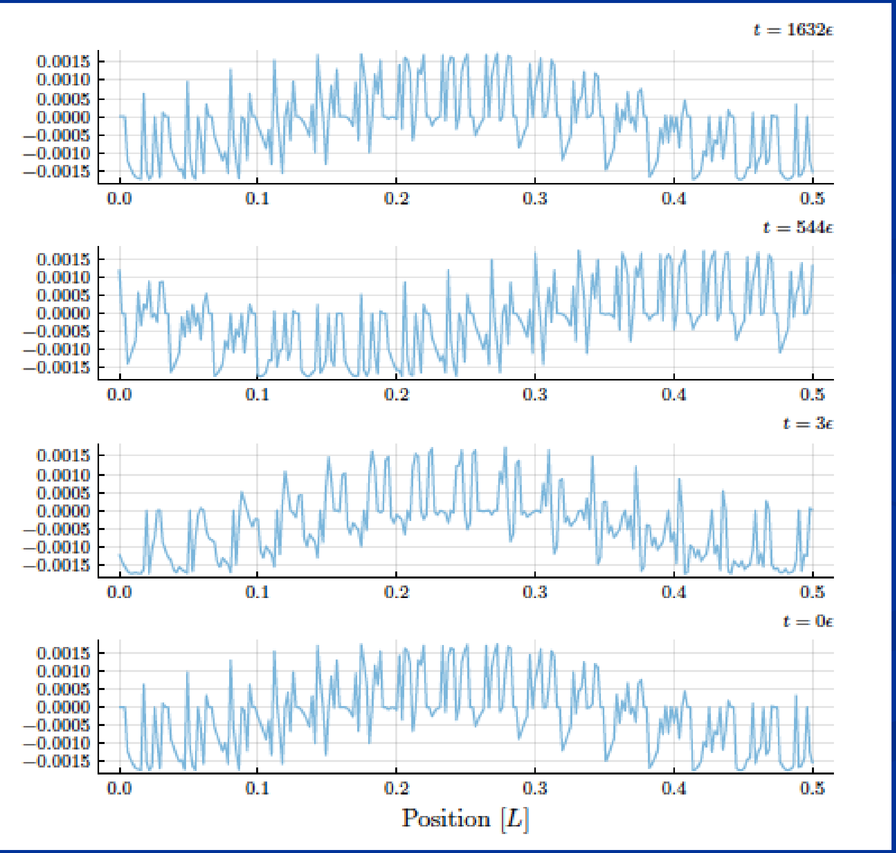
Quantum field theories from classical probabilities

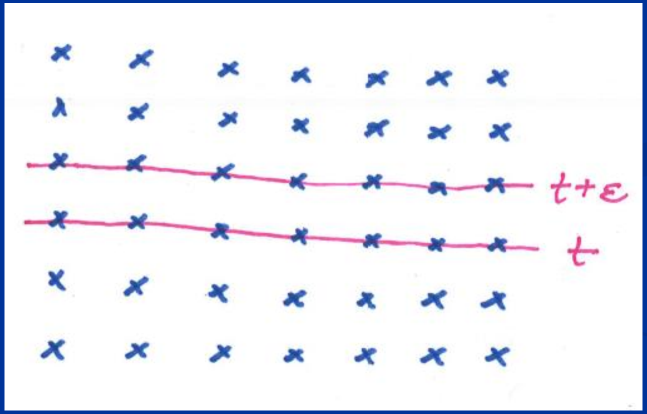


Generalised Ising model

- Overall probability distribution for the Universe covers all times and all space

s(t,x)=-1,1

n(t,x)=0,1

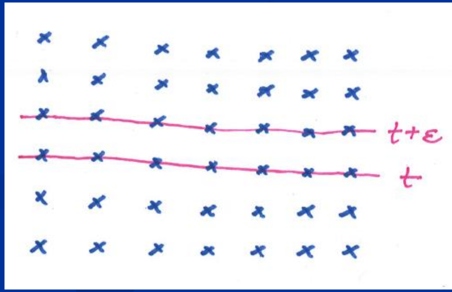


- Generalised Ising model on space-time lattice
- "Euclidean functional integral"

Static memory materials

Generalized Ising model: $w[s] = Z^{-1} \exp\big(-S[s]\big) b(s_{in}, s_f)$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$



Boundary term : $b(s_{in}, s_f) = \bar{f}_f(s_f) f_{in}(s_{in})$

Asymmetry : $\sigma \ll 1$

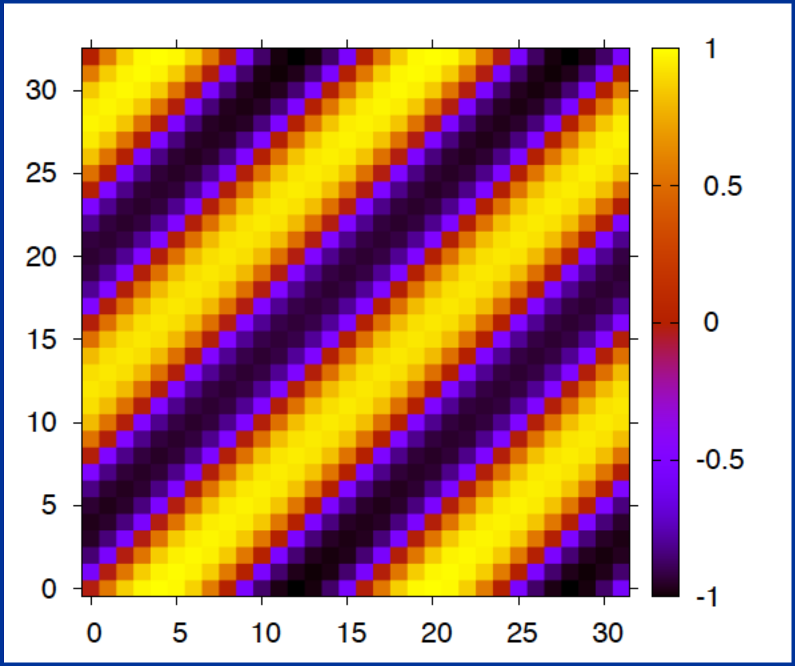
static memory material

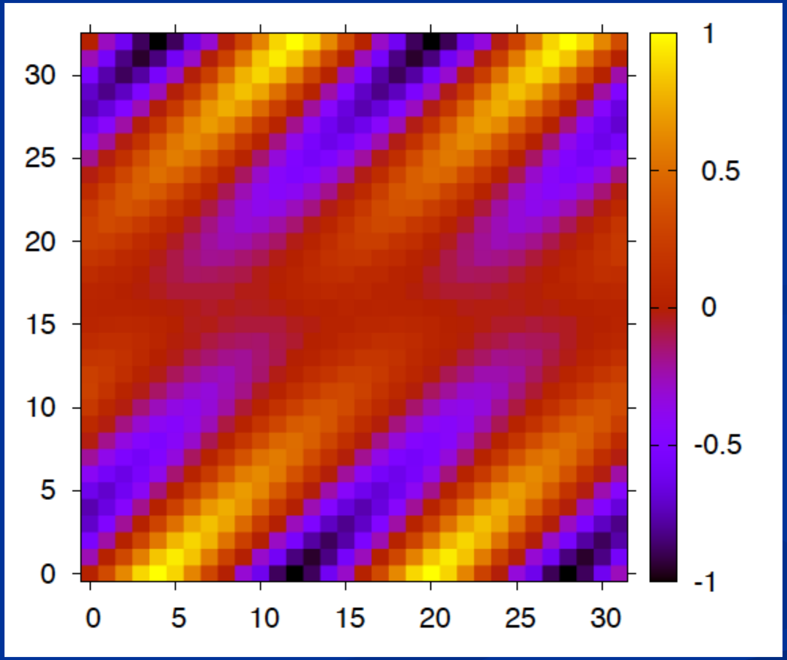
- general equilibrium classical statistics: transport of information from one layer to the next

- simulation, with D. Sexty

Classical interference

Depending on boundary conditions :





Positive interference

Negative interference

Probabilistic cellular automata

Generalized Ising model:

$$w[s] = Z^{-1} \exp\big(-S[s]\big) b(s_{in}, s_f)$$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$

limit : β to infinity , σ to zero :
only one possibility for change , unique jump

probabilistic aspects only in boundary term :

$$b(s_{in}, s_f) = \bar{f}_f(s_f) f_{in}(s_{in})$$

Majorana – Weyl automaton

$$S(\omega) = \sum_{t=t_{\mathrm{in}}}^{t_{\mathrm{f}}-\varepsilon} \mathcal{L}(t)$$

$$\mathcal{L}_{\mathrm{MW}} = \lim_{\beta\to\infty} \left(-\beta \sum_{x} \{ s(t+\varepsilon, x+\varepsilon) s(t,x) - 1 \} \right)$$

Probabilistic cellular automata with deterministic invertible updating are classical probabilistic systems

No loss of information during evolution

They are discrete quantum systems

Majorana – Weyl automaton

$$S(\omega) = \sum_{t=t_{\mathrm{in}}}^{t_{\mathrm{f}}-\varepsilon} \mathcal{L}(t)$$

$$\mathcal{L}_{\mathrm{MW}} = \lim_{\beta\to\infty} \left(-\beta \sum_{x} \{ s(t+\varepsilon, x+\varepsilon) s(t,x) - 1 \} \right)$$

Describes Quantum Field Theory for a massless Weyl fermion
in one space and one time dimension

Cellular automaton

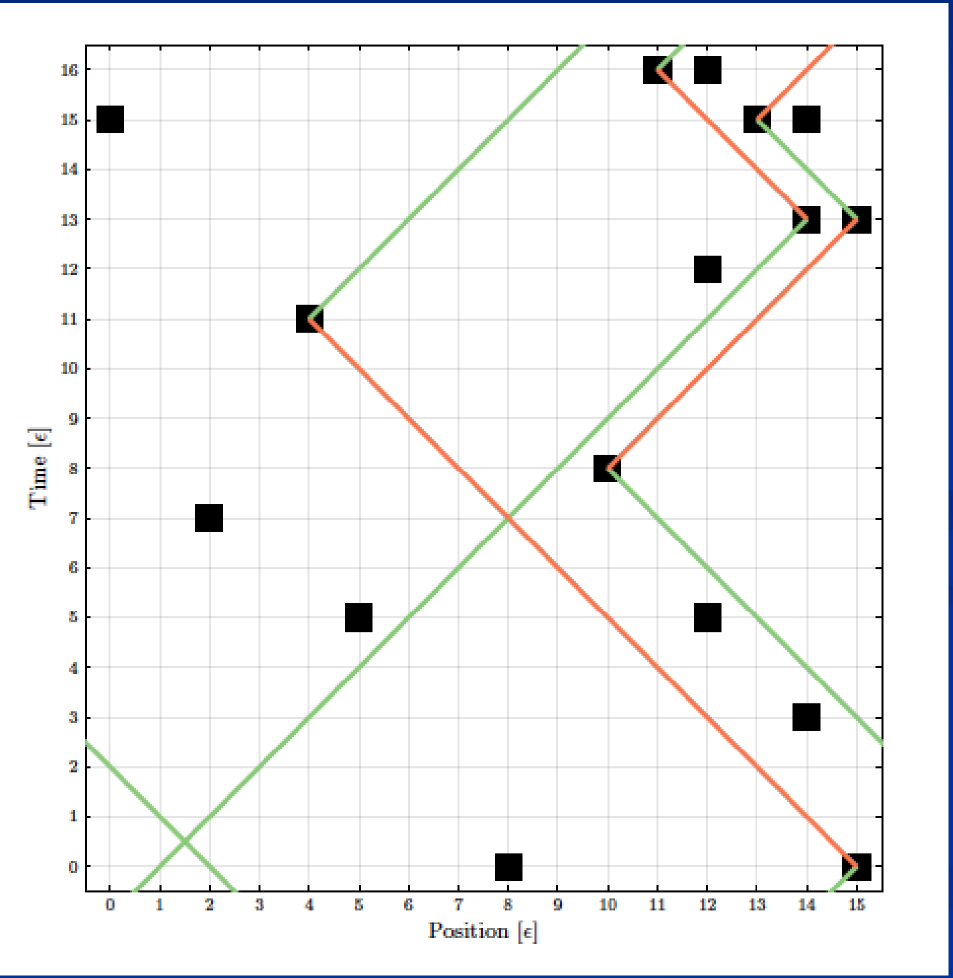
- Deterministic manipulation of bit configurations
- Updating rule of bit configurations in sequential steps

- Updating of a cell depends only on some neighboring cells
- Repetition (at least after certain number of time steps)

(Classical computer is an automaton without repetition)

- Generalisation : updating of real numbers

Updating rule for random automaton



Four types of bits

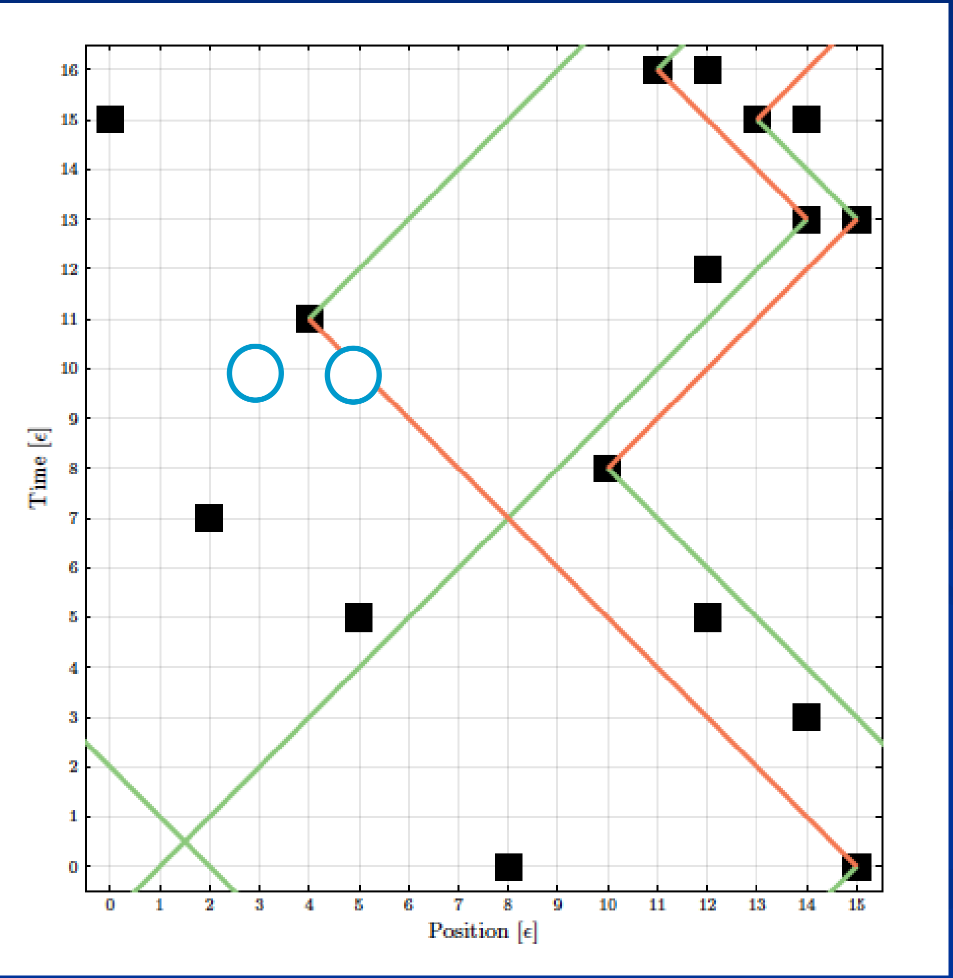
right- and left- movers

red and green

At randomly distributed scattering points: occupied bits change direction and color.

Repetition of distribution after certain number of time steps

Cellular automaton



Updating of bits
in cell (t, x)
is influenced only
by the cells
(t-ε, x-ε) and
(t-ε, x+ε).

Causal structure of
QFT with light
cones

Probabilistic cellular automaton

Probability distribution for initial configurations

deterministic updating

Probabilistic cellular automata are very general

- All deterministic classical field equations can be interpreted as automata
- Newton automaton

$$v(t+\varepsilon) = v(t) + \varepsilon F(z(t))$$

$$z(t+\varepsilon) = z(t) + \varepsilon v(t+\varepsilon)$$

- Evolution equation for probabilitiy distribution in phase space is the Liouville equation

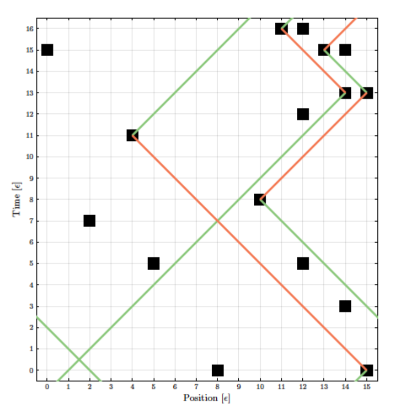
Probabilistic cellular automaton

Probabilistic initial condition: Specify at initial time t_{in} for each bit configuration $\overline{\rho}$ a probability $p_{\overline{\rho}}(t_{\text{in}})$

Evolution: every given configuration $\overline{\rho}$ at t_{in} propagates at t to a configuration $\tau(t, \overline{\rho})$

$$\Rightarrow \quad p_\tau(t) = p_{\overline{\rho}(\tau)}(t_{\text{in}})$$

Updating rule: specifies $\tau\big(t+\varepsilon, \rho(t)\big)$



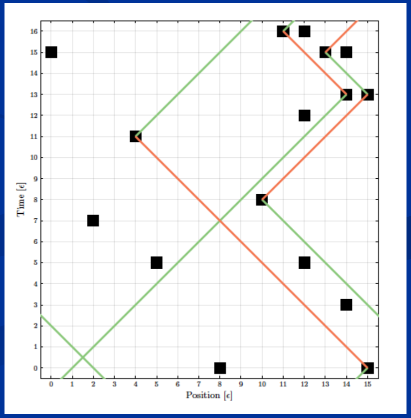
Overall probability distribution

Follow trajectory of some initial configuration

Probabilities are equal for each point on trajectory

Probabilities for arbitrary bit configurations in time and space. They differ from zero only for configurations that can be reached by allowed trajectories

$$p[\chi'] = Z^{-1} \exp\left(-S[\chi']\right)$$



Classical probabilistic system

Wave function for probabilistic cellular automaton

Local time probability distribution: at every time t a bit configuration τ occurs with probability $p_\tau(t)$, which equals the probability for the initial bit configuration from which it originates.

Real wave function q(t): probability amplitude

$$p_\tau(t) = (q_\tau(t))^2$$

$$q_\tau(t) q_\tau(t) = 1$$

q(t) is a unit vector

Deterministic and probabilistic cellular automaton

- Deterministic CA : sharp wave function

$$q_\rho(t_{\text{in}}) = \delta_{\rho,\overline{\rho}}$$

- Probabilistic CA : arbitrary wave function

Step evolution operator

- Evolution for basic time step is encoded in the step evolution operator

$$q(t+\varepsilon) = \widehat{S}(t)q(t)$$

$$q_\tau(t+\varepsilon) = \widehat{S}_{\tau\rho}(t)q_\rho(t)$$

- Contains the updating rule for CA

$$\widehat{S}_{\tau\rho}(t) = \delta_{\tau,\overline{\tau}(\rho)} = \delta_{\overline{\rho}(\tau),\rho}$$

$$q_\tau(t+\varepsilon) = q_{\overline{\rho}(\tau)}(t)\,, \quad p_\tau(t+\varepsilon) = p_{\overline{\rho}(\tau)}(t)$$

Unique jump matrix

- Step evolution operator for cellular automata is unique jump matrix
- In every row and column: precisely one element +1 or -1, all other elements zero

$\widehat{S}_{\tau\rho}(t)$ is orthogonal and therefore unitary

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Hamiltonian

- Define H by $\widehat{S} = \exp\left(-i\varepsilon H\right)$

- H is Hermitian and piecewise constant

- Interpolating continuous time evolution

  $q(t_2) = U(t_2, t_1) q(t_1)$ $U(t_1, t_2) = \exp\left(-i(t_1 - t_2)H\right)$

- Schrödinger equation $i\partial_t q = Hq$

- Solution agrees with discrete evolution for t= $t_{in} + m\varepsilon$

Complex structure

Suitable set of two discrete transformations for complex conjugation and multiplication by i

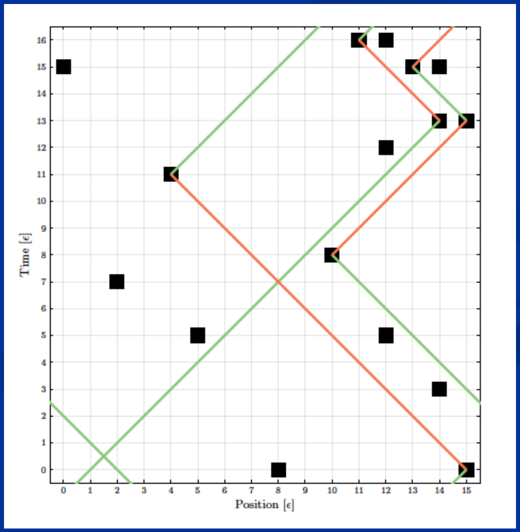
Configurations at given t with single occupied bit: (x, γ)
Wave function for a single occupied bit: $q_\gamma(t, x)$

red and green correspond to
real and imaginary parts of complex wave function

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \psi_R = q_1 + iq_2, \quad \psi_L = q_3 + iq_4.$$

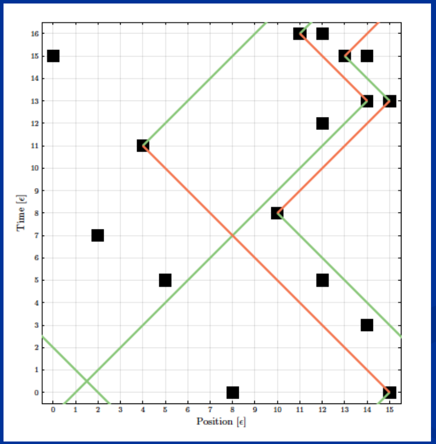
Free massless Dirac fermions

- Without scattering: automaton describes quantum field theory for free massless Dirac fermions in one space and one time dimension
- Arbitrary number of fermions
- Half filled vacuum with all negative energy states filled
- Single fermion state: $\psi_\alpha(t, x)$



Numerical solution for random automaton

Simulate simple system by following trajectories numerically



random automaton with single occupied bit

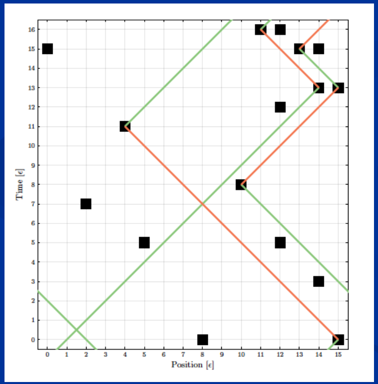
(not one-particle excitation of particle physics vacuum)

Quantum Systems from Random Probabilistic Automata

A. Kreuzkamp[1] and C. Wetterich[1]

Conserved Hamiltonian

- Time translation invariance (after several time steps) implies conserved Hamiltonian

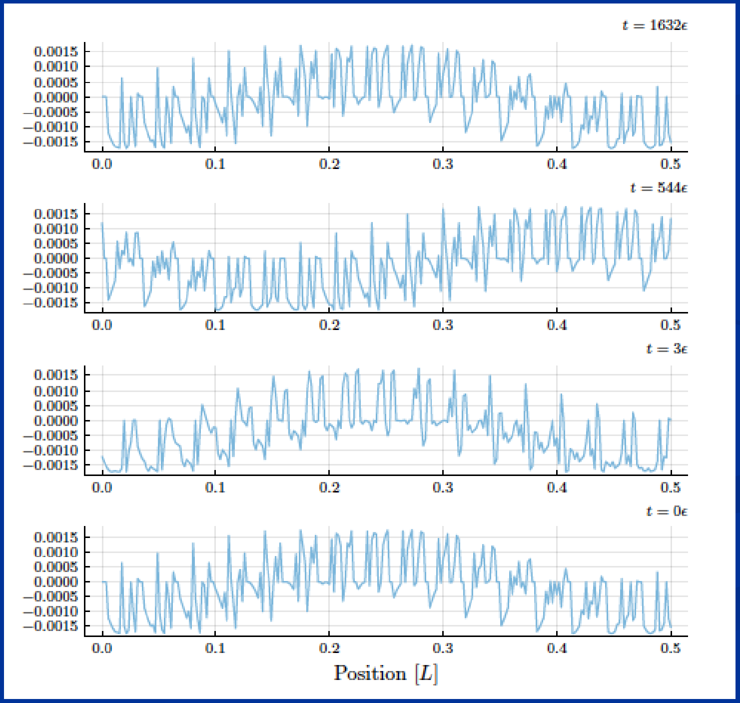


- Explicit form of H not known for random automaton

Periodic evolution of probability distribution for energy eigenstates

simple random automaton with periodicity in space: energy eigenstate found explicitly

evolution



Fermions

- quantum objects
- wave function totally antisymmetric

  (Pauli principle)
- anticommutator for annihilation and creation operators
- anticommuting Grassmann variables
- functional integral or partition function for many body systems or quantum field theories is Grassmann functional integral

Fermions are Ising spins or bits

- Fermionic occupation numbers $n = 0 , 1$
- Classical bits
- Ising spins $s = 2n - 1$
- Bit configurations = multi-particle states of fermions

Fermionic wave function

- Occupation number basis for multi-fermion systems:

- To each bit configuration one associates a component of the wave function
- Occupation numbers for different space points and species

Probabilistic cellular automata : generalized Ising models are fermionic quantum field theories

- Wave function in same Hilbert space
- If step evolution operator for automaton is the same as for the fermionic quantum field theory: Both are equivalent

fermion operators

- annihilation operator at j : a(j)

  (0,1,0,1,0,0,1) -> (0,1,0,0,0,0,1)

  (0,1,0,0,0,0,1) -> 0

- creation operator at j :

  (0,1,0,1,0,0,1) -> 0

  (0,1,0,0,0,0,1) -> (0,1,0,1,0,0,1)

annihilation and creation operators

- fermionic operators

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$a^2 = (a^T)^2 = 0, \quad \{a, a^T\} = 1.$$

$$\hat{n} = a^T a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- fermionic operators at each position j

$$\{a(j), a(j')\} = \{a^\dagger(j), a^\dagger(j')\} = 0$$
$$\{a(j), a^\dagger(j')\} = \delta_{jj'}$$

$$a(j) = \tau_3 \otimes \tau_3 \otimes \tau_3 \otimes a \otimes 1 \otimes 1 \otimes \dots 1,$$
$$a^\dagger(j) = \tau_3 \otimes \tau_3 \otimes \tau_3 \otimes a^T \otimes 1 \otimes 1 \otimes \dots 1$$

- fermion number at position j

$$\hat{n}(j) = a^\dagger(j) a(j) = 1 \otimes 1 \otimes \dots \hat{n} \otimes 1 \otimes 1 \otimes \dots 1.$$

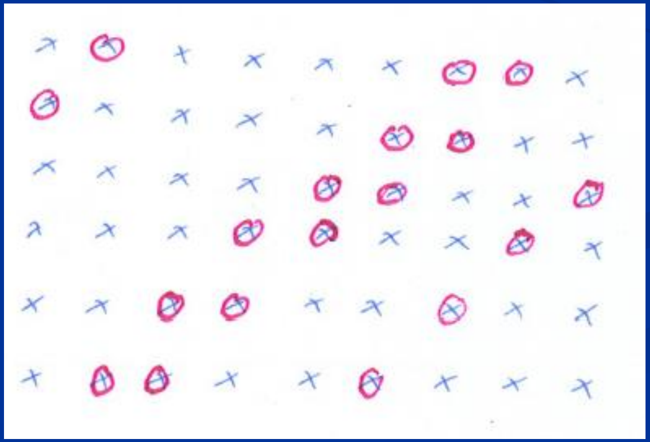
Majorana – Weyl automaton

$$\mathcal{L}_{\text{MW}} = \lim_{\beta\to\infty}\left(-\beta\sum_x \{s(t+\varepsilon, x+\varepsilon)s(t,x) - 1\}\right)$$

$$S(\omega) = \sum_{t=t_{\text{in}}}^{t_{\text{f}}-\varepsilon} \mathcal{L}(t)$$

Updating transports every bit configuration one position to the right

(0,1,1,0,0,0,1,0,1,0) -> (0,0,1,1,0,0,0,1,0,1) (periodic)



Annihilation and creation operators

Step evolution operator can be written in terms of fermionic annihilation and creation operators

$$\{a^\dagger_\gamma(x), a_\delta(y)\} = \delta_{\gamma\delta}\delta_{xy} \qquad \{a_\gamma(x), a_\delta(y)\} = \{a^\dagger_\gamma(x), a^\dagger_\delta(y)\} = 0$$

Fermionic operators in Fourier space

$$a(k) = \sum_j D(k,j)a(j)\,, \quad a^\dagger(k) = \sum_j a^\dagger(j)D^{-1}(j,k) \qquad D(k,j) = \frac{1}{\sqrt{N_x}}\exp\left\{-\frac{2\pi i}{N_x}kj\right\}$$

$$\{a(k'), a(k)\} = \{a^\dagger(k'), a^\dagger(k)\} = 0$$
$$\{a^\dagger(k'), a(k)\} = \delta_{kk'}\,.$$

Hamiltonian

$$\widehat{S} = \exp\{-i\varepsilon H^{(R)}\}, \quad H^{(R)} = \sum_k \frac{2\pi k}{N_x \varepsilon}\hat{n}(k)$$

$$\hat{n}(k) = a^\dagger(k)a(k)$$

$$\begin{aligned}
H^{(R)} &= \sum_j \sum_{m>0} \frac{i\pi(-1)^m}{L\sin\left(\frac{\pi m\varepsilon}{L}\right)} a^\dagger(j)[a(j+m) - a(j-m)] \\
&\approx \sum_x \sum_y{}' \frac{i\pi\varepsilon}{L\sin\left(\frac{\pi(y-x)}{L}\right)} a^\dagger(x) \left\{ \frac{a(y+\varepsilon) - a(y)}{\varepsilon} \right. \\
&\quad + \frac{a(2x-y) - a((2x-y)-\varepsilon)}{\varepsilon} \\
&\quad \left. - \frac{\pi}{L}\mathrm{cotg}\left(\frac{\pi(y-x)}{L}\right)[a(y+\varepsilon) - a(2x-(y+\varepsilon)] \right\}
\end{aligned}$$

simplicity in Fourier space

- the possibility to perform basis transformations is a major advantage of the use of wave functions
- not possible on the level of probability distributions

Energy

- Hamiltonian is a conserved quantity
- can be interpreted as quantum energy
- eigenvalues are positive and negative

$$\widehat{S} = \exp\{-i\varepsilon H^{(R)}\}\,, \quad H^{(R)} = \sum_k \frac{2\pi k}{N_x \varepsilon} \hat{n}(k)$$

- general property for real step evolution operator

vacuum

- vacuum state : all modes with negative energy are filled (Dirac vacuum)

- reorder

$$H^{(R)} = \frac{2\pi}{L}\left(\sum_{k>0} k a^\dagger(k) a(k) + \sum_{k<0} (-k)\left(a(k) a^\dagger(k) - 1\right)\right)$$

- antiparticles

$$b^\dagger(k) = a(-k)\,, \quad b(k) = a^\dagger(-k)$$

- $H\text{-}E_0$ positive

$$H^{(R)} - E_0 = \frac{2\pi}{L}\sum_{k>0}\left\{k\left(a^\dagger(k)a(k) + b^\dagger(k)b(k)\right)\right\}$$

$$E_0 = \frac{2\pi}{L}\sum_{k<0} k = -\frac{(N_x^2 - 1)\pi}{4N_x\varepsilon}$$

- vacuum condition

$$a(k)\,|0\rangle_p = 0\,, \quad b(k)\,|0\rangle_p = 0$$

vacuum

- vacuum is simple in momentum space

$$a(k)\,|0\rangle_p = 0\,, \quad b(k)\,|0\rangle_p = 0$$

- complicated in position space

- antiparticles arise naturally

half filling

- total fermion number is conserved

$$\hat{N} = \sum_j \hat{N}(j)$$

- vacuum is a state with half filling

$$\langle 0| \hat{N} |0\rangle = \frac{N_x}{2} \qquad \langle \hat{N}(j) \rangle = \langle 0| \hat{N}(j) |0\rangle = \frac{1}{2}$$

one-particle excitation

- apply fermionic operators to vacuum

$$\varphi_R^{(1)}(t) = \sum_k \tilde{\varphi}(t,k)\left(a^\dagger(k) + a(k)\right)|0\rangle$$

$$= \left[\sum_{k>0} \tilde{\varphi}(t,k)a^\dagger(k) + \sum_{k<0} \tilde{\varphi}(t,k)a(k)\right]|0\rangle$$

- one-particle state describes particle and antiparticle
- energy of one-particle state is positive

$$\left(H^{(R)} - 2E_0\right)\varphi_R^{(1)}(t) = \frac{2\pi}{L}\sum_{k>0} k\left[\tilde{\varphi}(t,k)a^\dagger(k) + \tilde{\varphi}(t,-k)a(-k)\right]|0\rangle$$

$$= \frac{2\pi}{L}\sum_k |k|\tilde{\varphi}(t,k)\left(a^\dagger(k) + a(k)\right)|0\rangle$$

one-particle wave function

$$\varphi_R^{(1)}(t) = \sum_k \tilde{\varphi}(t,k)\left(a^\dagger(k) + a(k)\right)|0\rangle$$

⇧

in momentum space

Momentum observable for single fermion state

Fourier transform

$$\psi(x) = N_x^{-\frac{1}{2}} \sum_q \exp(iqx)\psi(q),$$
$$\psi(q) = N_x^{-\frac{1}{2}} \sum_x \exp(-iqx)\psi(x).$$

Momentum operator

$$P(q,q') = q\tilde{\delta}_{q,q'}$$

Continuum limit

$$\tilde{P}(x,x') = -i\partial_x \delta(x-x')$$

Momentum distribution

$$w(q) = \psi^\dagger(q)\psi(q), \quad \langle f(P)\rangle = \sum_q f(q)w(q)$$

Expectation value

$$\langle f(P)\rangle = \sum_{q,q'} \psi^\dagger(q)(f(P))(q,q')\psi(q)$$

QFT

the complete formalism of quantum field theory can be constructed for this simple automaton

continuous field operators and their commutation relations

$$\Psi_+(t,x) = \sum_k u(t,x;\,k)\,a(k)$$

$$u(t,x;\,k) = \frac{1}{\sqrt{L}}\exp\left\{\frac{2\pi i k}{L}(x-t)\right\}$$

$$\{\Psi^\dagger_\pm(t,x),\Psi_\pm(t',x')\} = \tilde\delta(x'-x-t'+t)$$
$$\{\Psi_\pm(t,x),\Psi_\pm(t',x')\} = 0\,.$$

Feynman propagator

$$G_F(t,x;t',x') = \begin{cases} W_+(t-t',x-x') & \text{for} \quad t>t' \\ -W_-(t-t',x-x') & \text{for} \quad t<t' \end{cases}$$

$$W_+(t,x) = \langle 0|\,\Psi_+(t,x)\Psi^\dagger_+(0,0)\,|0\rangle_p - \frac{1}{2L} \;= \tilde\delta_+(x-t)$$

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{L\to\infty}\tilde\delta_+(x-t) &= \delta_+(x-t) \\ &= \frac{1}{2\pi}\int_0^\infty \mathrm{d}p\, e^{ip(x-t)} \\ &= -\frac{1}{2\pi i}\lim_{\tilde\varepsilon\to 0}\left(\frac{1}{x-t+i\tilde\varepsilon}\right)\end{aligned}$$

Dirac automaton

left movers and right movers

$$S = -\beta \sum_{t,x} \{ s_R(t+\varepsilon, x+\varepsilon) s_R(t,x) + s_L(t+\varepsilon, x-\varepsilon) s_L(t,x) - 2 \} .$$

$$\widehat{S} = \exp\{ -i\varepsilon (H^{(R)} + H^{(L)}) \}$$

$$H^{(R)} = \sum_k \frac{2\pi k}{N_x \varepsilon} a_R^\dagger(k) a_R(k) ,$$

$$H^{(L)} = -\sum_k \frac{2\pi k}{N_x \varepsilon} a_L^\dagger(k) a_L(k)$$

Dirac equation

Dirac automaton conserves Lorentz symmetry

evolution equation

$$\tilde{\varphi}^{(1)}(t+\varepsilon,x)=\begin{pmatrix}\tilde{\varphi}_R(t+\varepsilon,x)\\ \tilde{\varphi}_L(t+\varepsilon,x)\end{pmatrix}=\begin{pmatrix}\tilde{\varphi}_R(t,x-\varepsilon)\\ \tilde{\varphi}_L(t,x+\varepsilon)\end{pmatrix}$$

discrete derivatives

$$\partial_t\tilde{\varphi}(t,x)=\frac{1}{2\varepsilon}\left(\tilde{\varphi}(t+\varepsilon,x)-\tilde{\varphi}(t-\varepsilon,x)\right)$$

$$\partial_x\tilde{\varphi}(t,x)=\frac{1}{2\varepsilon}\left(\tilde{\varphi}(t,x+\varepsilon)-\tilde{\varphi}(t,x-\varepsilon)\right)$$

continuum limit

$$i\partial_t\tilde{\varphi}^{(1)}(t,x)=-i\partial_x\tau_3\tilde{\varphi}^{(1)}(t,x)$$

Dirac equation

$$\gamma^\mu\partial_\mu\tilde{\varphi}^{(1)}(t,x)=0$$

$$\gamma^0=-i\tau_2\,,\quad \gamma^1=\tau_1$$

Conserved momentum

for single particle states of Dirac automaton:

momentum is a conserved quantity

$$\overline{H}_f = P\tau_3.$$

the expectation values

do not depend on time

$$\langle f(P) \rangle = \sum_{q,q'} \psi^\dagger(q)(f(P))(q,q')\psi(q)$$

momentum eigenstates : plane waves

They require probabilistic automaton with smooth wave functions

Statistical observables

- Momentum is a statistical observable
- No fixed value for given bit configuration
- Characterizes properties of probabilistic information (similar to temperature)
- Does not commute with position operator
- Bell's inequalities do not apply to pair position and momentum since no classical correlation function can be defined

quantum energy

- Hamilton operator is associated to statistical observable

- no fixed value for a given configuration of Ising spins
- does not commute with position operator
- classical correlation for quantum energy and position does not exist

- Bell's inequality does not apply to this pair of observables

Quantum mechanics from classical statistics

- Probabilistic cellular automata are classical statistical systems.
- Probabilistic cellular automata are discrete quantum systems.
- Quantum mechanics emerges from a classical statistical system.
- No go theorems (Bell etc.) do not apply to all pairs of observables.

Outlook: How to find overall classical probability distribution for quantum particle in an arbitrary potential ?

Top down approach: Find automaton for interesting QFT. Construct vacuum and one-particle excitation. Find continuum limit. Realistic setting, but hard to implement.

Bottom up approach: Explicit construction of probabilistic automaton is already done for

- quantum particle in harmonic potential
- single qubit with arbitrary time-dependent Hamiltonian

Classical probability distribution not the one from QFT,

but useful conceptually. No contradictions.

Can quantum physics be described by classical probabilities ?

" No go " theorems

Bell , Clauser , Horne , Shimony , Holt

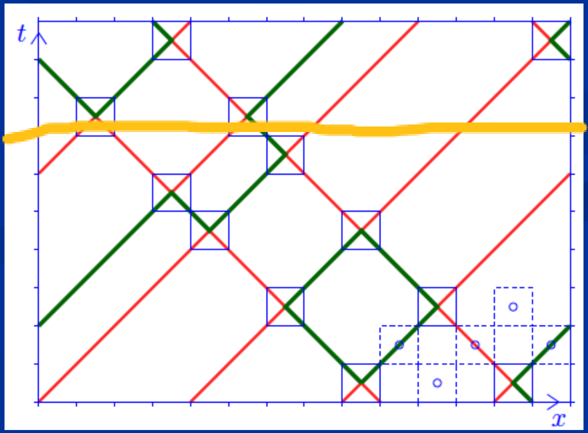
implicit assumption : use of classical correlation function for all correlations between measurements

Kochen , Specker

assumption : unique map from quantum operators to classical observables

Overall view on quantum mechanics

- Quantum mechanics from quantum field theory
- Functional integral : variables for all times (fields, bit configurations, paths)
- Local time physics :
  Focus on hypersurfaces
  labeled by t



Quantum mechanics

- Projection on local time subsystem (Feynman)

Wave function, operators,

linear evolution law, $\psi(t+\varepsilon)=U(t)\,\psi(t)$

superposition of solutions,

formalism of quantum mechanics

Conclusion

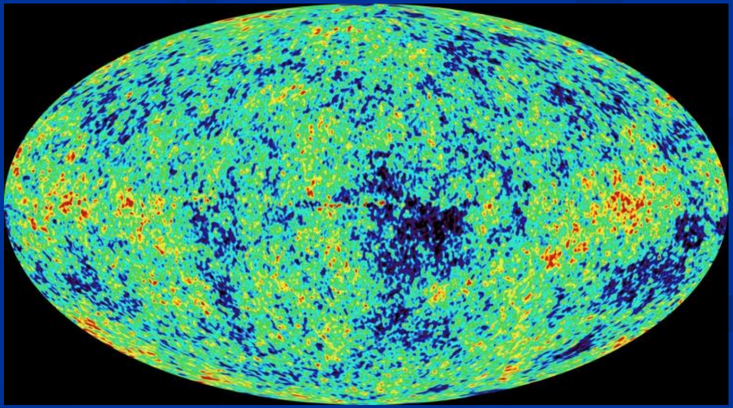
- Particular quantum field theories for interacting fermions are equivalent to classical statistical models of particular probabilistic cellular automatons.
- Large family of quantum models – not all models!
- Examples for quantum mechanics from classical statistics
- Useful for simulating fermionic models and understanding of statistical properties of cellular automata?

The probabilistic world

- Physicists describe the Universe by a probability distribution for events at all times and positions
- Classical statistics
- Quantum mechanics follows by focus on time-local subsystems

Fundamental probabilities

- Probabilities are starting point for mathematical description of Nature
- Not related to knowledge of observer



Observables and events

- Yes/no decisions: particle hits detector or not
- Ising spins s=+1 or -1
- Bits n=1 or n=0
- Fermionic occupation numbers n=1,0
- Events (states) are spin configurations

Overall probability distribution

$$p_\omega = \mathcal{B}_{\rm f}(\omega) \exp\left\{ - S(\omega) \right\} \mathcal{B}_{\rm in}(\omega)$$

$\mathcal{B}_{\rm in}$ and $\mathcal{B}_{\rm f}$ boundary terms

$p_\omega \geq 0$ $\sum_\omega p_\omega = 1$

$\sum_\omega$ sum over spin configurations

Observables A, expectation value $\langle A \rangle = \sum_\omega A_\omega p_\omega$

Particle wave duality

Particle aspect:

- Bits: yes/no decisions
- Possible measurement values 1 or 0

Discrete spectrum of observables

Wave aspect :

Continuous probabilistic information

(wave function)

Is this all useful?

- Quantum formalism offers new insights for the dynamics of probabilistic cellular automata
- New forms of correlated computing
- Clarification of origin of quantum concepts – demystification
- Probabilistic realism is a philosophical concept
- Restrictions on fundamental theory?

end