Origin of wave function and operators in quantum mechanics

quantum mechanics

- wave function (density matrix)
- operators
- Schrödinger equation

quantum axioms

can one understand this ?

quantum mechanics from quantum field theory

- fundamental setting : QFT
- particles are excitations of vacuum
- particle properties depend on vacuum
- atoms are excitations of vacuum
- qubits are excitations of vacuum

- QFT : functional integral over fluctuations
- quantum effective action similar to free energy in classical statistics

quantum operators

- non – commuting operators play central role in quantum mechanics
- functional integral involves fields, no operators
- where do the operators come from ?
- focus on time-local subsystem : Feynman

Quantum mechanics can arise from classical statistics !

Quantum formalism for classical statistics can be useful for understanding how information propagates in probabilistic systems

wave function

complex function $\psi(t,x)$

normalized $\int d^3x \; \psi^*(t,x)\psi(t,x) = 1$

probability to fine particle at x $\psi^*(t,x)\psi(t,x) = p(x)$

operators

operators are associated to observables A

$$\hat{A} = \vec{x}, \; -i\partial_x = -i\frac{d}{dx} = -i\vec{\nabla}$$

expectation value

$$\langle A \rangle = \int d^3x \, \psi^*(t,x) \hat{A} \, \psi(t,x)$$

time evolution

Schrödinger equation

$$i\,\partial_t \psi = H \psi \qquad \partial_t = \frac{d}{dt} \qquad \hbar = 1$$

$$i\,\partial_t \psi(t,x) = \int_{x'} H(t;x,x')\,\psi(t,x')$$

evolution operator U

$$\psi(t+\varepsilon) = U(t)\,\psi(t)$$

$$\psi(t+\varepsilon,x) = \int_{x'} U(t,x,x')\,\psi(t,x')$$

Unitary evolution in quantum mechanics

solution of Schrödinger equation :

$$\psi(t+\epsilon) = U(t+\epsilon, t)\psi(t)$$

$\psi(t)$ wave function

$U(t+\epsilon, t)$ evolution operator

unitary evolution of wave function

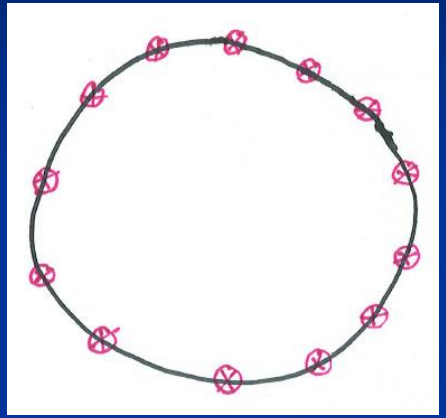
$$\psi(t+\epsilon) = U(t+\epsilon, t)\psi(t)$$

discrete quantum mechanics

use discrete time steps and

discrete space points

$$\psi(t+\epsilon) = U(t+\epsilon, t)\psi(t)$$



N space points:

U is unitary

N x N matrix

$$U(t, x, x')$$

$$U^{+}(t)U(t) = 1$$

$$\sum_x \psi^*(t,x)\psi(t,x) = 1$$

$$\sum_x \psi^*(t+\epsilon,x)\psi(t+\epsilon,x)$$

$$= \sum_x \sum_{x'} \sum_{x''} \psi^*(t,x')U^{+}(x',x)U(x,x'')\psi(t,x'')$$

$$= 1$$

quantum structures
arise by focus on
time - local subsystem of
overall probability (weight) distribution

overall weight distribution

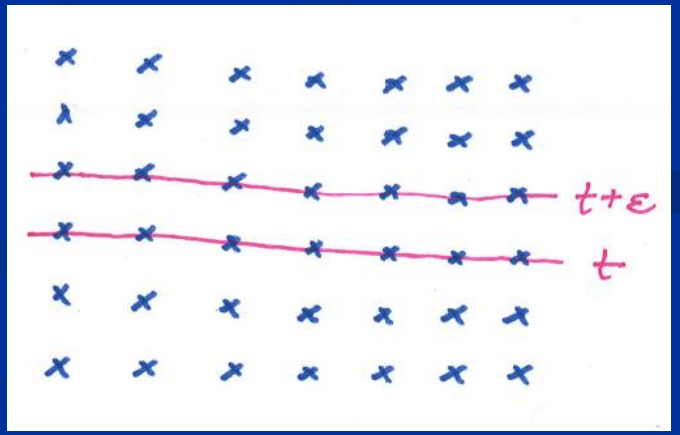
- basic setting : overall weight distribution for possible events at all times and locations
- classical statistics : weights are probabilities (real and positive)
- general quantum systems : weights can be complex
- set of basis events are ordered in time

example for events : particle detectors at t,x fire or not

n= 1 or 0 , discrete

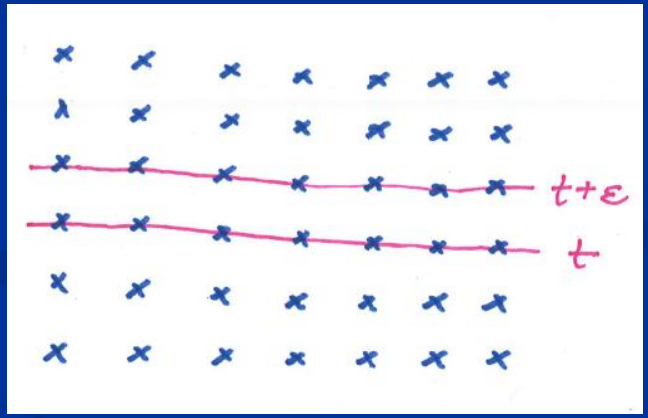
time – local subsystem

- basic setting : weight distribution for the whole universe, from the past to the future
- we are interested at a given time t, and in the evolution with t
- overall weight distribution has redundant information
- focus on time-local subsystem



time – local subsystem

- select time – hypersurfaces labeled by t
- local observables : events at t
- time-local probabilistic information for computation of expectation values of local observables
- evolution law : compute time-local probabilistic information at t+ε from the one at t
- time-local probabilities are not sufficient
- one needs probability amplitudes or wave functions

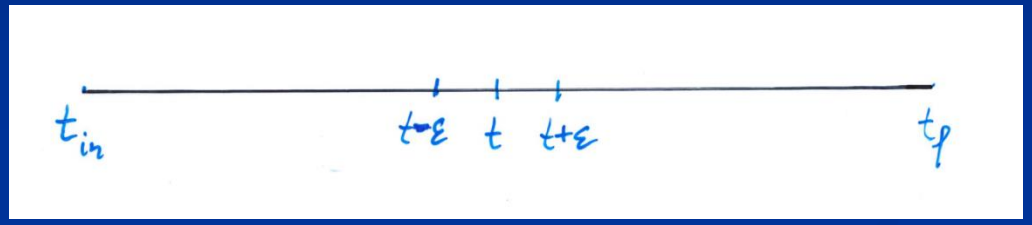


Ising chain

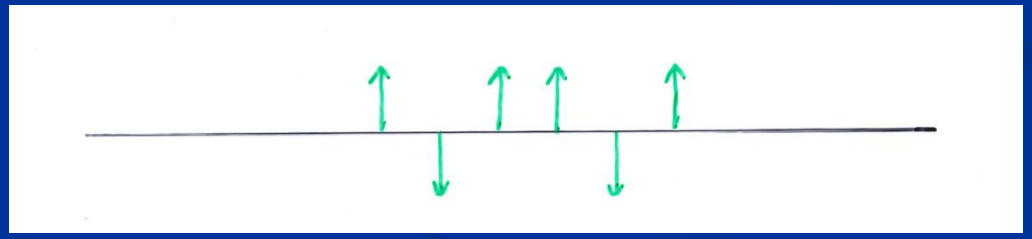
simple model in classical statistics

discrete variables

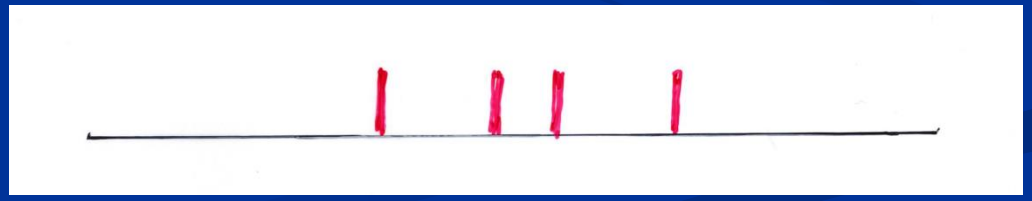
wire with
discrete points t



Ising spins
$s = 1, -1$

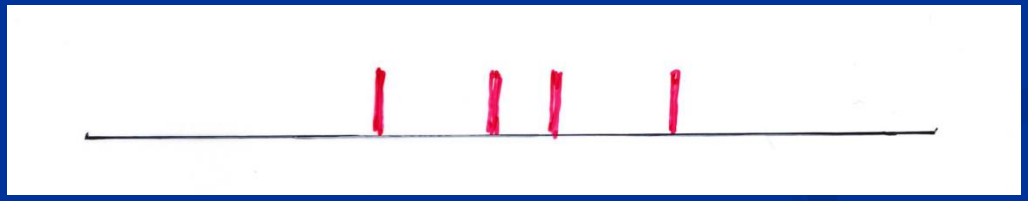


occupation numbers
$n = 0, 1$ (fermions)



classical statistical probability distribution

- {n} : configuration of occupation numbers for all n(t) or n(m)



$[\, n_s \,] = [0,0,1,0,1,1,0,1,0,1,1,1,1,0,\ldots]$

- w[n] : probability distribution

classical statistics

variables $n = \frac{s+1}{2}, \qquad s = 2n-1$

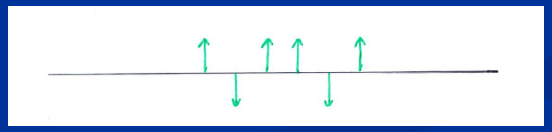
configurations $\tau \leftrightarrow \{s_\gamma\} \leftrightarrow \{n_\gamma\}$

probabilities $p_\tau \leftrightarrow p[s] \leftrightarrow p[n]$

expectations values $\langle A \rangle = \sum_\tau p_\tau A_\tau$

functional integral $\langle A \rangle = \sum_\tau p_\tau A_\tau = \int \mathscr{D}s \; p[s] A[s]$

$\sum_\tau = \int \mathscr{D}s = \prod_{\gamma=1}^{M} \left(\sum_{s_\gamma = \pm 1} \right)$



partition function Z $w_\tau \geq 0$ $Z = \sum_\tau w_\tau$ $p_\tau = Z^{-1} w_\tau$ $\langle A \rangle = Z^{-1} \sum_\tau w_\tau A_\tau$

action

$$w[s] = \exp(-\mathscr{S})\,\mathcal{B} \qquad Z = \int \mathscr{D}s\, w[s] \qquad p[s] = Z^{-1} w[s]$$

action $$\mathscr{S} = \sum_{m=0}^{\mathscr{M}-1} \mathscr{L}(m)$$

Ising chain $$\mathscr{L}(m) = \beta\left(\kappa s(m+1)\, s(m) + 1\right) \qquad \kappa = \pm 1$$

expectation value $$\langle A \rangle = \sum_{\tau} p_\tau A_\tau = \int \mathscr{D}s\, p[s] A[s]$$

local factor

probability distribution is product of local factors (with boundary terms)

$$w[s] = \exp(-\mathscr{S})\,\mathcal{B}$$

$$\mathscr{S} = \sum_{m=0}^{\mathscr{M}-1} \mathscr{L}(m)$$

$$\mathscr{L}(m) = \beta\left(\kappa\, s(m+1)\, s(m) + 1\right)$$

local factor involves neighboring Ising spins

$$\mathcal{K}(m) = \exp\left(-\mathscr{L}(m)\right)$$

boundary term

$$w[s] = \prod_{m=0}^{\mathscr{M}-1} \mathcal{K}(m)\,\mathcal{B}$$

boundary term $\mathcal{B}$ depends on spins at initial and final sites

$$\mathcal{B}(s_{\mathrm{f}}, s_{\mathrm{in}}) = \mathcal{B}_{\mathrm{f}}(s_{\mathrm{f}})\,\mathcal{B}_{\mathrm{in}}(s_{\mathrm{in}})$$

$$s_{\gamma,\mathrm{in}} = s_{\gamma}(0)$$

$$s_{\gamma,\mathrm{f}} = s_{\gamma}(\mathscr{M})$$

boundary problem

for given expectation values of boundary spins,
initial and final,

what is the expectation value of spin

in the bulk s(t) ?

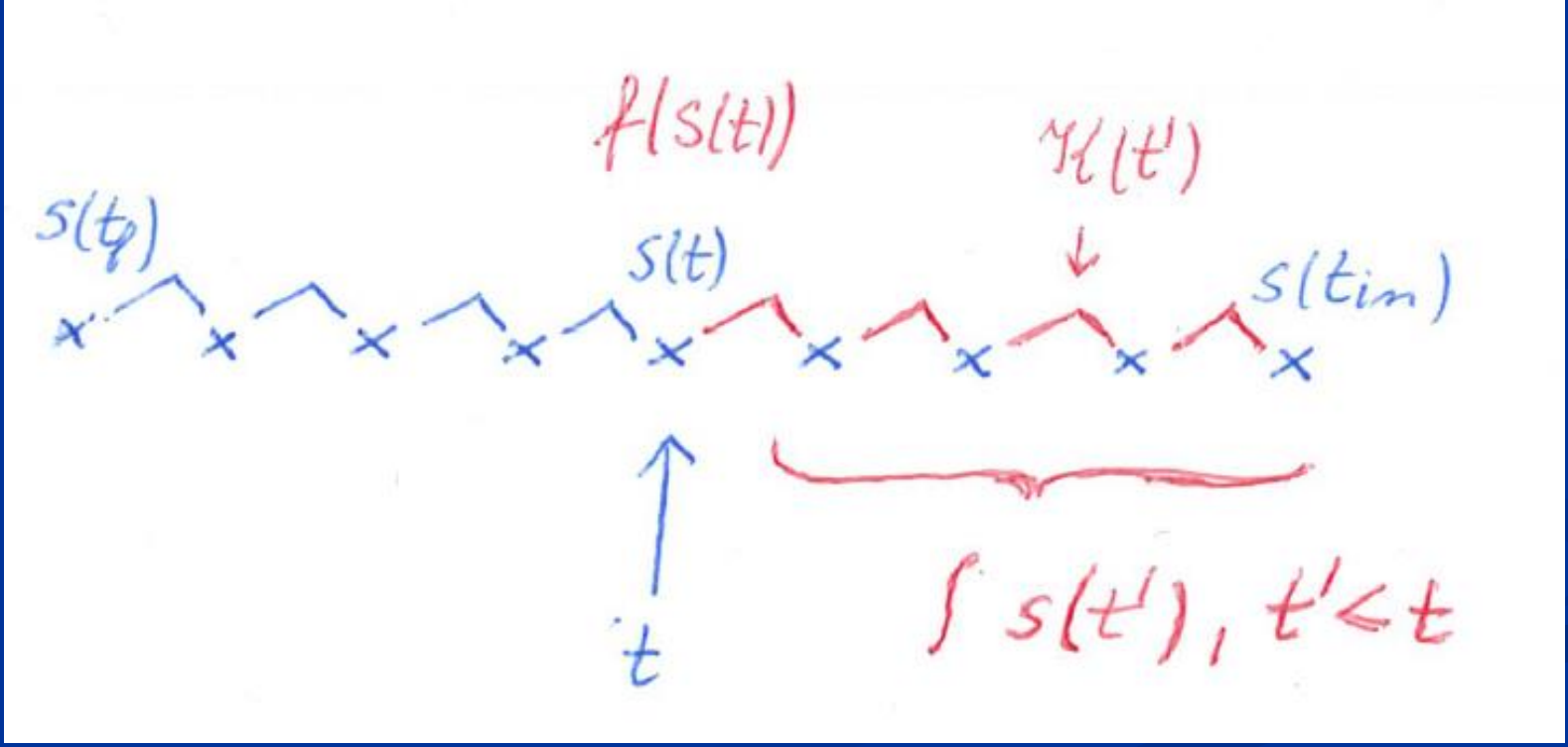
Wave function in classical statistical equilibrium systems

quantum formalism for information transport in classical statistics

- *Why wave function ?*
- *What determines evolution equation ?*
- *Non – commuting operators ?*

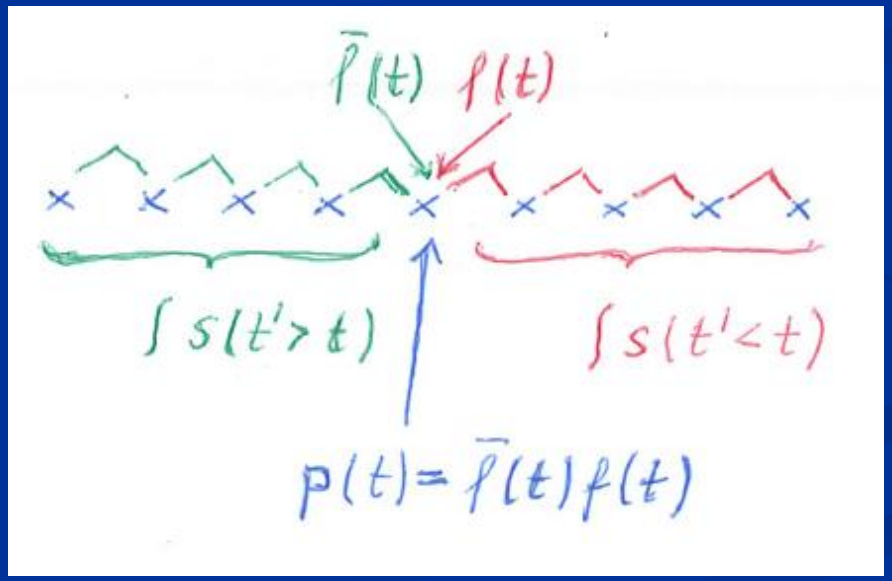
wave function

- integrate out the past



time-local probabilities

- time-local probabilities are bilinear in wave function and conjugate wave function
- wave functions = probability amplitudes



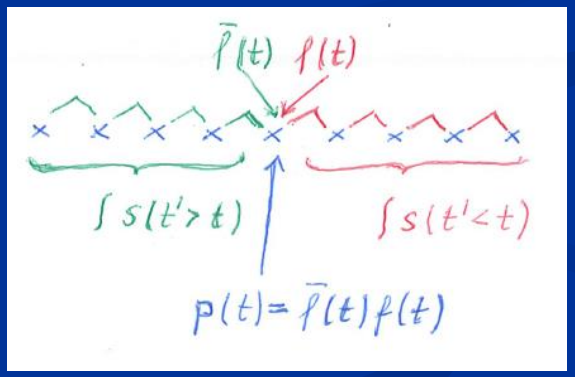
Partition function

$$Z = \sum_{\{n\}} \bar{f}_f\big(n(t_f)\big) K[n] f_{in}\big(n(t_{in})\big)$$

$$Z = \int \mathcal{D}n \; w[n] = 1$$

$$K[n] = \prod_t \mathcal{K}\big(n(t+\epsilon), n(t)\big) = \prod_t \mathcal{K}(t)$$

hypersurface located at t
can be used to
split K into two parts :



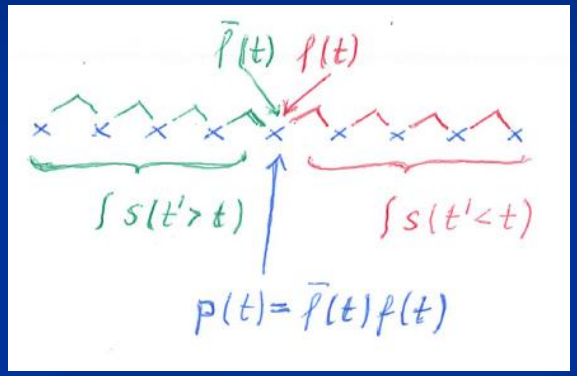
$$K[n] = K_>(t) K_<(t)$$

$$K_<(t) = \prod_{t'=t_{in}}^{t-\epsilon} \mathcal{K}(t') \;, \; K_>(t) = \prod_{t'=t}^{t_f-\epsilon} \mathcal{K}(t)$$

wave functions

$$f(t) = \int \mathcal{D}n(t' < t) K_<(t) f_{in}(t_{in})$$

$$\bar{f}(t) = \int \mathcal{D}n(t' > t) \bar{f}_f(t_f) K_>(t)$$



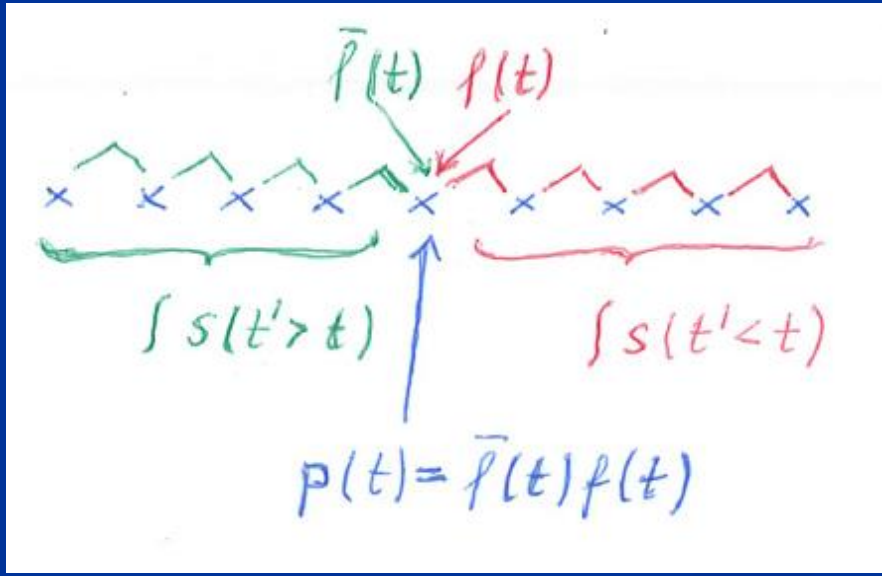
$$Z = \int dn(t) \bar{f}(t) f(t) = 1$$

Local probability distribution

$$p(t) = \bar{f}(t) f(t)$$

time-local probabilities

wave functions contains more time-local probabilistic information than the time-local probability distribution !



wave functions can be represented as real vectors

- single spin : f(t) can take two values, one for n=1, the other for n=0
- two- component real vector
- two spins : f(t) has four real values, for (n_1,n_2) taking the values (1,1), (1,0), (0,1), (0,0)
- four- component real vector

occupation number basis for wave function

$$f(t) = \tilde{q}_\tau(t) f_\tau(t)$$

$$\bar{f}(t) = \bar{q}_\tau(t) f_\tau(t)$$

$f_\tau(t)$: set of basis functions, depend on n[t]

occupation number basis for wave function

$$f(t) = \tilde{q}_\tau(t) f_\tau(t) \qquad \bar{f}(t) = \bar{q}_\tau(t) f_\tau(t)$$

four state system : only two occupation numbers

$$f_1 = n_1 n_2 \ , \ f_2 = (1 - n_1) n_2,$$
$$f_3 = n_1 (1 - n_2) \ , \ f_4 = (1 - n_1)(1 - n_2)$$

$$n_1 f_1 = f_1 \ , \ n_1 f_2 = 0 \ , \ n_1 f_3 = f_3 \ , \ n_1 f_4 = 0.$$
$$n_2 f_1 = f_1 \ , \ n_2 f_2 = f_2 \ , \ n_2 f_3 = 0 \ , \ n_2 f_4 = 0.$$

occupation number basis for wave function

$$f(t) = \tilde{q}_\tau(t) f_\tau(t) \qquad \bar{f}(t) = \bar{q}_\tau(t) f_\tau(t)$$

$$n_\gamma = 0, 1 \qquad \gamma = 1 \ldots M \qquad N = 2^M \text{ local states}$$

$$\tau = (0, 1, 1, 0, 1 \ldots)$$

$$f_\tau = (1 - n_1) n_2 n_3 (1 - n_4) n_5 \ldots$$

$$n_\gamma f_\tau = (n_\gamma)_\tau f_\tau$$

$f_1 = n_1 n_2$, $f_2 = (1 - n_1) n_2$,
$f_3 = n_1 (1 - n_2)$, $f_4 = (1 - n_1)(1 - n_2)$

occupation number basis for wave function

$$f(t) = \tilde{q}_\tau(t) f_\tau(t) \qquad \bar{f}(t) = \bar{q}_\tau(t) f_\tau(t)$$

$$n_\gamma = 0, 1 \qquad \gamma = 1 \ldots M \qquad N = 2^M \text{ local states}$$

product rule $$f_\tau f_\rho = f_\tau \delta_{\tau\rho}$$

integration rule $$\int dn f_\tau = \prod_\gamma \sum_{n_\gamma = 0,1} f_\tau = 1$$

sum rule $$\sum_\tau f_\tau = 1$$

“Evolution” in classical statistical equilibrium systems

Evolution

- Time-local information available at t
- How is it transferred to t + ϵ ?
- Evolution equation
- Simple for wave function
- Local probabilities are insufficient

quantum formalism for information transport (preview)

- go from one t to the next
- described by generalized Schrödinger equation for classical wave function

$$\tilde{q}(t+\epsilon) = S\tilde{q}(t)$$

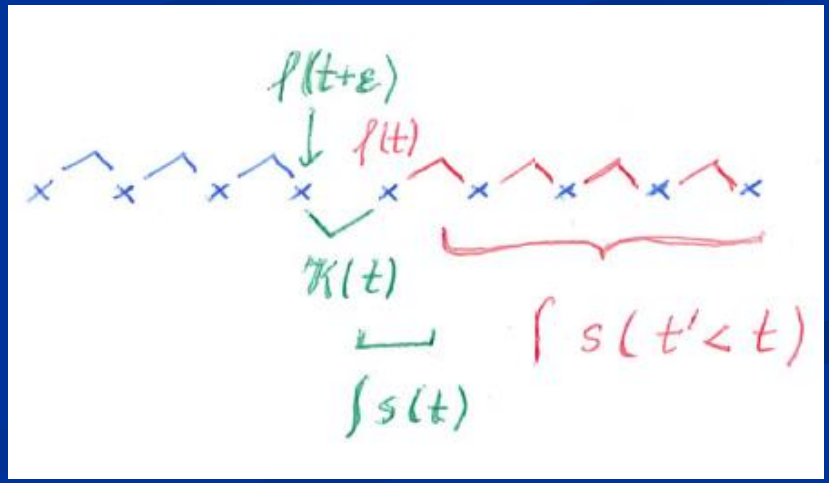
$$\partial_t \tilde{q} = \frac{d\tilde{q}}{dt} = W\tilde{q} \qquad W = \frac{1}{2\epsilon}(S - S^{-1})$$

evolution of wave function

integrate over n(t)

$$f(t+\epsilon) = \int dn(t)\mathcal{K}(t)f(t)$$

- linear evolution law
- superposition principle



step evolution operator

$$\mathcal{K}(m) = \exp(-\mathcal{L}(m))$$

$$\mathcal{K}(m) = \hat{T}_{\tau\rho}(m) h_{\tau}(m+1) h_{\rho}(m)$$

transfer matrix T

step evolution operator S equals transfer matrix with normalization such that largest eigenvalue equals one

evolution of wave function

$$f(t+\epsilon) = \int dn(t)\mathcal{K}(t)f(t)$$

$$f(t) = \int \mathcal{D}n(t' < t)K_<(t)f_{in}(t_{in})$$

$$K_<(t) = \prod_{t'=t_{in}}^{t-\epsilon} \mathcal{K}(t')$$

$$\mathcal{K}(m) = \hat{S}_{\tau\rho}(m)h_\tau(m+1)h_\rho(m)$$

$$\begin{aligned}
\tilde{f}(m+1) &= \tilde{q}_\tau(m+1)h_\tau(m+1) \\
&= \int \mathscr{D}n(m)\mathcal{K}(m)\tilde{f}(m) \\
&= \int \mathscr{D}n(m)\hat{S}_{\tau\sigma}(m)h_\tau(m+1)h_\sigma(m)\tilde{q}_\rho(m)h_\rho(m) \\
&= \hat{S}_{\tau\sigma}(m)\delta_{\sigma\rho}\,\tilde{q}_\rho(m)h_\tau(m+1) \\
&= \hat{S}_{\tau\rho}(m)\tilde{q}_\rho(m)h_\tau(m+1)\,,
\end{aligned}$$

$$\tilde{q}_\tau(t+\epsilon) = S_{\tau\rho}(t)\tilde{q}_\rho(t)$$

generalized discrete Schrödinger equation

$$\tilde{q}_\tau(t+\epsilon) = S_{\tau\rho}(t)\tilde{q}_\rho(t)$$

quantum mechanics

$$\psi(t+\epsilon) = U(t+\epsilon, t)\psi(t)$$

evolution of conjugate wave function

$$\bar{q}_\tau(t) = \bar{q}_\rho((t+\epsilon))S_{\rho\tau}(t)$$

$$\bar{q}_\tau(t+\epsilon) = \bar{q}_\rho(t)(S^{-1}(t))_{\rho\tau} = (S^T(t))^{-1}_{\tau\rho}\bar{q}_\rho(t)$$

quantum formalism for information transport

extended quantum formalism for problem of information transport

differences to quantum mechanics:

- real wave function and conjugate wave function instead of unique complex wave function
- evolution not necessarily unitary

Particle wave duality

Particle aspect:

- Bits: yes/no decisions
- Possible measurement values 1 or 0

Discrete spectrum of observables

Wave aspect :

Continuous probabilistic information

(wave function)

operators for observables

local observables take fixed values $A_\tau(m)$

for configurations at each discrete time value m

one can associate to each local observable an operator

the expectation value is given by the quantum rule

$$\langle A[n(m)]\rangle = \langle \bar{q}(m)|\hat{A}(m)|\tilde{q}(m)\rangle$$
$$= \bar{q}_\tau(m)\hat{A}_{\tau\rho}(m)\tilde{q}_\rho(m)$$

diagonal operators

in occupation number basis the local observables are represented by diagonal operators

$$\hat{A}_{\rho_m \rho_{m'}}(m) = \sum_{\tau} A_{\tau}(m) \delta_{\rho_m \tau} \delta_{\rho_{m'} \tau}$$

$$\begin{aligned} \langle A[n(m)] \rangle &= \langle \bar{q}(m) | \hat{A}(m) | \tilde{q}(m) \rangle \\ &= \bar{q}_{\tau}(m) \hat{A}_{\tau\rho}(m) \tilde{q}_{\rho}(m) \end{aligned}$$

operators for observables

expand in basis functions

$$A[n(m)] = A_\tau(m) h_\tau(m)$$

$$\begin{aligned}\langle A[n(m)]\rangle &= \int \mathscr{D}n A[n(m)] \prod_{m'=0}^{\mathscr{M}-1} \mathcal{K}(m) f_{\text{in}} \bar{f}_{\text{f}} \\ &= \int \mathscr{D}n(m) A[n(m)] \bar{f}(m) \tilde{f}(m)\,.\end{aligned}$$

$$\begin{aligned}\langle A[n(m)]\rangle &= \int \mathscr{D}n(m) \bar{q}_\tau(m) A_\sigma(m) \tilde{q}_\rho(m) h_\tau(m) h_\sigma(m) h_\rho(m) \\ &= \bar{q}_\tau(m) \hat{A}_{\tau\rho}(m) \tilde{q}_\rho(m)\,,\end{aligned}$$

$$\begin{aligned}\hat{A}_{\tau\rho}(m) &= \int \mathscr{D}n(m)\, A_\sigma(m) h_\tau(m) h_\sigma(m)\, h_\rho(m) \\ &= A_\tau(m) \delta_{\tau\rho}\,.\end{aligned}$$

quantum rule for local observables

depend only on configurations of occupation numbers at given t

$$\langle A(t) \rangle = \int \mathcal{D}n \quad A(t, [n(t)]) \quad w[n]$$

sum over configurations at all t

local probabilities

local observable can be computed from local probabilities

$$p(t,[n(t)]) = \prod_{t' \neq t} \int \mathcal{D}n(t')\, w[n]$$

$$\langle A(t) \rangle = \int \mathcal{D}n(t)\, A(t,[n(t)])\, p(t,[n(t)])$$

$$\langle A(t) \rangle = \int \mathcal{D}n \;\; A(t,[n(t)]) \;\; w[n]$$

classical wave function

$$\psi(t;[n(t)]) = \int \mathcal{D}n(t'<t) \prod_{t'<t} \mathcal{K}(t')\, \rho_{in}(t_{in})$$

classical wave function integrates the past half

$$\bar{\psi}(t;[n(t)]) = \int \mathcal{D}n(t'>t) \prod_{t'\geq t} \mathcal{K}(t')\, \bar{\rho}_{f}(t_{\varphi})$$

conjugate wave function integrates the future half

Local probability and wave functions

$$p\Big(t;\,[n(t)]\Big) \;=\; \bar{\psi}\Big(t;\,[n(t)]\Big)\,\psi\Big(t;\,[n(t)]\Big)$$

quantum rule for expectation value

$$\langle A(t) \rangle = \int \mathcal{D}n(t)\, \bar{\psi}(t,[n(t)])\, A(t,[n(t)])\, \psi(t,[n(t)])$$

$$\langle A(t) \rangle = \int \mathcal{D}n(t)\, A(t,[n(t)])\, p(t,[n(t)])$$

$$\langle A \rangle = \bar{\psi}\, A'\, \psi$$

generalization : local chain

$$w[n] = \bar{f}_f\big(n(t_f)\big)\, K[n]\, f_{in}\big(n(t_{in})\big)$$

$$K[n] = \exp\{-S_{cl}[n]\}$$

$$S_{cl}[n] = \sum_t \mathcal{L}(t)$$

$$K[n] = \prod_t \mathcal{K}\big(n(t+\epsilon), n(t)\big) = \prod_t \mathcal{K}(t)$$

quantum mechanics vs classical statistics

- this formalism works for all weight distributions which can be written as local chains

- quantum mechanics : complex weight function
- classical statistics : real weight function, positive probabilities

orthogonal evolution in classical statistics

- for orthogonal step evolution operators the evolution is the same for the wave function and the conjugate wave function
- they can be identified

only one real wave function q(t)

$$\bar{q}_\tau(t+\epsilon) = \bar{q}_\rho(t)(S^{-1}(t))_{\rho\tau} = (S^T(t))^{-1}_{\tau\rho}\bar{q}_\rho(t)$$

orthogonal evolution

- no information is lost for orthogonal step evolution operator
- the wave functions of classical statistical systems do not approach equilibrium for increasing t

Hamiltonian

- Define H by $\widehat{S} = \exp\left(-i\varepsilon H\right)$
- H is Hermitian and piecewise constant
- Interpolating continuous time evolution

  $q(t_2) = U(t_2, t_1)q(t_1)$ $\quad U(t_1, t_2) = \exp\left(-i(t_1 - t_2)H\right)$
- Schrödinger equation $i\partial_t q = Hq$
- Solution agrees with discrete evolution for t= $t_{\text{in}} + m\varepsilon$

complex structure

- quantum mechanics can always be written in real formulation

$$\psi(t,x) = q_R(t,x) + iq_I(t,x)$$

- unitary evolution operator becomes orthogonal in real formulation
- what are conditions to write orthogonal evolution for real wave function as unitary evolution for complex wave function?
- requires complex structure which is compatible with evolution

complex structure

- complex structure requires two discrete transformations acting on real wave function

$$q'_\tau = (K_c)_{\tau\rho} q_\rho \,, \quad q''_\tau = (I)_{\tau\rho} q_\rho \qquad K_c^2 = 1 \,, \quad I^2 = -1 \,, \quad \{K_c, I\} = 0$$

- define eigenfunctions of K $\quad K_c q_R = q_R \,, \quad K_c q_I = -q_I$
- define complex wave function $\quad \psi = q_R + i q_I$
- it obeys $\quad K_c(\psi) = \psi^* \quad I(\psi) = i\psi$

- particular basis $\quad q = \begin{pmatrix} q_R \\ q_I \end{pmatrix}, \quad K_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

complex operators

- operators in real formulation which are compatible with complex structure have to obey a condition

$$\hat{A} = 1 \otimes A_R + \tilde{I} \otimes A_I = A_R + I A_I = \begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix} \qquad [\hat{A}, I] = 0$$

- if operators obey this condition one has for $A = A_R + iA_I$

$$q' = \hat{A} q \rightarrow \psi' = A \psi$$

- step evolution operator has to be compatible with complex structure, example $\hat{S} = \begin{pmatrix} A_R & 0 \\ 0 & A_R \end{pmatrix}$

quantum mechanics from classical statistics

- orthogonal evolution operator
- complex structure compatible with evolution

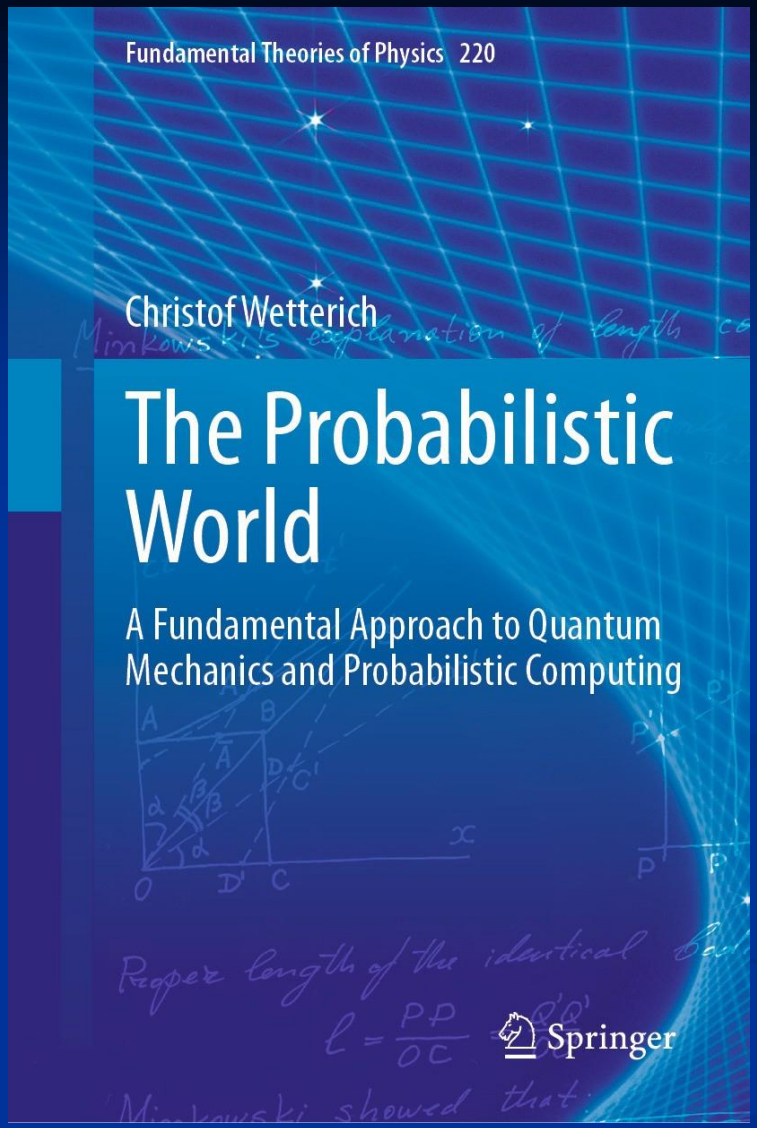
conclusions

- wave functions in quantum mechanics or classical statistics obtain by integrating out the past or future from overall weight distribution w
- linear evolution law
- operators

conclusion

- real formulation for general quantum systems : weight distribution w can take negative values, evolution is orthogonal

- classical statistics : w positive
- classical statistics can describe quantum mechanics if evolution is orthogonal and compatible with complex structure



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continuous evolution limit for classical wave function

- small change between t and t + ε

$$W(t) = \frac{1}{2\epsilon}\left(S(t) - S^{-1}(t-\epsilon)\right)$$

$$\partial_t \tilde{q}(t) = \frac{1}{2\epsilon}\left(\tilde{q}(t+\epsilon) - \tilde{q}(t-\epsilon)\right) = W(t)\tilde{q}(t)$$

generalized Schrödinger equation

evolution equation for conjugate wave function

$$\partial_t \bar{q}(t) = -\tilde{W}^T(t)\bar{q}(t)$$

$$\tilde{W}(t) = \frac{1}{2\epsilon}\left[S(t-\epsilon) - S^{-1}(t)\right]$$

$$t\text{-independent } S \ : \tilde{W} = W$$

complex structure for general classical probabilistic systems

$$\begin{pmatrix} W_1, & W_2 \\ -W_2, & W_1 \end{pmatrix}$$

$$\psi = \tilde{q}_R + i\tilde{q}_I$$

$$\bar{\psi} = \bar{q}_R - i\bar{q}_I$$

$$i\partial_t \psi = G\psi \ , \ G = \hat{H} + i\hat{J}$$

$$-i\partial_t \bar{\psi} = G^T \bar{\psi} = (\hat{H}^* + i\hat{J}^*)\bar{\psi}$$

$$\partial_t(\bar{\psi}^T \psi) = 0$$

$$\partial_t(\psi^\dagger \psi) = 2\psi^\dagger \hat{J} \psi$$

norm is not conserved, in general

quantum systems

- unitary evolution , J=0
- conserved norm
- conjugate wave function equals wave function

$$\bar{q} = \tilde{q}$$

$$\bar{\psi} = \psi^*$$

evolution factor for Ising chain

$$\mathcal{K}(t) = \exp\left\{\beta s(t+\epsilon)s(t) + \frac{\gamma}{2}\big(s(t+\epsilon)+s(t)\big)\right\}$$

$$\beta = J/(k_B T)$$
$$\gamma = H/(k_B T)$$

$$n(t) = (1+s(t))/2$$

$$= \exp\left\{\beta(2n(t+\epsilon)-1)(2n(t)-1)\right\}$$
$$\times \exp\left\{\gamma\big(n(t+\epsilon)+n(t)-1\big)\right\}$$

$$f_1 = n,\ f_2 = 1-n$$

$$= \exp\{\beta(f_1(t+\epsilon)-f_2(t+\epsilon))(f_1(t)-f_2(t))\}.$$
$$\times \exp\left\{\frac{\gamma}{2}\big(f_1(t+\epsilon)-f_2(t+\epsilon)+f_1(t)-f_2(t)\big)\right\}$$

evolution factor and transfer matrix

$$\mathcal{K}(t) = e^{\beta+\gamma} f_1(t+\epsilon) f_1(t) + e^{\beta-\gamma} f_2(t+\epsilon) f_2(t) + e^{-\beta}\big(f_1(t+\epsilon) f_2(t) + f_2(t+\epsilon) f_1(t)\big),$$

transfer matrix $\bar{S} = \begin{pmatrix} e^{\beta+\gamma} & , & e^{-\beta} \\ e^{-\beta} & , & e^{\beta-\gamma} \end{pmatrix}$

multiply evolution factor with normalization constant such that Z=1

$$S = e^{-\varphi} \bar{S}$$

$$\mathcal{K}(t) = S_{\tau\rho}(t) f_\tau(t+\epsilon) f_\rho(t)$$

step evolution operator

$$S = \begin{pmatrix} e^{\beta+\gamma-\varphi} & , & e^{-\beta-\varphi} \\ e^{-\beta-\varphi} & , & e^{\beta-\gamma-\varphi} \end{pmatrix}$$

free energy density

$$\varphi(\beta,\gamma) = \ln\left[e^{\beta}\cosh\gamma + \sqrt{e^{-2\beta} + \sinh^2\gamma e^{2\beta}}\right]$$

$$\tilde{q}_\tau(t_{in} + n\epsilon) = (S^n)_{\tau\rho}\tilde{q}_\rho(t_{in})$$

eigenvalues of step evolution operator

- largest eigenvalue : 1

- second eigenvalue : $\lambda_- = \det S = \dfrac{e^{4\beta} - 1}{\left(\frac{1}{g} + \cosh \gamma e^{2\beta}\right)^2}$ $g = \dfrac{1}{\sqrt{1 + \sinh^2 \gamma e^{4\beta}}}$

for large number of steps :

- wave function approaches eigenstate to largest eigenvalue : equilibrium wave function

$$\tilde{q}_\tau(t_{in} + n\epsilon) = (S^n)_{\tau\rho}\tilde{q}_\rho(t_{in})$$

approach to equilibrium

- equilibrium wave function $S\tilde{q}_* = \tilde{q}_*$

$$\tilde{q}_{*2} = \left(\frac{1}{g} - \sinh\gamma e^{2\beta}\right)\tilde{q}_{*1} \qquad g = \frac{1}{\sqrt{1+\sinh^2\gamma e^{4\beta}}}$$

- approach to equilibrium

$$\tilde{q}(t) = c_1\left(\tilde{q}_* + \exp\left(-\frac{t-t_{in}}{\xi}\right)\delta\tilde{q}(t_{in})\right)$$

- correlation length $\xi = \dfrac{\epsilon}{\ln(1/\lambda_-)}$

solution of initial value problem

$$\bar{q}_{*2} = \left(\frac{1}{g} - \sinh \gamma e^{2\beta} \right) \bar{q}_{*1}$$

$$\bar{q}_{*1}\tilde{q}_{*1} + \bar{q}_{*2}\tilde{q}_{*2} = 1$$

$$n_* = \langle n \rangle = \bar{q}_{*1}\tilde{q}_{*1} = \frac{1}{2N_1}$$

$$N_1 = 1 + \sinh^2 \gamma e^{4\beta} - \frac{1}{g} \sinh \gamma e^{2\beta}$$

equilibrium spin

$$s_* = 2n_* - 1 = \frac{1}{N_1} - 1 = \frac{\Delta}{1+\Delta}$$

$$\Delta = \sinh \gamma e^{2\beta} \left(\frac{1}{g} + \sinh \gamma e^{2\beta} \right)$$

$$\langle n(t) \rangle = n_* + \left(n(t_{in}) - n_* \right) \exp \left(-\frac{t - t_{in}}{\xi} \right)$$

information transport

- *initial value problem solved in term of classical wave functions*
- *information loss in the bulk*
- *related to unique largest eigenvalue*