} = -\sum_{i=m, c, t} V_{ib} V_{is}^{*}$, we obtain:

$$\mu \approx -F'(0) \sum_{i=m, c} V_{ib} V_{is}^{*} \frac{M_i^2 - M_t^2}{M_W^2} \sim \left( V_{ub} V_{us}^{*} + V_{cb} V_{cs}^{*} \right) \frac{M_t^2}{M_W^2} \frac{\varepsilon^2}{\varepsilon^2}.$$ 

Notice that for $M_{ij}^2 \to 0$, $M \to 0$. Together with the CKM suppression ($\sim \varepsilon^2$), this feature is called the "GIM mechanism" (Glashow, Iliopoulos, Maiani).

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$^\$Although $M_t < M_W$ is not true, the main conclusions from this simple derivation hold for $M_t > M_W$. 
Yes, there are FCNCs in the SM, induced by charged currents at the loop level.

**Summary flavor structure of SM interactions:**

- Weak charged currents: $\sim V_{ij}$; CKM suppression
- Weak neutral currents: $\sim 1$; FCNC $\sim \frac{g^2}{16\pi^2} V_{ij} V_{ik}^{*}$; GIM suppression

QED, QCD: flavor-universal $\sim 1$;
Yukawa: flavor-diagonal $\sim \left( \begin{array}{ccc} y_{e,\mu,\tau} & 0 & 0 \\ 0 & y_{\nu_e,\nu_\mu} & 0 \\ 0 & 0 & y_{b,s,t} \end{array} \right)$.

**Flavor violation beyond the standard model**

Recall, in the SM the Yukawa couplings are the only source of flavor violation. Beyond the SM, a priori the flavor structure can be completely arbitrary. For instance, consider a second Higgs $H'$ coupling only to right-handed $D_R$ quarks,

$$L_A = - \overline{D_L} \lambda_{ij} D_R^{\dagger} H' + h.c.$$  

In the mass eigenbasis, this new interaction is in general not flavor-diagonal,

$$(\hat{\lambda})_{ij} = (V_{dL}^{\dagger} \lambda_{ij} V_{dR})_{ij} \equiv (\lambda_{FC})_{ij}.$$  

The new interaction $(\lambda_{FC})_{ij}$ induces scalar FCNCs, for instance, $(\lambda_{FC})_{32} \overline{b_L} s_R H'$, at tree level.
In general, FCNCs are strongly constrained by measurements. This has triggered the idea to copy the flavor structure of the SM in new interactions.

**Strategy:** "minimal flavor violation" (MFV)

Restore the SM flavor symmetry $G_F$ by treating the Yukawa couplings $Y_u, Y_d$ as auxiliary fields, "spurius", transforming under $SU(3)_c \times SU(3)_u \times SU(3)_d$ as

$$Y_u \sim (3, \overline{3}, 1); \quad Y_d \sim (3, 1, \overline{3}),$$

while the quark fields transform as

$$Q_L \sim (3, 1, 1); \quad U_R \sim (1, 3, 1); \quad D_R \sim (1, 1, 3).$$

We can form the flavor-symmetric interactions

$$\mathcal{L}_{MFV} = \bar{Q}_L (\lambda_{MFV})_{ij} \bar{Q}_{R} H^+ + h.c.,$$

with

$$\lambda_{MFV} = a Y_d + b Y_u Y_u^T Y_d + ...$$

In the mass eigenbasis, we obtain

$$a \bar{d}_L \tilde{y}_d \tilde{d}_R h^+_R + b \bar{d}_L \tilde{v}_{uR} \tilde{v}_{dL} (\tilde{y}_u)^2 \tilde{v}_{uR} \tilde{v}_{dL} \tilde{y}_d \tilde{d}_R h^+_R + h.c.$$

With $\tilde{y}_u \sim \text{diag}(0, 0, y_t)$, the $b \to s$ FCNC can be written as

$$\bar{d}_L \lambda_{MFV}^{(3)} \bar{d}_R \tilde{h}^+_R \sim y_s \tilde{y}_t \tilde{v}_{uR} \tilde{v}_{dL} \tilde{S}_R \tilde{h}^+_R.$$

In MFV, flavor-changing neutral currents thus feature the same CKM suppression as in the loop-induced SM process.