Lectures on quantum field theory 2

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ABSTRACT: Notes for lectures that introduce students of physics to quantum field theory with applications to high energy physics, condensed matter and statistical physics.

Contents

0.1 Organizational issues

There is a webpage to accompany this lecture: https://uebungen.physik.uni-heidelberg.de/vorlesung/20191/qft2. Exercises will be proposed every week and discussed in tutorial classes. The registration goes via the webpage above.

0.2 Literature

There is a large amount of literature on different aspects of quantum field theory. Here is only a fine selection.

Statistical field theory / renormalization group

- John Cardy, *Scaling and renormalization in statistical physics* (1996)

- Giorgio Parisi, *Statistical field theory* (1998)

- Jean Zinn-Justin, *Quantum field theory and critical phenomena* (2002)

- Crispin Gardiner, *Handbook of stochastic methods* (1985)

Relativistic quantum field theory

- Mark Srednicki, *Quantum field theory* (2007)

- Michael Peskin & Daniel Schroeder, *An introduction to quantum field theory* (1995)

- Steven Weinberg, *The quantum theory of fields I & II* (1998)

Non-relativistic quantum field theory / condensed matter

- Alexander Altland & Ben Simons, *Condensed matter field theory* (2010)

- Lev Pitaevskii & Sandro Stringari, *Bose-Einstein condensation* (2003)

- Crispin Gardiner & Peter Zoller, *The quantum world of ultra-cold atoms and light* (2014)

Group theory

- Anthony Zee, *Group theory in a nutshell for physicists* (2016)

0.3 Typos

Please send any typos to a.mazeliauskas@thphys.uni-heidelberg.de.

11 Generating functionals

The aim of this chapter is the development of efficient techniques for the computation of expectation values of fields, correlation functions and quantum field equations. One main tool will be the quantum effective action. It resembles the action in classical field theories in the sense that field equations follow by variation. They will be quantum field equations, which include all effects of quantum fluctuations. Correlation functions of the full quantum field theory follow from suitable functional derivatives of the quantum effective action.

11.1 Partition function for scalar $O(N)$-models

Action

The microscopic or classical action for $O(N)$-symmetric scalar field theories is given by

$$S = \int d^d x \, \frac{1}{2} \left\{ \partial^\mu \chi_a \partial_\mu \chi_a + V(\rho) \right\}, \tag{11.1}$$

with classical potential $V(\rho)$ depending on the $O(N)$ invariant bilinear

$$\rho = \frac{1}{2} \chi_a \chi_a. \tag{11.2}$$

We take a quartic potential with $\lambda > 0$,

$$V = m^2 \rho + \frac{\lambda}{2} \rho^2 = \frac{\lambda}{2} \left(\rho - \rho_0 \right)^2. \tag{11.3}$$

For $m^2 > 0$ the minimum is at the region $\rho = 0$ (symmetric or SYM regime), while for $m^2 < 0$ the minimum is at $\rho_0 > 0$ (spontaneously broken regime, SSB regime) with

$$\rho_0 = -\frac{m^2}{\lambda}. \tag{11.4}$$

This follows directly from

$$\frac{\partial V}{\partial \rho} = \partial_\rho V = V' = m^2 + \lambda \rho. \tag{11.5}$$

We work in d-dimensional euclidean space, $\partial^\mu = \delta^{\mu\nu} \partial_\nu$, with possible analytic continuation to Minkowski space. In momentum space the Fourier-modes $\chi(q)$ are given by

$$\chi(x) = \int_q e^{iq_\mu x^\mu} \chi(q) = \int \frac{d^d q}{(2\pi)^d} \, e^{iq_\mu x^\mu} \chi(q). \tag{11.6}$$

In order to have a well defined functional integral we limit the momentum range by an ultraviolet (UV) cutoff Λ,

$$q^2 < \Lambda^2. \tag{11.7}$$

In statistical physics Λ may be roughly associated with an inverse lattice distance. For particle physics we may associate Λ with some energy or momentum scale beyond which a given model remains no longer valid. The momentum cutoff means that no fluctuations with wavelength smaller than Λ^{-1} are taken into account. In this sense the microscopic action S is associated to a momentum scale Λ, $S = S_\Lambda$. The aim is a computation of macroscopic properties at length scales larger than Λ^{-1}.

Local sources

The partition function $Z[j]$ is a functional of local sources $j_a(x)$ or $j_a(q)$,

$$Z[j] = \int D\chi \exp\{-S[\chi] + \int_x j_a \chi_a\}..$$

(11.8)

For magnets this generalizes a constant magnetic field, corresponding to x-independent j, to and arbitrary inhomogeneous magnetic field. For particle physics the local sources are mainly a technical device. Only in certain cosmological situation they have direct physical meaning. An important point in the formal development is that the source term is linear in the microscopic fluctuating field χ_a.

The measure in the functional integral is defined as

$$\int D\chi = \prod_q \prod_a \int_{-\infty}^{\infty} d\chi_a(q).$$

(11.9)

in some suitable basis of real $\chi^a(q)$. The "regularization" $q^2 < \Lambda^2$ constrains the functional measure

$$\prod_q = \prod_{q, q^2 < \Lambda^2} .$$

(11.10)

We may further choose space or space-time to have finite volume by taking a d-torus, with circumferences L_μ. The Fourier modes are discrete in this case,

$$q_\mu = \frac{2\pi n_\mu}{L_\mu},$$

(11.11)

with n_μ integers. The partition function is then a well defined finite-dimensional integral. The limit $L_\mu \to \infty$ defines the (infinite-dimensional) functional integral. If we keep $L_0 = \beta = \frac{1}{T}$ fixed with $L_i \to \infty$, the functional integral describes quantum field theory on non-zero temperature T (Matsubara formalism).

11.2 Expectation values

For finite L and Λ a "functional derivative" is a standard partial derivative

$$\frac{\delta}{\delta j_a(q)} \equiv \frac{\partial}{\partial j_a(q)}.$$

(11.12)

For continuous functions $j_a(q)$ the functional derivative is defined as the limit $L \to \infty$ of the partial derivative. We use for functional derivatives the same notation as for partial derivatives. Functional derivatives with respect to $j_a(x)$ are given by the Fourier-transform.

Since S does not depend on $j_a(q)$ one has

$$\frac{\partial Z}{\partial j_a(x)} = \int D\chi e^{-S + \int_x j\chi} \chi_a(x).$$

(11.13)

We can therefore express the expectation value

$$\langle \chi_a(x) \rangle = Z^{-1} \int D\chi e^{-S + \int_x j\chi} \chi_a(x)$$

(11.14)

as

$$\langle \chi_a(x) \rangle = \frac{\partial \ln Z[j]}{\partial j_a(x)}.$$

(11.15)

This is the expectation value for an arbitrary local source $j_a(x)$. It therefore depends on $j_a(x)$. At the end one is often interested at the expectation value for vanishing source $j_a(x)$. This is obtained by setting $j = 0$ *after* the differentiation.

The two-point function,

$$\langle \chi_a(x)\chi_b(y)\rangle = \frac{1}{Z}\int D\chi \chi_a(x)\chi_b(y)e^{-S+\int j\chi}, \tag{11.16}$$

can be obtained as the second functional derivative

$$\langle \chi_a(x)\chi_b(y)\rangle = \frac{1}{Z}\frac{\partial^2 Z}{\partial j_a(x)\partial j_b(y)}. \tag{11.17}$$

This generalizes directly to arbitrary n-point functions. We conclude that the partition function $Z[j]$ contains the information about arbitrary n-point functions. They can be obtained by simple differentiation. The functional integral is already performed here, thus all quantum and thermal fluctuations are already included. At non-zero temperature and finite volume, Z corresponds to the canonical partition function in the presence of local sources. It contains the full thermodynamics, in particular also for zero sources or constant sources.

11.3 Connected correlation functions, Schwinger functional

The generating functional for the connected Greens functions (Schwinger functional) is defined as the logarithm of the partition function

$$W[j] = \ln Z[j]. \tag{11.18}$$

It is a kind of thermodynamic potential. The first functional derivative is the field expectation value

$$\langle \chi_a(x)\rangle = \frac{\partial W}{\partial j_a(x)}. \tag{11.19}$$

The second functional derivative of W yields the connected two point function or correlation function

$$G_{ab}(x,y) = \langle \chi_a(x)\chi_b(y)\rangle - \langle \chi_a(x)\rangle\langle \chi_b(y)\rangle = \langle(\chi_a(x) - \langle \chi_a(x)\rangle)(\chi_b(y) - \langle \chi_b(y)\rangle)\rangle = \frac{\partial^2 W}{\partial j_a(x)\partial j_b(y)}. \tag{11.20}$$

We may collect space indices x and internal indices into a collective index $\alpha' = (x, a)$. In momentum space on has similarly $\alpha' = (q, a)$. In this notation we can express relations independently of the chosen basis in field-space. The correlation function is a matrix

$$G_{\alpha\beta} = \langle \chi_\alpha \chi_\beta\rangle - \langle \chi_\alpha\rangle\langle \chi_\beta\rangle. \tag{11.21}$$

One has

$$\langle \chi_\alpha\rangle = \frac{\partial W}{\partial j_\alpha}. \tag{11.22}$$

and

$$\frac{\partial^2 W}{\partial j_a \partial j_\beta} = \frac{\partial}{\partial j_\alpha}\frac{\partial \ln Z}{\partial j_\beta} = \frac{\partial}{\partial j_\alpha}\left(\frac{1}{Z}\frac{\partial Z}{\partial j_\beta}\right) = \frac{1}{Z}\frac{\partial^2 Z}{\partial j_\alpha \partial j_\beta} - \frac{1}{Z^2}\frac{\partial Z}{\partial j_\alpha}\frac{\partial Z}{\partial j_\beta} = \langle \chi_\alpha \chi_\beta\rangle - \langle \chi_\alpha\rangle\langle \chi_\beta\rangle, \tag{11.23}$$

proving eq. (11.20).

Propagator

For simplicity, we take a single real scalar, $N = 1$, with straightforward generalization. The propagator

$$G(x, y) = \langle (\chi(x) - \langle \chi(x) \rangle) (\chi(y) - \langle \chi(y) \rangle) \rangle. \tag{11.24}$$

is in a statistical sense the mean square or variance. Quite generally, arbitrary statistical theories have the same structure as our euclidean functional integral formulation of quantum field theories. The statistical variables are χ_i or $\chi(x)$, and the probability distribution is

$$p[\chi] = \frac{1}{Z} e^{-S[\chi] + \int j\chi}, \tag{11.25}$$

with measure given by the functional measure $\int D\chi$ such that $\int D\chi p[\chi] = 1$. Much of the formalism developed in the lecture can be applied directly to arbitrary statistical theories. That explains why QFT methods based on functional integral appear in economics or biology, and inversely statistical methods as Monte-Carlo simulations have brought important progress in particle physics problems as QCD.

The propagator $G(x, y)$ is evaluated for arbitrary sources. In general $\langle \chi(x) \rangle$ is therefore not zero. In particular, the ground state of particle physics is characterized by spontaneous symmetry breaking where $\langle \chi(x) \rangle$ differs from zero, being nevertheless time-independent and homogeneous for consistency with Lorentz-symmetry. In cosmology, the ground state is replaced by the cosmological state, which is a time dependent, typically isotropic and homogeneous solution of the field equations. Again, $\langle \chi(t) \rangle$ differs from zero. Propagating particles are interpreted as excitations above the ground state or cosmological state. The definition of the propagator by the connected two-point function has the important property

$$\lim_{|x-y| \to \infty} G(x, y) \to 0. \tag{11.26}$$

This does not hold for the unconnected two point function. For example, for x-independent $\langle \chi \rangle$ on has

$$\lim_{|x-y| \to \infty} \langle \chi(x) \chi(y) \rangle = \langle \chi^2 \rangle. \tag{11.27}$$

Higher connected correlation functions

The naming "connected correlation functions" can be understood by taking higher derivatives of W

$$\frac{\partial^3 W}{\partial j(x_1) \partial j(x_2) \partial j(x_3)} = \langle \chi(x_1) \chi(x_2) \chi(x_3) \rangle \tag{11.28}$$

$$- \langle \chi(x_1) \rangle \langle \chi(x_2) \chi(x_3) \rangle - \langle \chi(x_2) \langle \chi(x_1) \chi(x_3) \rangle \tag{11.29}$$

$$- \langle \chi(x_3) \rangle \langle \chi(x_1) \chi(x_2) \rangle + 2 \langle \chi(x_1) \rangle \langle \chi(x_2) \rangle \langle \chi(x_3) \rangle. \tag{11.30}$$

or

$$\frac{\partial^4 W}{\partial j(x_1) \partial j(x_2) \partial j(x_3) \partial j(x_4)} = \langle \chi(x_1) \chi(x_2) \chi(x_3) \chi(x_4) \rangle \tag{11.31}$$

$$- \langle \chi(x_1) \chi(x_2) \rangle \langle \chi(x_3) \chi(x_4) \rangle - \langle \chi(x_1) \chi(x_3) \rangle \langle \chi(x_2) \chi(x_4) \rangle \tag{11.32}$$

$$- \langle \chi(x_1) \chi(x_4) \rangle \langle \chi(x_2) \chi(x_3) \rangle + \text{terms involving } \langle \chi(x_n) \rangle \tag{11.33}$$

$$\equiv \langle \chi(x_1) \chi(x_2) \chi(x_3) \chi(x_4) \rangle_c. \tag{11.34}$$

For $\langle \chi(x) \rangle = 0$ the connected four-point function subtracts from the four point function the "unconnected parts". Graphically, one has

$$\langle \chi(x_1)\chi(x_2)\chi(x_3)\chi(x_4)\rangle = \quad \text{(diagram)} \quad . \tag{11.35}$$

Only the connected part appears in scattering problems. The unconnected parts are "propagating particles somewhere in the world", that are not related to the scattering process. Remember: uncorrelated pieces do not contribute to scattering. Scattering amplitude: amputated connected Greens function.

Free theory (Gaussian theory)

For S quadratic in χ it follows that W is quadratic in j. In this case the Gaussian integral can be easily performed separately for every component, once the quadratic form is diagonalized

$$Z' = \int d\chi\, e^{-\frac{a}{2}\chi^2 + j\chi} = \int d\chi\, e^{-\frac{a}{2}\left(\chi - \frac{j}{a}\right)^2 + \frac{j^2}{2a}} = c_1 e^{\frac{j^2}{2a}}. \tag{11.36}$$

For a single mode one has

$$W' = \frac{j^2}{2a} + c_2, \tag{11.37}$$

with "'propagator"$\sim \frac{1}{a}$. The quadratic form generalizes to arbitrary quadratic S. Therefore all higher connected n-point functions vanish (no scattering).

Generating functionals

The functional $W[j]$ is a "generating functional". It generates the connected n-point functions by simple differentiation. It contains all the necessary information about the fluctuation effects in a quantum field theory, or more generally, in a probabilistic field theory. It is not very intuitive, however. This will be different for the effective action, which is another thermodynamic potential, related to W by a Legendre transform.

12 Effective action and quantum field equations

12.1 Effective action

For a real scalar field $\chi(x)$ we have considered arbitrary sources $j(x)$ and computed the expectation value of $\chi(x)$ *in presence* of a local source $j(x)$

$$\langle\chi(x)\rangle = \frac{\delta W}{\delta j(x)} \equiv \varphi(x). \tag{12.1}$$

Here $\varphi(x)$ denotes the expectation value in the presence of sources and depends on $j(x)$. It plays the role of a *macroscopic* scalar field. Consider now $\varphi(x)$ as a functional of sources. For a finite number of degrees of freedom φ_α, j_α, the φ_α are functions of j_β

$$\varphi_\alpha = \varphi_\alpha(j_\beta). \tag{12.2}$$

If φ_α depends monotonically on j_β one can invert $\varphi_\alpha(j_\beta)$ and compute $j_\beta(\varphi_\alpha)$. One can then employ a Legendre transform for the definition of the effective action

$$\boxed{\Gamma[\varphi] = -W[j[\varphi]] + \int_x j(x)\varphi(x)}. \tag{12.3}$$

Here j is considered as a functional of φ. The functional Γ depends on $\varphi(x)$. In practice we will need only the formal properties and do not explicitly perform the Legendre transform.

12.2 Quantum field equation

$$\boxed{\frac{\delta\Gamma}{\delta\varphi(x)} = j(x)}. \tag{12.4}$$

In particular for $j(x) = 0$ one has $\frac{\delta\Gamma}{\delta\varphi(x)} = 0$. This is the exact quantum field equation. Comparing with the classical field equation:

$$\frac{\delta S}{\delta\chi(x)} = 0, \tag{12.5}$$

one concludes that Γ replaces S once all quantum effects are included. More generally, $j(x)$ may be an electromagnetic current or the energy-momentum tensor in gravity.

Proof:

$$\frac{\delta\Gamma}{\delta\varphi(x)} = -\frac{\delta W}{\delta\varphi(x)} + \int_y \frac{\delta j(y)}{\delta\varphi(x)}\varphi(y) + j(x) \tag{12.6}$$

$$\frac{\delta W}{\delta\varphi(x)} = \int_y \frac{\delta W}{\delta j(y)}\frac{\delta j(y)}{\delta\varphi(x)} = \int_y \varphi(y)\frac{\delta j(y)}{\delta\varphi(x)}. \tag{12.7}$$

The first two terms on the r.h.s. of eq. (12.6) cancel and one remains with eq. (12.4).

12.3 Inverse propagator

The inverse propagator is given by the second functional variation of the effective action,

$$\Gamma^{(2)}(x,y) = \frac{\delta^2\Gamma}{\delta\varphi(x)\delta\varphi(y)}. \tag{12.8}$$

We can interpret $\Gamma^{(2)}$ as an (infinite dimensional) matrix. For a finite number variables φ_α it is a finite matrix $\Gamma^{(2)}_{\alpha\beta} = \frac{\partial^2\Gamma}{\partial\varphi_\alpha\partial\varphi_\beta}$, and we take, as usual, the limit of infinitesimal many variables. As an important relation, we will show that $\Gamma^{(2)}$ is the inverse of the propagator matrix G,

$$\boxed{\Gamma^{(2)} = G^{-1}}. \tag{12.9}$$

In particular, in coordinate space $\Gamma^{(2)}$ is the inverse of the propagator $G(x,y) = \frac{\delta W}{\delta j(x)\delta j(y)}$, which is also considered as matrix,

$$\int_y \Gamma^{(2)}(x,y)G(y,z) = \delta(x-z). \tag{12.10}$$

Proof :

$$\left(\Gamma^{(2)}G\right)_{\alpha\gamma} = \frac{\partial^2\Gamma}{\partial\varphi_\alpha\partial\varphi_\beta}\frac{\partial^2 W}{\partial j_\beta\partial j_\gamma} = \frac{\partial}{\partial\varphi_\alpha}j_\beta\frac{\partial}{\partial j_\beta}\varphi_\gamma = \frac{\partial j_\beta}{\partial\varphi_\alpha}\frac{\partial\varphi_\gamma}{\partial j_\beta} = \frac{\partial\varphi_\gamma}{\partial\varphi_\alpha} = \delta_{\alpha\gamma}. \tag{12.11}$$

The matrix identity $\Gamma^{(2)}W^{(2)} = 1$ is valid in an arbitrary basis. In momentum space one has in case of translation symmetry

$$W^{(2)}_{qq'} = G(q)\delta(q-q') \leftrightarrow \Gamma^{(2)}_{qq'} = G^{-1}(q)\delta(q-q'). \tag{12.12}$$

Classical theory:

$$G_0 = \left(S^{(2)}\right)^{-1} \quad \text{classical propagator.} \tag{12.13}$$

Quantum field theory:

$$G = \left(\Gamma^{(2)}\right)^{-1} \quad \text{full propagator, including all quantum effects!.} \tag{12.14}$$

Momentum space

In momentum space the Fourier modes $\varphi(q)$ are complex. it is convinent to choose a convention for the matrix elements

$$\Gamma^{(2)}(q,q') = \frac{\delta^2\Gamma}{\delta\varphi^*(q)\delta\varphi(q')} = \frac{\delta j(q)}{\delta\varphi(q')} \tag{12.15}$$

$$W^{(2)}(q,q') = \frac{\delta^2 W}{\delta j^*(q)\delta j(q')} = \frac{\delta\varphi(q)}{\delta j(q')}, \tag{12.16}$$

such that

$$\int\frac{d^d q''}{(2\pi)^d}\,\Gamma^{(2)}(q,q'')W^{(2)}(q'',q') = (2\pi)^d\delta^d(q-q'). \tag{12.17}$$

For Fourier modes we employ a complex matrix convention: $\begin{pmatrix} & \\ & \end{pmatrix}\begin{matrix}\varphi \\ \varphi^* \end{matrix}$, such that the propagator $\varphi^* \quad \varphi$

is diagonal in momentum space.

12.4 Generating functional for one-particle irreducible (1PI) vertices

The higher functional derivatives of the effective action generate the one-particle irreducible (1PI) n-point vertices.

Cubic vertex

For the cubic vertex we take a φ-derivative of $\Gamma^{(2)} = (W^{(2)})^{-1}$,

$$\frac{\delta^3\Gamma}{\delta\varphi(x_1)\delta\varphi(x_2)\delta\varphi(x_3)} = \frac{\delta}{\delta\varphi(x_1)}\left(W^{(2)}\right)^{-1}(x_2,x_3) \tag{12.18}$$

$$= \int_{y_1}\frac{\delta j(y_1)}{\delta\varphi(x_1)}\frac{\delta}{\delta j(y_1)}\left(\frac{\delta^2 W}{\delta j(x_2)\delta j(x_3)}\right)^{-1}$$

$$= -\int_{y_1,y_2,y_3}\frac{\delta j(y_1)}{\delta\varphi(x_1)}\left(W^{(2)}\right)^{-1}(x_2,y_2)\frac{\delta^3 W}{\delta j(y_1)\delta j(y_2)\delta j(y_3)}\left(W^{(2)}\right)^{-1}(y_3,x_3).$$

Here we employ the matrix identity

$$\frac{\partial}{\partial t} A^{-1} = -A^{-1} \frac{\partial A}{\partial t} A^{-1}, \text{ where } A \text{ corresponds to } W^{(2)} \text{ and } t \text{ correspond to } j(y_1). \qquad (12.19)$$

One finally obtains

$$\frac{\delta^3 \Gamma}{\delta\varphi(x_1)\delta\varphi(x_2)\delta\varphi(x_3)} = -\int_{y_1,y_2,y_3} \Gamma^{(2)}(y_1,x_1)\Gamma^{(2)}(y_2,x_2)\Gamma^{(2)}(y_3,x_3)\frac{\delta^3 W}{\delta j(y_1)\delta j(y_2)\delta j(y_3)} \qquad (12.20)$$

$$= -\int_{y_1,y_2,y_3} \underbrace{\langle \chi(y_1)\chi(y_2)\chi(y_3)\rangle_c}_{\text{connected 3-point function}} \underbrace{G^{-1}(y_1,x_1)G^{-1}(y_2,x_2)G^{-1}(y_3,x_3)}_{\text{amputated}}.$$

Consider the Yukawa model with the interaction given by $\mathcal{L} \sim y\varphi\bar\psi\psi$. Replacing $\varphi(x_2)\varphi(x_3)$ by $\bar\psi(x_2), \psi(x_3)$ one has in lowest order: $\Gamma^{(3)} = S^{(3)} = y$.

Quartic vertex

Taking one further functional derivatives yields

$$\frac{\delta^4 \Gamma}{\delta\varphi(x_1)\delta\varphi(x_2)\delta\varphi(x_3)\delta\varphi(x_4)} = -\Big\{ \frac{\delta^4 W}{\delta j(y_1)\delta j(y_2)\delta j(y_3)\delta j(y_4)} \qquad (12.21)$$

$$\times\, G^{-1}(y_1,x_1)G^{-1}(y_2,x_2)G^{-1}(y_3,x_3)G^{-1}(y_4,x_4) - X \Big\},$$

where

$$X = -\Gamma^{(3)}(z_1,x_1,x_4)G^{-1}(y_2,x_2)G^{-1}(y_3,x_3)\frac{\delta^3 W}{\delta j(z_1)\delta j(y_2)\delta j(y_3)} - 2 \text{ other terms} \qquad (12.22)$$

$$= G^{-1}(z_1,z_2)G^{-1}(y_1,x_1)G^{-1}(y_4,x_4)\frac{\delta^3 W}{\delta j(z_2)\delta j(y_1)\delta j(y_2)}$$

$$\times\, G^{-1}(y_2,x_2)G^{-1}(y_3,x_3)\frac{\delta^3 W}{\delta j(z_1)\delta j(y_2)\delta j(y_3)} + 2 \text{ other permut.}$$

Here we have no longer written the integration over coordinates explicitly. Considering them as indices one integrates (sums) over all coordinates which appear twice, similar to the standard convention for index summations. One can use a graphical representation of X :

$$(12.23)$$

Amputated connected four point function

The amputated connected four point function can be expressed in terms of all the 1PI vertices,

$$\langle \chi(y_1)\chi(y_2)\chi(y_3)\chi(y_3)\rangle_c G^{-1}(y_1,x_1)\dots G^{-1}(y_4,x_4) = -\Big\{ \Gamma^{(4)}(x_1,x_2,x_3,x_4) + X \Big\}, \qquad (12.24)$$

with

$$X = \frac{\delta^3 W}{\delta j(z_2)\delta j(y_1)\delta j(y_4)}\frac{\delta^3 W}{\delta j(z_1)\delta j(y_2)\delta j(y_3)}G^{-1}(z_1,w_1)G(w_1,w_2)G^{-1}(w_2,z_2) \qquad (12.25)$$

$$\times\, G^{-1}(y_1,x_1)G^{-1}(y_4,z_4)G^{-1}(y_2,x_2)G^{-1}(y_3,x_3)$$

$$= \Gamma^{(3)}(x_1,x_4,w_2)\Gamma^{(3)}(x_2,x_3,w_1)G(w_1,w_2).$$

where we have used
$$G^{-1}(z_1, z_2) = G^{-1}(z_1, w_1)G(w_1, w_2)G^{-1}(w_2, z_2). \tag{12.26}$$

Graphical representation X :

$$X = \Gamma^{(3)}(w_1, x_2, x_3) \underset{w_1}{\overset{G(w_1, w_2)}{\bigcirc\!-\!\bigcirc}} \Gamma^{(3)}(w_2, x_1, x_4) \ . \tag{12.27}$$

The part X corresponds to a one particle reducible Feynman diagram. This means that it can be cut into two parts by cutting one internal line.

One particle irreducible graphs

The 1PI-vertices can be constructed from all Feynman diagrams that can *not* be cut into two parts by cutting one internal line. For the example of the Yukawa model, the connected 4-point function has a reducible and an 1PI-part

reducible $\tag{12.28}$

1PI. $\tag{12.29}$

Only 1PI graphs contribute to Γ (much less calculation).

12.5   Transition amplitude and effective action

For $n \to m$ scattering we need the transition amplitude $\mathcal{M}$. It is related to the amputated connected $n + m$-point function by

$$\boxed{\mathcal{M}\, \delta\left(\sum p_f - \sum p_i\right) = (\Gamma^{(n+m)} + X)_M}\,. \tag{12.30}$$

Here p_i are the incoming momenta and p_f the outgoing ones. The index $(\cdots)_M$ reminds us that correlation functions are evaluated in Minkowski space for vanishing sources. They can be found by analytic continuation from the euclidean correlation functions $\Gamma^{(n+m)}$. The part X arises from the one-particle reducible contributions, which can be constructed as trees from lower $\Gamma^{(p)}$, $p < n + m$.

An example is $2 \to 2$ scattering

$$\mathcal{M}(p_1, p_2; p_3, p_4) = \quad + \quad + \quad + \quad . \tag{12.31}$$

The internal line corresponds to the *full* propagator $\left(\Gamma^{(2)}\right)^{-1}$. We observe that for known 1PI-vertices only tree diagram appear. There are no more loops, since the fluctuating effects are already incorporated into the computation of the effective action.

Classical approximation

In the classical approximation the effective action is given by the microscopic classical action, $\Gamma = S$. No fluctuation effect are included in this case. In this approximation the 1PI-vertices and correlation function are simply the classical ones.

$$\Gamma_E^{(n)} = S_E^{(n)}. \tag{12.32}$$

More precisely, this holds in euclidean space. In Minkowski space there is an additional historic minus sign for the definition of the classical action, such that

$$\Gamma_M^{(n)} = -S_M^{(n)}. \tag{12.33}$$

Summary

- The (quantum) effective action Γ replaces the classical action. Once Γ is known, only tree diagrams have to be evaluated for the computation of $\mathcal{M}$!

- The full propagator and the full vertices in tree diagrams are given by the propagator $G = \left(\Gamma^{(2)}\right)^{-1}$ and the 1PI-vertices $\Gamma^{(n \geq 3)}$.

- The quantum effects of fluctuations changes S to Γ. S : microscopic action, Γ : macroscopic action. The effective action includes all fluctuation effects.

- Computation of the transition amplitude $\mathcal{M}$:
    1. compute Γ
    2. draw all tree diagrams
    3. insert full propagator for lines and full vertices

12.6   Functional integral for effective action

We want to represent the effective action Γ directly as a functional integral, without explicit use of a Legendre transform. This will be an implicit representation in the form of a functional differential equation.

We start with

$$e^{-\Gamma} = e^{-(-W + \int j\varphi)} = e^{W - \int j\varphi} = Z e^{-\int j\varphi} = \int D\chi \, e^{-S + \int (j\chi - j\varphi)}. \tag{12.34}$$

We use $j = \frac{\partial \Gamma}{\partial \varphi}$ in order to obtain

$$e^{-\Gamma} = \int D\chi e^{-S + \int \frac{\partial \Gamma}{\partial \varphi}(\chi - \varphi)}. \tag{12.35}$$

Finally, we consider fluctuations around the macroscopic field

$$\chi' = \chi - \varphi. \tag{12.36}$$

This yields the "background field identity"

$$\boxed{\exp\left(-\Gamma[\varphi]\right) = \int D\chi' \exp\left\{-S[\varphi + \chi'] + \int_x \frac{\partial \Gamma}{\partial \varphi(x)} \chi'(x)\right\}.} \tag{12.37}$$

Formally this is a functional differential equation. It can be used for the definition of the effective action without the use of the Legendre transform.

Classical approximation

The classical approximation holds if S is quadratic in χ. For

$$S = \frac{1}{2} A_{\alpha\beta}(\varphi_\alpha + \chi'_\alpha)(\varphi_\beta + \chi'_\beta) \tag{12.38}$$

one finds

$$\Gamma[\varphi] = S[\varphi] + c. \tag{12.39}$$

This can be proven by showing that $\Gamma = S + c$ solves the differential equation (12.37). Using $\frac{\partial \Gamma}{\partial \varphi_\alpha} = A_{\alpha\beta}\varphi$ one has

$$\exp\left(-\frac{1}{2} A_{\alpha\beta}\varphi_\alpha\varphi_\beta - c\right) = \int D\chi' \exp\left\{-\frac{1}{2} A_{\alpha\beta}(\varphi_\beta + \chi'_\beta) + A_{\alpha\beta}\varphi_\beta\chi'_\alpha\right\}. \tag{12.40}$$

The φ dependent terms can drop out and one ends with

$$\exp(-c) = \int D\chi' \exp\left\{-\frac{1}{2} A_{\alpha\beta}\chi'_\alpha\chi'_\beta\right\}. \tag{12.41}$$

Indeed, c does not depend on φ. It drops out from all functional derivatives with respect to φ and can be neglected for this purpose. Then the classical approximation is simply

$$\Gamma[\varphi] = S[\varphi]. \tag{12.42}$$

One has to do be more careful if $A_{\alpha\beta}$ depends on external parameters as T or μ, or if the integral over χ' depends on external parameter. Thus c may play a role for thermodynamics.

Perturbation theory

Interactions correspond to terms in S that are not quadratic in χ, e.g. cubic or quartic terms. If the couplings characterizing the interaction are small, one expects some kind of perturbation expansion in the small couplings,

$$\Gamma = S + \text{perturbation corrections}. \tag{12.43}$$

12.7 Quantum vertices

The effective action Γ contains new vertices that are not present in the classical action S. Example: photon-photon interaction. Classical Maxwell theory; no photon-photon interaction, Maxwell equations are linear. Quantum 1PI – four point function

$$\tag{12.44}$$

One loop contribution to photon-photon scattering

$$\Gamma^{(4)} \sim \alpha^2 \sim e^4, \tag{12.45}$$

For very small very small momenta below m_e one finds

$$\Gamma^{(4)} \sim \frac{q^4}{m_e^4} \quad . \tag{12.46}$$

The correction is small, but observable by precisions measurements. Recently light-by-light scattering has been observed by ATLAS experiment in heavy ion collisions at the Large Hadron Collider.

Another example is $g - 2$, as generated by 1PI-diagrams of the type

$$\tag{12.47}$$

The corresponding piece of the effective action is

$$\Gamma \sim \int_x \bar{\psi} \left[\gamma^\mu, \gamma^\nu\right] F_{\mu\nu} \psi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{12.48}$$

12.8 Classical field theory

A new classical field theory can be based on the quantum effective action Γ. Take vanishing sources $j = 0$.

- The field equation $\frac{\delta \Gamma}{\delta \varphi} = 0$ is exact! It contains already all quantum effects.

- Symmetries restrict the possible form of Γ. At a given order in the derivative expansion only few invariant are possible.

  The effective action is the basic concept for the use of gravitational field equations in a quantum context. Just imagine that Γ exists, even if quantum gravity not known. Expanding in the number of derivatives the invariance under general coordinates transformations (diffeomorphism symmetry) implies

  $$\Gamma = \int_x \sqrt{g} \left(-\frac{M^2}{2} R + \lambda\right) + \mathcal{O}(R^2). \tag{12.49}$$

  with metric $g_{\mu\nu}$, R the curvature scalar and $g = \det g_{\mu\nu}$. This effective action has two parameters: the reduced Planck mass M is defined by $M^2 = M_p^2/8\pi$ and related to Newton's coupling constant G_N by $M_p^2 = G_N^{-1}$. The other parameter is the cosmological constant λ.

Except for cosmology it can be neglected. General relativity is derived from the field equation that are originated by variations of the effective (Einstein-Hilbert) action. In vacuum it reads

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{12.50}$$

The energy momentum tensor $T_{\mu\nu}$ is a source term.

$$M^2 \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = T_{\mu\nu}. \tag{12.51}$$

- For photons the effective action with up to two derivatives is given by

$$\Gamma = \frac{1}{4} \int_x F_{\mu\nu} F^{\mu\nu}, \tag{12.52}$$

The field equations in vacuum are

$$\partial_\mu F^{\mu\nu} = 0. \tag{12.53}$$

Possible sources are electromagnetic currents

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{12.54}$$

Quantum fluctuation of electron leads to an extension of Maxwell equations, non-linear electrodynamics

$$\Gamma = \int_x \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + c \frac{\alpha^2}{m_e^4} \left(F^{\mu\nu} F_{\mu\nu} \right)^2 + \ldots \right\}. \tag{12.55}$$

- Why should one bother about QFT? One could use classical field theory based on Γ and simply "measure" the parameters appearing in the most general form of the field equations. Scattering amplitudes and cross section would all be given by classical (tree) approximation.

The crucial point is that QFT *predicts* constants like c. This is intimately related to the issue of renormalization.

12.9 Effective potential

The part if the effective action for scalars that involves no derivative is the effective potential U

$$\Gamma = \int_x U + \cdots \tag{12.56}$$

For the O(N)-symmetry scalar model U can only depend on

$$\rho = \frac{1}{2} \varphi_a \varphi_a. \tag{12.57}$$

The exact quantum field equation for homogeneous fields $\varphi_a(x) = \varphi_a$ are

$$\frac{\partial U}{\partial \varphi_a} = 0, \quad \frac{\partial U}{\partial \rho} \varphi_a = 0. \tag{12.58}$$

The contribution from derivative terms in Γ vanish for constant φ_a.

There is always a solution $\varphi_a = 0$, but this may not be the absolute minimum of U. For reasons of stability the solution should be at least a local minimum. For a minimum at $\rho_0 \neq 0$ one has spontaneous symmetry breaking, see next section.

Omitting fluctuations effects one has $U(\rho) = V(\rho)$. In this limit the effective potential equals the microscopic potential. Quantum fluctuation induce a map $V(\rho) \to U(\rho)$. For the classical statistic of magnets one has the typical situation illustrated in Figure 1
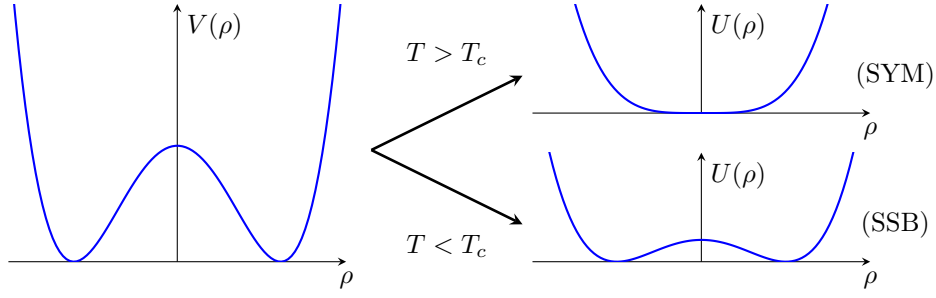
Figure 1. Illustration of spontaneous symmetry breaking

12.10 Thermodynamics

For thermal equilibrium at temperature T and chemical potential μ the partition function $Z(\mu, T)$ is the grand canonical partition function. One evaluates the effective action at its minimum $\varphi_{0,a}(x)$,

$$\Gamma_{\min}(\mu, T) = \Gamma[\varphi_{0,a}(x), \mu, T]. \tag{12.59}$$

This is proportional to the Gibbs free energy Φ_G

$$\Gamma_{\min}(\mu, T) = T^{-1}\Phi_G(\mu, T). \tag{12.60}$$

(We set $k_{\mathrm{B}} = 1$.) All the thermodynamics can be computed from the effective action. This is a good example for demonstrating that Γ contains the macrophysics for given microphysics specified by the classical action S.

For constant φ_0 one has

$$\Gamma_{\min} = \Omega_4 U(\varphi_0) = \frac{\Omega_3}{T} U(\varphi_0), \tag{12.61}$$

with

$$\Omega_4 = \int d^4x, \quad \Omega_3 = \int d^3\vec{x}. \tag{12.62}$$

One infers

$$U_0 = U(\varphi_0) = \frac{1}{\Omega_3}\Phi_G = \frac{1}{\Omega_3}(F - \mu N), \tag{12.63}$$

with F the Helmholtz free energy and N the mean of particle number. In particular, one has

$$-\frac{\partial U_0}{\partial \mu} = \frac{N}{\Omega_3} = n, \quad \text{(particle density)}, \tag{12.64}$$

$$U_0 = -p, \quad \text{(pressure)}. \tag{12.65}$$

13 Higgs mechanism and superconductivity

Once the effective action is computed or a given form is assumed, many properties of the system follow from the field equations and the correlation functions. Often one knows only the symmetries of Γ, the generic form of the effective potential, and uses an expansion in the number of derivatives $\partial_\mu \varphi$. The derivative approximation is motivated by the interest in macroscopic wave lengths, for which smooth fields often (not always!) play a dominant role. The validity of the derivative expansion may depend on the appropriate choice of macroscopic fields. In general the macroscopic fields can be more complicated than simply $\varphi(x) = \langle \chi(x) \rangle$. An example are antiferomagnets.

13.1 Symmetry of effective action

Consider a complex scalar field $\chi(x)$, with $S[\chi]$ invariant under $\chi(x) \to e^{i\alpha}\chi(x)$. This constitutes a global $U(1)$ symmetry and induces a conserved charge. The symmetry of phase transformations is the same as two-dimensional rotations, $U(1) = SO(2)$. Let us take a classical action which is $U(1)$ invariant,

$$S = \int \left(\partial^\mu \chi^* \partial_\mu \chi + \bar{m}^2 \chi^* \chi + \frac{\bar{\lambda}}{2}(\chi^* \chi)^2 \right). \tag{13.1}$$

If the functional measure $\int D\chi$ is invariant under $U(1)$ transformations, it follows that the effective action $\Gamma[\varphi]$ is also invariant under $U(1)$ transformations, where $\varphi(x) \to e^{i\alpha}\varphi(x)$. *Proof:* The partition function

$$Z[j] = \int D\chi e^{-S[\chi] + \int_x (j^*\chi + j\chi^*)}, \tag{13.2}$$

is invariant under the global $U(1)$ symmetry with $j(x) \to e^{i\alpha}j(x)$. therefore also $W[j]$ is invariant and $\varphi(x) = \frac{\delta W}{\delta j^*(x)}$ transforms as $\varphi(x) \to e^{i\alpha}\varphi(x)$, as one expects for $\varphi(x) = \langle \chi(x) \rangle$. As a consequence, one finds that $\int_x (j^*\varphi + j\varphi^*)$ is invariant. This establishes the invariance of $\Gamma = -W + \int_x (j^*\varphi + j\varphi^*)$. This generalizes to all symmetry transformations: If $S[\chi]$ is invariant under some symmetry transformation $\chi_\alpha \to \hat{s}_{\alpha\beta}\varphi_\beta$, and the functional measure $\int D\chi$ is invariant under the transformation as well, it follows that $\Gamma[\varphi]$ is invariant under $\varphi_\alpha \to \hat{s}_{\alpha\beta}\varphi_\beta$.

> The effective action has the same symmetries as the classical action.
> This holds unless there exists an "anomaly" in the functional measure.

13.2 Landau theories

In condensed matter physics, the precise microscophysics is often not known, and the transition from microphysics to macrophysics (computation of effective action) is very difficult. In addition, very different microphysical systems often give similar macroscopic phenomena. This is called universality. An example is superconductivity.

A useful approach is a guess for the effective action. From comparison with experiment and general considerations one makes an assumption on what are the relevant macroscopic degrees of freedom $\varphi(x)$, without necessarily knowing the microscopic origin. Examples are spin waves for antiferomagnetism, or a complex scalar field $\varphi(x)$ for superconductivity. The miscroscopic degrees of freedom are electrons, and the macroscopic field may represent Cooper pairs or similar composite objects. A second central ingredient is an assumption about the symmetries of the effective action. Third, one employs a derivative expansion, typically up to two derivatives $\partial_\mu \varphi$. This restricts the effective action already severely. For the example of the scalar $O(N)$ model one remains with three functions of $\rho = \varphi_a \varphi_a / 2$,

$$\Gamma = \int_x \left\{ U(\rho) + \frac{1}{2}Z(\rho)\partial^\mu \varphi_a \partial_\mu \varphi_a + \frac{1}{4}Y(\rho)\partial^\mu \rho \partial_\mu \rho \right\}. \tag{13.3}$$

Making further assumptions, as a polynomial expansion of $U(\rho)$ around its minimum and constant Z and Y, one ends with a few parameters. These parameters may be fixed by comparison with experiment. For thermodynamics, they can depend on T and μ. This approach is very successful to gain physical insight without knowledge of the microphycis. The aim of QFT is to do better by computing the free couplings or relations between them.

Often the most important quantity is the effective potential. For the scalar $O(N)$ model one may write

$$U(\rho) = m^2 \rho + \frac{\lambda}{2}\rho^2 + \dots. \tag{13.4}$$

In lowest order one has two couplings m^2 and λ. One further takes $Z(\rho) = 1$, $Y(\rho) = 0$. We will concentrate first on $N = 2$. The symmetry $SO(2) = U(1)$ is an abelian symmetry. We employ a simple complex field $\varphi(x)$, $\rho = \varphi^*\varphi$. Our "Landau theory" is

$$\Gamma = \int_x \left\{ \partial^\mu \varphi^* \partial_\mu \varphi + m^2 \varphi^* \varphi + \frac{\lambda}{2}(\varphi^*\varphi)^2 \right\}, \tag{13.5}$$

here the quartic coupling λ determines the strength of the interaction.

13.3 Spontaneous symmetry breaking

Spontaneous symmetry breaking is a key concept in condensed matter and particle physics. It extends to other branches of science as well. The basic ingredient is an effective action that has a given symmetry, while the solution of the field equation breaks this symmetry. The most important example is an effective potential with a minimum at $\varphi \neq 0$. In a euclidean setting the stable solution of the field equations is the "ground state". It typically corresponds to minimum of Γ. We include the possibility of a local minimum would corresponds to a metastable state. A positive kinetic term is minimized by a constant $\varphi(x) = \varphi_0$, $\partial_\mu \varphi = 0$. The minimum of Γ corresponds then to a minimum of the effective potential U.

There two general possibilities for $U(\rho)$:

- minimum at $\rho_0 = 0 \to \varphi_0 = 0$ is invariant under $U(1)$. "Symmetric phase".
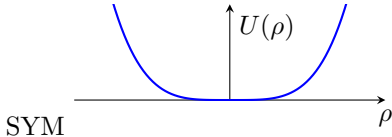
- minimum at $\rho_0 > 0$, $\varphi = \varphi_0 \neq 0$ is not invariant under $U(1)$. "phase with spontaneous symmetry breaking" or "spontaneously broken phase".
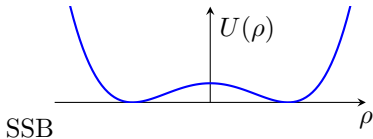
Here we have picked some (arbitrarily) particular direction for φ and rotation symmetry around $\varphi = 0$ the axes is understood. A potential with this shape is often called "mexican hat potential", since it is rotation symmetric around the axis $\varphi = 0$. The phase of φ_0 is not determined! Every phase of φ_0 is equivalent, but the ground state must pick up a fixed direction! An example is a rotationally symmetric stick under the influence of gravity. The rotation symmetric state of a vertical stick is unstable, and the ground state of a horizontal stick lying on the floor breaks rotation symmetry spontaneously. Other examples are magnets for which the expectation value of the spin in a Weiss domain singles out some direction.
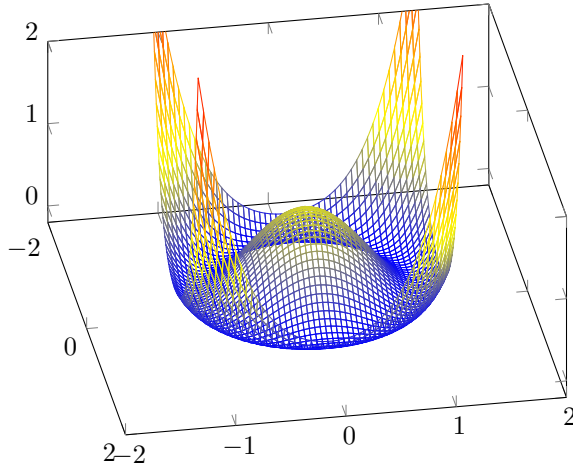
$\varphi_0 \neq 0$: *spontaneous symmetry breaking*

13.4 Goldstone bosons

Massless fields play an important role since they dominate the long-distance behaviour. Here the naming "massless" originates from a Lorentz-invariant setting, where the euclidean propagator $G = (q^2 + m^2)^{-1}$ becomes very large for $q^2 \to 0$ if $m^2 = 0$. In condensed matter physics the "massless modes" correspond to gapless excitations. The momentum dependent part in the inverse propagator $G^{-1}(q) = P(\omega, \vec{q}) + m^2$, with $P(0,0) = 0$, differs for $\vec{p} \to 0$ strongly if there is a gap, i.e. $m^2 > 0$, or if excitations are gapless ($m^2 = 0$). Sometimes the situation is even more complicated since the gap may depend on the direction of $\vec{q}$. Spontaneous breaking of a continuous global symmetry always introduces massless particles or gapless modes.

> If the ground state leads to spontaneous breaking of a continuous global symmetry, massless scalar excitations have to be present. They are called "Goldstone bosons".

This follows from symmetry arguments and is intuitively clear:

- A flat direction in potential (valley in Mexican hat potential) is dictated by invariance of U with respect to the continuous symmetry

- The vanishing of the mass term $m^2 = \frac{\partial^2 U}{\partial \varphi^2}$ in this direction, follows directly.

For real fields φ_a the mass matrix reads

$$M_{ab}^2 = \left. \frac{\partial^2 U}{\partial \varphi_a \partial \varphi_b} \right|_{\varphi_0}. \tag{13.6}$$

Its eigenvalues m_i^2 determine the particle masses. The inverse propagator in the ground state,

$$\Gamma_{ab}^{(2)}(q, q') = \left(q^2 \delta_{ab} + M_{ab}^2 \right) \delta(q - q'), \tag{13.7}$$

is easily diagonalized. In the diagonal form one has $G^{-1}(q) = q^2 + m_i^2$ and identifies m_i with the mass of a scalar particle.

For an example with $U(1)$ symmetry we employ

$$U = \frac{1}{2} \lambda \left(\rho - \rho_0 \right)^2, \quad \rho = \varphi^* \varphi = \frac{1}{2} \left(\varphi_1^2 + \varphi_2^2 \right), . \tag{13.8}$$

where the real fields φ_1, φ_2, are related to the complex field $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$. We can choose $\varphi_{1,0} = \sqrt{2}\varphi_0$, $\varphi_{2,0} = 0$, $\rho = \varphi_0^2$ since the direction of the expectation value is arbitrary. The mass matrix is easily computed with

$$\frac{\partial U}{\partial \varphi_1} = \frac{\partial U}{\partial \rho} \frac{\partial \rho}{\partial \varphi_1} = \lambda(\rho - \rho_0)\varphi_1, \tag{13.9}$$

$$\frac{\partial U}{\partial \varphi_2} = \frac{\partial U}{\partial \rho} \frac{\partial \rho}{\partial \varphi_2} = \lambda(\rho - \rho_0)\varphi_2, \tag{13.10}$$

on has

$$M_{11}^2 = \frac{\partial^2 U}{\partial \varphi_1^2} = \lambda(\rho - \rho_0) + \lambda \varphi_1^2, \tag{13.11}$$

$$M_{22}^2 = \frac{\partial^2 U}{\partial \varphi_2^2} = \lambda(\rho - \rho_0) + \lambda \varphi_2^2, \tag{13.12}$$

$$M_{12}^2 = M_{21}^2 = \frac{\partial^2 U}{\partial \varphi_1 \partial \varphi_2} = \lambda \varphi_1 \varphi_2. \tag{13.13}$$

The mass matrix for $\rho = \rho_0$,

$$M^2 = \lambda \begin{pmatrix} \varphi_1^2 & \varphi_1 \varphi_2 \\ \varphi_1 \varphi_2 & \varphi_2^2 \end{pmatrix}, \tag{13.14}$$

has a zero eigenvalue for an arbitrarily direction of φ_0. In particular, the evaluation at $\varphi_1 = \sqrt{2}\varphi_0, \varphi_2 = 0$ yields a diagonal matrix

$$M_{ab}^2 = \begin{pmatrix} 2\lambda\rho_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{13.15}$$

The radial mode φ_1 has mass $m^2 = 2\lambda\rho_0$, while the Goldstone mode φ_2 is massless, $m^2 = 0$.

The massless field is called "Goldstone boson". This Goldstone boson is the origin of super-fluidity! An example is superfluid He^4. For a non-relativistic spin zero complex field $\varphi(x)$ the $U(1)$ symmetry $\varphi \to e^{i\alpha}\varphi$ is related to particle number conservation. The field equation for φ is the "Gross-Pitaevskii equation". For the relativistic case it is a Klein-Gordon equation with interaction,

$$\partial^\mu \partial_\mu \varphi - \lambda(\varphi^* \varphi - \rho_0)\varphi = 0. \tag{13.16}$$

13.5 Higgs mechanism

Local continuous symmetry

Assume that the effective action is invariant under local phase transformations of a complex scalar field, $\varphi(x) \to e^{i\alpha(x)}\varphi(x) = \varphi'(x)$ (local $U(1)$ symmetry), where α depends on x. This is the gauge symmetry of electromagnetism! In contrast for the case of a global symmetry α is independent of x.

Covariant derivative

The derivative term does not transform homogeneously under local gauge transformations

$$\partial_\mu \varphi'(x) = \partial_\mu \left(e^{i\alpha(x)}\varphi(x) \right) = \underbrace{e^{i\alpha(x)}\partial_\mu \varphi(x)}_{\text{homogeneous}} + \underbrace{\partial_\mu \alpha(x) e^{i\alpha(x)}\varphi(x)}_{\text{inhomogeneous}}. \tag{13.17}$$

Therefore $\partial^\mu \varphi^* \partial_\mu \varphi$ is not invariant under local gauge transformations.

To fix this we introduce the *covariant derivative* for the transformation $\varphi \to e^{i\alpha(x)Q}\varphi$,

$$D_\mu \varphi = (\partial_\mu - ieQA_\mu)\varphi, \tag{13.18}$$

where we take here $Q = 1$. (For the electron one has $Q = -1$.) Simultaneously the gauge field transforms as

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha(x) \tag{13.19}$$

Then

$$(D_\mu\varphi)' = \partial_\mu\left(e^{i\alpha(x)}\varphi(x)\right) - ieA'_\mu e^{i\alpha(x)}\varphi(x) = e^{i\alpha(x)}D_\mu\varphi. \tag{13.20}$$

transforms homogeneously. Consequently, $(D_\mu\varphi)^*(D^\mu\varphi)$ is invariant! For a local continuous symmetry (gauge symmetry) we need to replace partial derivatives ∂_μ with covariant derivatives D_μ.

Landau type theory

Take the following effective action

$$\Gamma = \int_x \left\{ (D_\mu\varphi)^* D^\mu\varphi + \frac{\lambda}{2}(\varphi^*\varphi - \rho_0)^2 + \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \right\}. \tag{13.21}$$

First consider a constant background scalar field $\varphi = \varphi_0$, where φ_0 is taken real, and A_μ is the gauge field for the photon, and $F_{\mu\nu}$ is the correspondingly field strength. For this configuration of the scalar field the relevant terms in the effective action are given by

$$D_\mu\varphi = -ieA_\mu\varphi_0, \quad (D_\mu\varphi)^*D_\mu\varphi = e^2\varphi_0^2 A_\mu A^\mu,$$
$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \quad F_{\mu\nu}F^{\mu\nu} = 2\partial^\mu A^\nu \partial_\mu A_\nu - 2\partial^\nu A^\mu \partial_\mu A_\nu$$
$$\int_x F^{\mu\nu}F_{\mu\nu} = \int_x \left(-2A^\nu\partial^2 A_\nu - 2\partial_\mu A^\mu \partial_\nu A^\nu\right). \tag{13.22}$$

We choose a gauge fixing $\partial_\mu A^\mu = 0$. Then in momentum space the effective action takes the simple form

$$\Gamma = \frac{1}{2}\int_q (q^2 + 2e^2\varphi_0^2)A^\nu(-q)A_\nu(q). \tag{13.23}$$

Higgs mechanism I: photon mass

The field equation for the photon replaces Maxwell equations in vacuum. In momentum space it reads

$$(q^2 + 2e^2\varphi_0^2)A_\nu(q) = 0. \tag{13.24}$$

The solutions are plane waves with q^2 obeying

$$q^2 + m^2 = 0, \quad m^2 = 2e^2\varphi_0^2. \tag{13.25}$$

This is the dispersion relation for a massive field with an effective photon mass m,

$$q_0^2 = m^2 + \vec{q}^2 = E^2. \tag{13.26}$$

> Higgs mechanism! Photon acquires a mass through spontaneous breaking of a gauge symmetry.

Coulomb potential $\Rightarrow$ Yukawa potential

$$\frac{1}{\vec{q}^2} \leftrightarrow \frac{1}{|r|} \quad \frac{1}{\vec{q}^2 + m^2} \leftrightarrow \frac{1}{|r|}e^{-m|r|} \quad \text{3d Fourier transform.} \tag{13.27}$$

The electromagnetic interaction becomes a short range interaction!
$\Rightarrow$ superconductivity;
similar: origin of $W^\pm, Z$ mass

> superconductivity = Higgs mechanism for electromagnetism.

13.6 Superconductivity

Field equations for (relativistic) superconductor

The field equations obtained by functional variation of the effective action Γ with respect to the sclaar and gauge fields,

$$\frac{\partial \Gamma}{\partial \varphi^*(x)} = 0 : \qquad -D_\mu D^\mu \varphi + \lambda(\varphi^* \varphi - \rho_0)\varphi = 0$$

$$\frac{\partial \Gamma}{\partial A_\mu} = 0 : \qquad \partial_\nu F^{\mu\nu} + 2e^2 \varphi^* \varphi A^\mu + ie(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*) = 0. \tag{13.28}$$

This follows directly from

$$(D^\mu \varphi)^*(D_\mu \varphi) = (\partial^\mu + ieA^\mu)\varphi^*(\partial_\mu - ieA_\mu)\varphi$$
$$= \partial^\mu \varphi^* \partial_\mu \varphi + e^2 \varphi^* \varphi A^\mu A_\mu + ieA^\mu \varphi^* \partial_\mu \varphi - ieA^\mu \partial_\mu \varphi^* \varphi. \tag{13.29}$$

We define the current

$$j^\mu = -ie(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*). \tag{13.30}$$

Maxwell's equations are modified by an additional term $\sim \varphi^* \varphi A^\mu$

$$\boxed{\partial_\nu F^{\mu\nu} + 2e^2 \varphi^* \varphi A^\mu = j^\mu}. \tag{13.31}$$

Higgs mechanism I: photon "eats" Goldstone boson

Massless photons have two degrees of freedoms, namely two polarizations. Massive spin one particles on the other hand have three degrees of freedom because of three spin states. Where does this come from?

If $\varphi_0 = 0$: Goldstone boson. Spontaneous symmetry breaking of a *global* continuous symmetry produces a Goldstone boson, while breaking of a *local* continuous symmetry results in a massive photon, but no Goldstone boson.

Sometimes it is convenient to reparameterize the scalar field φ in terms of non-linear fields. One such non-linear parameterization is

$$\varphi(x) = \sigma(x)e^{i\pi(x)}, \tag{13.32}$$

where σ, ϕ are real scalar fields. In the low energy effective action of QCD, σ corresponds to the sigma resonance (but very broad and not really observed) and π is the pion.

In this parameterization, the gauge transform shifts the pion field $\pi(x) \to \pi(x) + \alpha(x)$, while $\sigma(x)$ is invariant under gauge transformations.

We will now derive the effective action for the new fields. First we rewrite derivatives of φ in terms of the non-linear fields

$$\partial_\mu \varphi = \partial_\mu \sigma e^{i\pi} + i\partial_\mu \pi \sigma e^{i\pi}$$
$$(\partial^\mu \varphi)^* \partial_\mu \varphi = (\partial^\mu \sigma e^{-i\pi} - i\partial^\mu \pi \sigma e^{-i\pi})(\partial_\mu \sigma e^{i\pi} + i\partial_\mu \pi \sigma e^{i\pi})$$
$$= (\partial^\mu \sigma - i\partial^\mu \pi \sigma)(\partial_\mu \sigma + i\partial_\mu \pi \sigma) \tag{13.33}$$
$$= \partial^\mu \sigma \partial_\mu \sigma + \sigma^2 \partial^\mu \pi \partial_\mu \pi$$

Finally the potential terms do not depend on π at all, i.e. $U(\varphi^* \varphi) = U(\sigma^2)$ For a Global $U(1)$ symmetry π is the Goldstone boson— it is a massless excitation with no potential term and only the kinetic term in the action. Meanwhile σ is the radial mode and has mass $m^2 = \left.\frac{\partial^2 U}{\partial \sigma^2}\right|_{\sigma_0}$.

We next investigate how the non-linear fields couple to a gauge field of models with a local $U(1)$ symmetry. The covariant derivatives read

$$D_\mu \varphi = \partial_\mu \varphi - ieA_\mu \sigma e^{i\pi}$$
$$(D_\mu \varphi)^* = \partial_\mu \varphi^* + ieA_\mu \sigma e^{-i\pi} \tag{13.34}$$

So the kinetic term can be rewritten as follows

$$
\begin{aligned}
(D^\mu \varphi)^*(D_\mu \varphi) &= \partial^\mu \sigma \partial_\mu \sigma + \sigma^2 \partial^\mu \pi \partial_\mu \pi + e^2 \sigma^2 A^\mu A_\mu \\
&\quad + ieA^\mu \sigma e^{-i\pi}(\partial_\mu \sigma + i\sigma \partial_\mu \pi)e^{i\pi} - ieA^\mu \sigma e^{i\pi}(\partial_\mu \sigma - i\sigma \partial_\mu \pi)e^{-i\pi} \\
&= \partial^\mu \sigma \partial_\mu \sigma + \sigma^2 \partial^\mu \pi \partial_\mu \pi + e^2 \sigma^2 A^\mu A_\mu \\
&\quad + ieA^\mu \sigma(\partial_\mu \sigma + i\sigma \partial_\mu \pi - \partial_\mu \sigma + i\sigma \partial_\mu \pi) \\
&= \partial^\mu \sigma \partial_\mu \sigma + \sigma^2 \partial^\mu \pi \partial_\mu \pi + e^2 \sigma^2 A^\mu A_\mu - 2e\sigma^2 A^\mu \partial_\mu \pi \\
&= (\partial^\mu \pi - eA^\mu)(\partial_\mu \pi - eA_\mu)\sigma^2 + \partial^\mu \sigma \partial_\mu \sigma
\end{aligned}
\tag{13.35}
$$

We can cross-check if gauge symmetry still holds. If $\pi \to \pi + \alpha$ then $\partial_\mu \pi \to \partial_\mu \pi + \partial_\mu \alpha$, while $eA_\mu \to eA_\mu + \partial_\mu \alpha$. Indeed the gauge symmetry is conserved since the pion field only appears in the combination $\partial_\mu \pi - eA_\mu$! Actually we can use the gauge symmetry to make the pion field constant, i.e. $\partial_\mu \pi = 0$. Then $\pi(x)$ disappears from the effective action and the field equations, i.e. the photon "eats" it and becomes massive and we are left with the following action

$$
\Gamma = \int_x \left\{ \partial^\mu \sigma \partial_\mu \sigma + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + e^2 \sigma^2 A^\mu A_\mu + U(\sigma^2) \right\}.
\tag{13.36}
$$

It describes a massive gauge field and a massive scalar.

13.7 Electroweak symmetry breaking

The same phenomenon happens for the spontaneous breaking of the electroweak symmetry $SU(2) \times U(1)$. The standard model invovles a scalar doublet $\varphi(x) = \begin{pmatrix} \varphi^\pm(x) \\ \varphi^0(x) \end{pmatrix}$ and gauge bosons: a triplet $\vec{W}_\mu(x)$ and a singlet $Y_\mu(x)$. The fields $W_\mu^\pm(x), Z_\mu(x), A_\mu(x)$ are linear combinations of $\vec{W}_\mu$ and Y_μ. After spontaneous symmetry breaking $SU(2) \times U(1) \to$

$$
\langle \varphi \rangle = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}.
\tag{13.37}
$$

The $W^\pm$ and Z bosons acquire mass but the photon remains massless. The scalar $\varphi^\pm$ disappears from the spectrum, much like π before.
σ : Higgs scalar $\to$ LHC!

13.8 Redundancy

Local gauge theories are "redundant" descriptions. For every generator of the gauge groups, there is one degree of freedom on which nothing depends. It is eliminated by gauge fixing. Different gauge fixings eliminates different fields. For electromagnetism, one may choose

$$
\begin{aligned}
\partial_\mu A^\mu \quad &: \quad \text{eliminates longitudinal photons} \\
\pi = 0 \quad &: \quad \text{eliminates Goldstone bosons}
\end{aligned}
\tag{13.38}
$$

Of course one can not apply both conditions simultaneously. All gauge fixings are physically equivalent even though the gauge fixed actions might look different.
The reason for a redundant description is the locality of the gauge covariant action.

13.9 Chiral symmetry breaking in QCD

In the limit of vanishing masses for the up and down quarks, $m_u = m_d = 0$, the effective action for QCD has a chiral global $SU(2)_L \times SU(2)_R$ symmetry. The kinetic term for the two light quarks takes the following form

$$
\Gamma_{\text{kin}} = i \int_x \bar{\psi} \gamma^\mu D_\mu \psi.
\tag{13.39}
$$

with

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}. \tag{13.40}$$

where u, d fields carry color indices. The covariant derivative

$$D_\mu = \partial_\mu - ig T_Z A_\mu^Z, \quad Z = 1 \dots 8. \tag{13.41}$$

it is also invariant under local $SU(3)$ (color) gauge transformation. Here A_μ^Z are the eight gluon fields and T_Z are the generators of the $SU(3)$ transformations. In the chiral basis,

$$\psi_L = \frac{1 + \gamma_5}{2} \psi, \quad \psi_R = \frac{1 - \gamma_5}{2} \psi, \quad \bar{\psi}_L = \bar{\psi} \frac{1 - \gamma_5}{2}, \quad \bar{\psi}_R = \bar{\psi} \frac{1 + \gamma_5}{2}. \tag{13.42}$$

The kinetic term involves independent species of the left-handed ψ_L and right-handed ψ_R quarks

$$\Gamma_{\text{kin}} = i \int_x \left(\bar{\psi}_L \gamma^\mu D_\mu \psi_L + \bar{\psi}_R \gamma^\mu D_\mu \psi_R \right). \tag{13.43}$$

It is invariant under independent global symmetry transformations

$$SU(2)_L : \quad \psi_L \to e^{\frac{i}{2} \vec{\alpha} \vec{\tau}} \psi_L, \qquad \bar{\psi}_L \to \bar{\psi}_L e^{-\frac{i}{2} \vec{\alpha} \vec{\tau}} \tag{13.44}$$

$$SU(2)_R : \quad \psi_R \to e^{\frac{i}{2} \vec{\beta} \vec{\tau}} \psi_R, \qquad \bar{\psi}_R \to \bar{\psi}_R e^{-\frac{i}{2} \vec{\beta} \vec{\tau}}. \tag{13.45}$$

This "chiral symmetry" is violated by quark mass terms. In the massless limit $m_u, m_d \to 0$ this symmetry is restored. The classical action is invariant, since both the covariant kinetic term for the quarks and the gluon kinetic term are invariant. As a consequence, the effective action Γ is also invariant.

At low energies below $500\,\text{MeV}$ the gluons no longer play a dynamic role. On the other hand, experiment tells us that there are mesons as bound states. A Landau-type theory at low energies may involve quarks and mesons. It has to be invariant under the chiral $SU(2)_L \times SU(2)_R$ symmetry in the limit of massless quarks. An invariant coupling between mesons and quarks, which does not involve derivatives, has the form

$$\Gamma_y = y \int_x \left(\bar{\psi}_R \phi \psi_L - \bar{\psi}_L \phi^\dagger \psi_R \right), \tag{13.46}$$

where ϕ is a complex 2×2 matrix in the $(2, 2)$ representation of $SU(2)_L \times SU(2)_R$

$$\begin{aligned} \phi &\to e^{\frac{i \vec{\beta} \vec{\tau}}{2}} \phi e^{-\frac{i \vec{\alpha} \vec{\tau}}{2}}, \\ \phi^\dagger &\to e^{\frac{i \vec{\alpha} \vec{\tau}}{2}} \phi^\dagger e^{-\frac{i \vec{\beta} \vec{\tau}}{2}}. \end{aligned} \tag{13.47}$$

One can also have a kinetic and a potential term for ϕ. For example

$$\Gamma = \int_x \left\{ \text{Tr}\, \partial^\mu \phi^\dagger \partial_\mu \phi + U \left(\text{Tr}\, (\phi^\dagger \phi), \text{Tr}\, (\phi^\dagger \phi \phi^\dagger \phi) \right) \right\}. \tag{13.48}$$

If the potential minimum occurs for the diagonal

$$\phi = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \tag{13.49}$$

where σ_0 is real, the chiral symmetry is spontaneously broken. What remains unbroken is the isospin symmetry $SU(2)_V$ given by $\vec{\alpha} = \vec{\beta}$. The quarks acquire a "constituent" mass from chiral symmetry breaking

$$L_y \to \int_x y \sigma_0 \bar{\psi} \gamma^5 \psi. \tag{13.50}$$

that is $\tilde{m}_q = y\sigma_0$. This is the basis for the non-relativistic quark model. The nucleons have a mass of $m_N \approx 3\tilde{m}_q$. Note that the mass obtained through chiral symmetry breaking is much larger than the "current masses" m_u, m_d obtained through the Higgs mechanism, which is often (incorrectly) attributed with the generation of all mass

There are three Goldstone bosons from the spontaneous chiral symmetry breaking, namely the pions $\pi^\pm, \pi^0$. In the limit $m_u, m_d \to 0$, they would be exactly massless. For non-zero current quark masses they aquire mass and $m_\pi \sim m_u, m_d$.

14 Saddle point approximation and perturbation theory

In this chapter we will start a computation of the effective action Γ. We assume that interactions are small, and that some type of perturbation expansion in the small couplings should be possible. We recall that in the absence of interactions the microscopic action $S[\chi]$ is quadratic in χ, and we have shown that $\Gamma[\varphi] = S[\varphi]$.

14.1 Saddle point approximation

Background field identity

Our starting point is the functional integral representation of Γ (background field identity)

$$\Gamma[\varphi] = -\ln \int D\chi' \exp\left\{ -S[\varphi + \chi'] + \int_x \frac{\delta\Gamma}{\delta\varphi(x)} \chi'(x) \right\}. \tag{14.1}$$

where φ is the background field, χ' is the quantum fluctuation field. We take out the classical contribution

$$\Gamma[\varphi] = \underbrace{S[\varphi]}_{\text{classical contribution}} \underbrace{-\ln \int D\chi' \exp\left\{ -(S[\varphi + \chi'] - S[\varphi]) + \int_x \frac{\delta\Gamma}{\delta\varphi(x)} \chi'(x) \right\}}_{\text{fluctuation contribution}}. \tag{14.2}$$

This is similar to the free energy: there are energy and entropy contributions. Every term beyond the classical action is called the fluctuation contribution or "loop contribution".

$$\Gamma[\varphi] = S[\varphi] + \Gamma_l[\varphi]. \tag{14.3}$$

Here Γ_l accounts for all loops in perturbation theory. The structure of our expression for $\Gamma_l[\varphi]$ is an implicit equation since $\frac{\delta\Gamma}{\delta\varphi}$ appears on the r.h.s.

Saddle point expansion

We expand $S[\varphi + \chi']$ around $\chi' = 0$

$$S[\varphi + \chi'] = S[\varphi] + \int_x \frac{\delta S}{\delta\varphi(x)} \chi'(x) + \frac{1}{2} \int_{x,y} S^{(2)}(x,y) \chi'(x) \chi'(y) + \dots. \tag{14.4}$$

The piece $\frac{\delta S}{\delta\varphi}$ cancels against the classical term in $\frac{\delta\Gamma}{\delta\varphi}$

$$\Gamma_l[\varphi] = -\ln \int D\chi' \exp\left\{ -\frac{1}{2} \int_{x,y} S^{(2)}(x,y) \chi'(x) \chi'(y) + \int_x \frac{\delta\Gamma_l}{\delta\varphi(x)} \chi'(x) + \dots \right\}. \tag{14.5}$$

We can now proceed to an iterative solution. The lowest order is the one loop contribution. In lowest order one neglects $\frac{\delta\Gamma_l}{\delta\varphi(x)}$ and higher order terms in the expansion, like $S^{(3)}$ etc. What remains is a Gaussian integral

$$\begin{aligned}
\Gamma_{1l} &= -\ln \int D\chi' \exp\left\{ -\frac{1}{2} \int_{x,y} S^{(2)}(x,y) \chi'(x) \chi'(y) \right\} \\
&= -\ln\left(\det(S^{(2)})^{-\frac{1}{2}} \cdot \text{const} \right) = \frac{1}{2} \text{Tr} \ln S^{(2)} + \text{const}.
\end{aligned} \tag{14.6}$$

Here we used the identity $\ln \det A = \text{Tr} \ln A$. Omitting the constant, one ends with the one loop approximation for Γ:

$$\boxed{\Gamma_{1l}[\varphi] = \frac{1}{2} \text{Tr} \ln S^{(2)}[\varphi].} \tag{14.7}$$

Some remarks

- The saddle point approximation requires $S^{(2)}$ to be positive semidefinite. Then the Gaussian integral for a euclidean functional integral is well defined.

- The second functional derivative of the classical action $S^{(2)}$ is the inverse of the classical propagator *in the presence* of a background field $\varphi(x)$.

- In momentum space the 1-loop effective action is

$$\Gamma_{1l} = \frac{1}{2} \underbrace{\int \frac{d^d q}{(2\pi)^d} \int \frac{d^d q'}{(2\pi)^d} (2\pi)^d \delta^{(d)}(q - q')}_{\text{Tr}} (\ln S^{(2)}[\varphi])(q, q'). \tag{14.8}$$

For example take $\varphi(x) = \varphi$ to be a constant, then

$$S^{(2)}[\varphi] = G^{-1}(\varphi, q)\delta(q - q'). \tag{14.9}$$

And then

$$\Gamma_{1l} = \frac{1}{2}\Omega_d \int_q \ln G^{-1}(\varphi, q). \tag{14.10}$$

Here Ω_d is the d-dimensional volume, which is finite on the torus T^d.

14.2 One loop effective potential

We employ the generic form of Γ

$$\Gamma = \int_x U + \text{derivative terms.} \tag{14.11}$$

The effective potential U depends on scalar fields and involves no derivatives. For the computation of U one evaluates Γ for homogeneous scalar fields. For $\partial_\mu \varphi = 0$ the derivate terms do not contribute.

We investigate first a real scalar field with classical action

$$S = \int_x \left\{ \frac{1}{2}\partial^\mu \chi \partial_\mu \chi + V \right\}. \tag{14.12}$$

with

$$V(\chi) = \frac{\bar{m}^2}{2}\chi^2 + \frac{\bar{\lambda}}{8}\chi^4. \tag{14.13}$$

With

$$\frac{\partial V}{\partial \chi} = \bar{m}^2 \chi + \frac{\bar{\lambda}}{2}\chi^3, \quad \frac{\partial^2 V}{\partial \chi^2} = \bar{m}^2 + \frac{3}{2}\bar{\lambda}\chi^2. \tag{14.14}$$

one finds in momentum space for $S^{(2)}$, evaluated at $\chi = \varphi = \text{const}$,

$$S^{(2)}(\varphi, q, q') = (q^2 + \bar{m}^2 + \frac{3}{2}\bar{\lambda}\varphi^2)\delta(q - q'). \tag{14.15}$$

and therefore the inverse propagator function is

$$G^{-1}(q) = q^2 + \bar{m}^2 + \frac{3}{2}\bar{\lambda}\varphi^2. \tag{14.16}$$

The propagator depends now on the microscopic field φ! This yields the one loop contribution to the effective potential

$$\Gamma_{1l} = \frac{\Omega_d}{2} \int_q \ln(q^2 + \bar{m}^2 + 3\bar{\lambda}\rho). \tag{14.17}$$

or

$$U_{1l} = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \ln(q^2 + \bar{m}^2 + 3\bar{\lambda}\rho). \tag{14.18}$$

with $\rho = \frac{1}{2}\varphi^2$. Writing the momentum cutoff Λ explicitly one has

$$U = \bar{m}_\Lambda^2 \rho + \frac{1}{2}\bar{\lambda}_\Lambda \rho^2 + \frac{1}{2} \int_{q, q^2 < \Lambda^2} \ln(q^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda \rho). \tag{14.19}$$

14.3 Ultraviolet and infrared divergences

At the end we want to take $\Lambda \to \infty$ and $\Omega \to \infty$. The questions is if the momentum integrals remain finite in these limits. We have implicitly assumed a continuous momentum integral $\int_q$. This is allowed only if the infrared limit $L \to \infty$ exists, with L the circumference of the torus in every space coordinate, $\Omega_d = L^d$.

We consider a polynomial expansion of U, with $U' = \frac{\partial U}{\partial \rho}$, and define

$$
\begin{aligned}
m^2 &= U'(0) = \bar{m}_\Lambda^2 + U'_{1l}(0) \\
\lambda &= U'(0) = \bar{\lambda}_\Lambda + U''_{1l}(0) \\
\nu &= U'''(0) = U'''_{1l}(0).
\end{aligned}
\tag{14.20}
$$

Note that ν is a six-point vertex not present in the classical actions. This is a "quantum vertex".

Feynman graphs

We want to evaluate

$$
\Delta m^2 = \frac{\partial}{\partial \rho} U_{1l}|_{\rho=0}.
\tag{14.21}
$$

For this purpose we take a derivative of U_{1l} with respect to ρ :

$$
\frac{\partial U_{1l}}{\partial \rho} = \frac{3\bar{\lambda}_\Lambda}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda \rho}.
\tag{14.22}
$$

This corresponds to a one-loop Feynman diagram.

$$
\frac{1}{q^2+\bar{m}_\Lambda^2+3\bar{\lambda}_\Lambda\rho} \quad .
\tag{14.23}
$$

Lines are given by the classical propagator $G = (q^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda \rho)^{-1}$, and the point denotes the classical vertex $\frac{\partial^4 V_{cl}}{\partial \chi^4} = 3\bar{\lambda}_\Lambda$. As usual, a closed line involves a trace, i.e. a momentum integral and a sum over contracted indices.

For $\rho = 0$ one finds

$$
\Delta m^2 = \frac{3\bar{\lambda}_\Lambda}{2} \int_q \frac{1}{q^2 + \bar{m}_\Lambda^2}.
\tag{14.24}
$$

A similar procedure for higher couplings takes higher ρ-derivatives,

$$
\frac{\partial^2 U_{1l}}{\partial \rho^2} = -\frac{9\bar{\lambda}_\Lambda^2}{2} \int_q \frac{1}{(q^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda \rho)^2} =
\tag{14.25}
$$

$$
\frac{\partial^3 U_{1l}}{\partial \rho^3} = 27\bar{\lambda}_\Lambda^3 \int_q \frac{1}{(q^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda \rho)^3} =
\tag{14.26}
$$

Evaluating them at $\rho = 0$, we have:

$$\Delta m^2 = \frac{3\bar{\lambda}_\Lambda}{2} \int_q \frac{1}{q^2 + \bar{m}_\Lambda^2}, \tag{14.27}$$

$$\Delta\lambda = -\frac{9\bar{\lambda}_\Lambda^2}{2} \int_q \frac{1}{(q^2 + \bar{m}_\Lambda^2)^2}, \tag{14.28}$$

$$\Delta\nu = 27\bar{\lambda}_\Lambda^3 \int_q \frac{1}{(q^2 + \bar{m}_\Lambda^2)^3}. \tag{14.29}$$

Regularization

For the UV-regularization we take here $q^2 < \Lambda^2$ in order to have finite integrals. The evaluation of the integrals is simple. We employ

$$x = q^2, \quad \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} f(q^2) = v_d \int_0^{\Lambda^2} dx\, x^{\frac{d}{2}-1} f(x), \tag{14.30}$$

where v_d is given by

$$v_d = \frac{1}{2^{d+1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}. \tag{14.31}$$

with

$$v_4 = \frac{1}{32\pi^2}, \quad v_3 = \frac{1}{8\pi^2}, \quad v_2 = \frac{1}{8\pi}. \tag{14.32}$$

The integrals can be computed for arbitrary and continuous dimension d. This allows for expansions close to the integer d, as $d = 4 - \epsilon$ expansions, or $d = 2 + \epsilon$ expansions. Such expansions are the basis of dimensional regularization.

14.4 One loop effective potential for $d = 4$

For the loop correction to the mass term, we need the integral

$$\int_q \frac{1}{q^2 + \bar{m}_\Lambda^2} = \frac{1}{16\pi^2} \int_0^{\Lambda^2} dx \frac{x}{x + \bar{m}_\Lambda^2} = \frac{1}{16\pi^2} \left(\Lambda^2 - \int_0^{\Lambda^2} dx \frac{\bar{m}_\Lambda^2}{x + \bar{m}_\Lambda^2} \right)$$
$$= \frac{1}{16\pi^2} \left(\Lambda^2 - \bar{m}_\Lambda^2 \ln \frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} \right), \tag{14.33}$$

resulting in

$$\Delta m^2 = \frac{3\bar{\lambda}_\Lambda}{32\pi^2} \left(\Lambda^2 - \bar{m}_\Lambda^2 \ln \frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} \right) \tag{14.34}$$

One concludes that bosonic fluctuations increase the mass term!
For the quartic coupling we can use a simple identity:

$$\int_q \frac{1}{(q^2 + \bar{m}_\Lambda^2)^2} = -\frac{\partial}{\partial \bar{m}_\Lambda^2} \int_q \frac{1}{q^2 + \bar{m}_\Lambda^2} \tag{14.35}$$

$$= \frac{1}{16\pi^2} \left(\ln \frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} + \bar{m}_\Lambda^2 \frac{\partial}{\partial \bar{m}_\Lambda^2} \ln \frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} \right) \tag{14.36}$$

$$= \frac{1}{16\pi^2} \left(\ln \frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} + \frac{\bar{m}_\Lambda^2}{\Lambda^2 + \bar{m}_\Lambda^2} - 1 \right). \tag{14.37}$$

Neglecting terms $\sim \Lambda^{-n}$ with $n > 0$ yields

$$\int_q \frac{1}{(q^2 + \bar{m}_\Lambda^2)^2} = \frac{1}{16\pi^2} \left(\ln \frac{\Lambda^2}{\bar{m}_\Lambda^2} - 1 \right), \tag{14.38}$$

and similarly

$$\int_q \frac{1}{(q^2 + \bar{m}_\Lambda^2)^3} = -\frac{1}{2}\frac{\partial}{\partial\bar{m}_\Lambda^2}\int_q \frac{1}{(q^2 + \bar{m}_\Lambda^2)^2} = \frac{1}{32\pi^2\bar{m}_\Lambda^2}. \tag{14.39}$$

We find the one loop correction to the quartic coupling of $\Delta\lambda$ and the six-point function ν

$$\Delta\lambda = -\frac{9\bar{\lambda}_\Lambda^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{\bar{m}_\Lambda^2} - 1\right) \tag{14.40}$$

$$\nu = \frac{27\lambda_\Lambda^3}{32\pi^2}\frac{1}{\bar{m}_\Lambda^2}. \tag{14.41}$$

Fluctuation effects

The one loop corrections increase the mass, reduce the 4-vertex coupling strength and generates new 6-point vertex,

$$m^2 > \bar{m}_\Lambda^2, \quad \lambda < \bar{\lambda}_\Lambda, \quad \nu > 0. \tag{14.42}$$

Note that $\Delta m^2 \sim \Lambda^2$ and $\Delta\lambda \sim -\ln\frac{\Lambda^2}{\bar{m}_\Lambda^2}$ are divergent for $\Lambda \to \infty$, while ν is independent of the cut-off.

Impact of fluctuation effects on mass term

Since the mass correction is positive, $\Delta m^2 > 0$, it is possible to have a negative bare mass term $\bar{m}_\Lambda^2 < 0$ but a positive renormalized mass term $m^2 > 0$. Then the system is in the SYM-phase, even for $\bar{m}_\Lambda^2 < 0$. This happens for Ising type models. One has local order but not global order. Strong fluctuation effects destroy the order!

In thermal equilibrium for $T \neq 0$ the fluctuation correction Δm^2 depends on T. This can lead to a phase transition as a function of temperature. The fluctuation contribution $\Delta m^2(T)$ is monotonically increasing with T.

The phase transition occurs at T_c. At the critical temperature the mass term vanishes $m^2(T_c) = 0$. For fermions the sign of the fluctuation effects is opposite. The contribution from fermion fluctuations amounts to $\Delta m^2 < 0$. For the standard model this may lead to top-quark induced electroweak symmetry breaking.

The dependence of $\Delta m^2 \sim \Lambda^2$ on Λ is called "quadratic divergence'. In the standard model, one has $m \approx 100\,\text{GeV}$, while grand unification or gravity scales are $\Lambda \sim 10^{15}\,\text{GeV}$ and $\Lambda \sim 10^{18}\,\text{GeV}$ respectively. With

$$m^2 = \bar{m}_\Lambda^2 + g^2\Lambda^2. \tag{14.43}$$

a small mass does not seem natural.

Separation of scales

$\Lambda \gg \bar{m}_\Lambda^2, m^2$ ($\bar{\lambda}_\Lambda \ll 1$:

- $\Delta\lambda$ logarithmically divergent for $\Lambda \to \infty$.

- ν independent of Λ

- The different quantities are dominated by rather different momentum ranges in the loop integral:

- Δm^2 dominated by modes $q^2 \approx \Lambda^2$ (UV-dominated, microphysics)

- $\Delta\lambda$ all modes contribute

- ν dominated by modes with $q^2 \approx \bar{m}_\Lambda^2$ (IR-dominated, independent of microphysics)

Predictivity of QFT

The φ^6 vertex ν can be predicted! One may add to the microscopic model a χ^6 coupling. By dimension counting it is of the form $\bar{\nu}_\Lambda \sim \frac{1}{\Lambda^2}$. The macroscopic coupling ν is dominated by the fluctuation contribution, and that $\bar{\nu}_\Lambda$ plays no role for $\Lambda \to \infty$. We may add to the classical potential a term

$$V_6 = \frac{1}{48} \frac{\gamma}{\Lambda^2} \varphi^6. \tag{14.44}$$

with dimensionless coupling γ. This yields for the ratio of the fluctuation contribution and the classical contribution

$$\frac{\text{fluctuation}}{\text{classical}} = \frac{27\bar{\lambda}^3}{32\pi^2} \frac{6}{\gamma} \frac{\Lambda^2}{\bar{m}^2}. \tag{14.45}$$

One infers that the fluctuation contribution dominates for large $\Lambda/\bar{m}$

One loop effective potential for $d = 3$

The computations for the classical statistics in three dimensions are similar. One now has

$$\Delta m^2 = \frac{3\bar{\lambda}_\Lambda}{8\pi^2} \int_0^{\Lambda^2} dx x^{\frac{1}{2}} \frac{1}{x + \bar{m}_\Lambda^2}. \tag{14.46}$$

and

$$\Delta\lambda = -\frac{9\bar{\lambda}_\Lambda^2}{8\pi^2} \int_0^{\Lambda^2} dx x^{\frac{1}{2}} \frac{1}{(x + \bar{m}_\Lambda^2)^2}. \tag{14.47}$$

Inspecting the momentum integrals, one finds that

$$\Delta m^2 \sim \Lambda \quad \text{UV dominated}, \tag{14.48}$$

$$\Delta\lambda \quad \text{IR dominated!}, \tag{14.49}$$

$$\Delta\lambda = -\frac{9\bar{\lambda}_\Lambda^2}{16\pi^2} \frac{1}{\sqrt{\bar{m}_\Lambda^2}}. \tag{14.50}$$

For $\bar{m}_\Lambda \to 0$ one observes an IR-divergence for $\Delta\lambda$. This is a major difficulty for perturbative calculations for $d = 3$ near phase transitions!

14.5 Perturbative renormalization

The strategy is to replace microscopic parameters $\bar{m}_\Lambda^2, \bar{\lambda}_\Lambda$ by macroscopic quantities m^2, λ. The microscopic or "bare" parameters $\bar{m}_\Lambda^2, \bar{\lambda}_\Lambda$ are not known. In contrast, the renormalized parameters m^2, λ can be determined by measurements, since they enter directly in the computation of cross sections etc.

For perturbation theory, one expands in the small renormalized coupling λ. In lowest order, one keeps only the corrections linear in $\lambda, \bar{\lambda}_\Lambda$. We concentrate here on four dimensions, $d = 4$.

(i) mass renormalization

$$m^2 = \bar{m}_\Lambda^2 + \frac{3\bar{\lambda}_\Lambda}{32\pi^2}\Lambda^2 - \frac{3\bar{\lambda}_\Lambda}{32\pi^2} \ln\frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2}, \tag{14.51}$$

$$\bar{m}_\Lambda^2 = m^2 - \frac{3\lambda}{32\pi^2}\Lambda^2 + \frac{3\lambda}{32\pi^2} \ln\frac{\Lambda^2 + m^2}{m^2} + \mathcal{O}(\lambda^2). \tag{14.52}$$

(ii) coupling renormalization

$$\lambda = \bar{\lambda}_\Lambda - \frac{9\bar{\lambda}_\Lambda^2}{32\pi^2}\left(\ln\frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} - 1\right) \approx \bar{\lambda}_\Lambda\left(1 - \frac{9}{32\pi^2}\lambda \ln\frac{\Lambda^2}{m^2}\right). \tag{14.53}$$

$$\bar{\lambda}_\Lambda = \frac{\lambda}{1 - \frac{9}{32\pi^2}\lambda \ln \frac{\Lambda^2}{m^2}}. \tag{14.54}$$
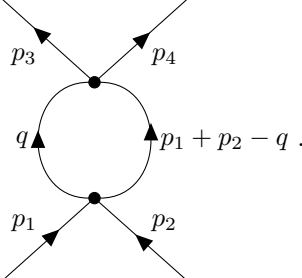
These relations express the bare parameters in terms of the renormalized couplings. For fixed λ the bare quartic coupling. $\bar{\lambda}_\Lambda$ diverges at a "Landau pole" when $\ln \frac{\Lambda^2}{m^2} = \frac{32\pi^2}{9\lambda}$. This indicates a limit of validity of the theory. Arbitrary high Λ are not possible for given λ !

In the opposite direction for fixed λ_Λ, Λ and $m^2 \to 0$ one has $\lambda \to 0$. This is called "triviality of φ^4 theory". For a given finite Λ one finds an upper bound for λ and therefore the mass, of the higgs boson in the standard model. For Λ a few TeV the bound is $m_H \leq 500\,\text{GeV}$.

(iii) The φ^6 coupling $\nu = \frac{27\lambda^3}{32\pi^2 m^2}$ is fixed in terms of λ and m^2. There is no additional free parameter. The theory is specified in terms of only two renormalized parameters m^2 and λ.

(iv) Momentum dependence of φ^4-vertex. The 1PI four point vertex depends on the momentum of of the incoming or outgoing particles

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = 3\tilde{\lambda}(p_1, p_2, p_3)\delta(p_1 + p_2 - p_3 - p_4). \tag{14.55}$$

The Feynman graph for the fluctuation contribution is given by

$$\tag{14.56}$$

In this notation, the quartic coupling λ computed previously corresponds to $\tilde{\lambda}(0,0,0) = \lambda$. The difference

$$\lambda(p_1, p_2, p_3) - \lambda(0,0,0) \tag{14.57}$$

is then found to be IR-dominated. It does not depend on Λ. In perturbation theory it is computable and found $\sim \lambda^2$. Cut-off corrections are $\sim \frac{1}{\Lambda^2}$ and vanish for $\Lambda \to \infty$. The difference is therefore predictable! The whole momentum dependence of $\Gamma^{(4)}$ and the associated cross sections are predicted in terms of the two parameters λ and m^2.

Lesson: IR-dominated quantities are predictable! Only a finite number of renormalized couplings is needed to be specified! Same strategy for QED: renormalized coupling e.

(v) Renormalizable theories miracle. Once all quantities are expressed in terms of renormalized couplings, all momentum integrals become ultraviolet finite for $\Lambda \to \infty$, even for higher loops. Theories with this property are called renormalizable theories. For renormalizable theories, the limit $\Lambda \to \infty$ can be taken! Similar for other regularizations, e.g. $\epsilon \to 0$ for dimensional regularizaton. Typical renormalized parameters are

$$\varphi^4\text{-theory} \quad m^2, \lambda \tag{14.58}$$
$$\text{QED} \quad m_e^2, e \tag{14.59}$$
$$\text{QCD} \quad m_q^2, g_s. \tag{14.60}$$

All other quantities are predictable!

(vi) Example QED, anomalous magnetic momentum of muon.

- Dirac equation. magnetic moment in Bohr units:

$$g = 2. \tag{14.61}$$

- Quantum corrections $g - 2 \neq 0$

$$+ \dots \tag{14.62}$$

$$\mathcal{L} = c\bar{\psi}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\psi. \tag{14.63}$$

Consistent with all all symmetries. Dimension of c is mass^{-1}, in unknown misroscopic theory $c \sim 1/\Lambda$.

- Quantum corrections to c are IR-dominated, cutoff dependence $\sim 1/\Lambda$.
- For $\Lambda \to \infty$ $g - 2$ is predictable! QED has been computed through five loops.
- Observation for muon magnetic moment differs from pure QED (e, μ, τ) at the level 10^{-7}.[1]

$$\left.\frac{g-2}{2}\right|_{\text{exp}} = 11659209.1(5.4)(3.3) \times 10^{-10}, \tag{14.64}$$

$$\left.\frac{g-2}{2}\right|_{\text{QED}} = 11658471.895(0.008) \times 10^{-10}. \tag{14.65}$$

The difference is

$$\frac{1}{2}(g_{\text{exp}} - g_{\text{QED}}) = 738 \times 10^{-10}. \tag{14.66}$$

Pure QED (without hadrons) is excluded at $> 100\sigma$ level. Hadronic ffects are

$$\Delta_h \frac{g-2}{2} = 693.1(3.3)(0.7) \times 10^{-10}. \tag{14.67}$$

Remaining discrepancy $\to$ new physics needed at cutoff scale $\Lambda \approx 100\,\text{GeV}$.

- Weak effects contribute to $g - 2$. Electroweak effects $(W^\pm, Z$ and H bosons)

$$\Delta_{EW} \frac{g-2}{2} = 15.36(0.1) \times 10^{-10}. \tag{14.68}$$

After this is accounted for, one is left with an interesting, but not conclusive 3.5σ deviation. Could it be new physics below TeV scale?

(vii) IR-dominated terms in Γ : "irrelevant operators". Keys for predictability in QFT if microphysics is not precisely known

- symmetries
- fluctuation domination of "irrelevant operators"

Additional predictivity if microphysics is known.

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[1]M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018).

15 Quantum field theory in thermal equilibrium

15.1 Grand canonical partition function

Recall from statistical mechanis that the grand canonical partition function is given by

$$Z(T, \mu, V) = \text{Tr} \left\{ e^{-\frac{1}{T}(H - \mu N)} \right\} \tag{15.1}$$

where T is the temperature, μ is the chemical potential, H is the Hamiltonian and N is the particle number operator. Related to the partition function is the grand canonical potential $\Omega(T, \mu, \mu, V)$ through the relation

$$Z = e^{-\Omega/T}. \tag{15.2}$$

Morevoer, one has with entropy S and pressure p

$$\Omega = E - TS - \mu N = -pV. \tag{15.3}$$

and the differential

$$d\Omega = -SdT - Nd\mu = -pdV. \tag{15.4}$$

This shows that thermodynamic quantities can be directly derived from Z or Ω, for example

$$S = -\left.\frac{\partial \Omega}{\partial T}\right|_{\mu, V} \qquad N = -\left.\frac{\partial \Omega}{\partial \mu}\right|_{T, V}. \tag{15.5}$$

Other observables follow from Legendre transforms, for example

$$E = \Omega + TS + \mu N = \Omega - T\frac{\partial \Omega}{\partial T} - \mu\frac{\partial \Omega}{\partial \mu}. \tag{15.6}$$

Exercise 15.1. In a homogeneous situation, it is convenient to work with pressure $p(T, \mu)$ as a thermodynamic potential. Show that for constant volume V one has

$$Z(T, \mu) = \exp\left(\frac{Vp(T, \mu)}{T}\right) = \exp\left[\int_0^{\frac{1}{T}} d\tau \int d^3x \; p(T, \mu)\right], \tag{15.7}$$

and that

$$dp = sdT + nd\mu = \left(\frac{S}{V}\right) dT + \left(\frac{N}{V}\right) d\mu. \tag{15.8}$$

Derive expresions for entropy density s, particle density n and energy density ϵ from $p(T, \mu)$ and its derivatives.

Exercise 15.2. **(advanced):** Derive also expresions for the heat capacity densities

$$c_v = \frac{C_v}{V} = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{V,N} \qquad c_p = \frac{C_p}{V} = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{p,N}. \tag{15.9}$$

and the thermal expansion coefficient

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p,V}. \tag{15.10}$$

in terms of $p(T, \mu)$ and its derivatives.

From these considerations, it becomes clear that it would be very useful to have a method to calculate the grand canonical partition function for matter described by a quantum field theory.

Note that equation (15.1) resembles a transition amplitude in a Euclidean quantum field theory. More specifically, we have previously derived functional integral representations for transition amplitudes between initial and final states. For *real* Minkowski time, they are of the form

$$\langle \phi_f | \, e^{-i(t_f - t_i)H} \, | \phi_i \rangle = \int \mathcal{D}\phi \; e^{iS_M[\phi]} \tag{15.11}$$

where the right hand side involves the Minkowski action S_M

$$S_M[\phi] = - \int_{t_i}^{t_f} dt \int d^3x \; \left\{ \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 \right\} \tag{15.12}$$

for the example of a complex scalar field. The functional integral has the boundary conditions

$$\phi(t_i, \vec{x}) = \phi_i(\vec{x}), \qquad \phi(t_f, \vec{x}) = \phi_f(\vec{x}). \tag{15.13}$$

They parameterise the initial and final state.

In order to use equation (15.11) to calculate the partition function in equation (15.1), we need to do a few things.

(i) To match $\exp(-i(t_f - t_i)H)$ with $\exp(-H/T)$, we must choose

$$t_f - t_i = -\frac{i}{T} \tag{15.14}$$

to be *imaginary*. For example, we can choose $t_i = 0$ and $t_f = -i/T = -i\beta$.

(ii) Taking the trace means to identify initial and final states (because the trace is cyclic) and to sum over them. In other words, we need to set

$$\phi_i(\vec{x}) = \phi_f(\vec{x}) = \phi(0, \vec{x}) = \phi(-i/T, \vec{x}) \tag{15.15}$$

and include a (functional) integral over $\phi(0, \vec{x})$. This leads to a functional integral without boundaries but with the periodic identification

$$\phi(0, \vec{x}) = \phi(-i/T, \vec{x}). \tag{15.16}$$

The imaginary time dimension is periodic, the geometry is like the one of a cylinder with times $t = 0$ and $t = -i/T$ identified.
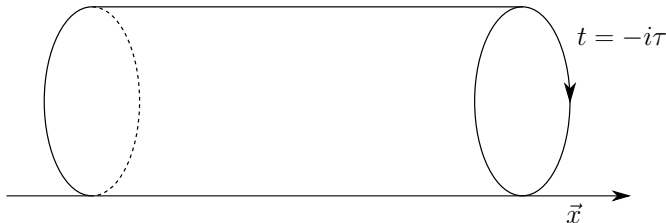

Figure 2. Imaginary time direction in a compactified dimension

(iii) It is convenient to introduce the imaginary or Euclidean time τ with $t = -i\tau$ where τ gets integrated from 0 to $\beta = 1/T$. Note that $dt = -id\tau$. Also $iS_M[\phi] = -S_E[\phi]$ with

$$S_E[\phi] = \int_0^{1/T} d\tau \int d^3x \ \left\{ \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi + \frac{\lambda}{2}(\phi^*\phi)^2 \right\} \tag{15.17}$$

and

$$\partial_\mu \phi^* \partial^\mu \phi = \frac{\partial}{\partial\tau}\phi^* \frac{\partial}{\partial\tau}\phi + \nabla\phi^* \nabla\phi \tag{15.18}$$

(iv) We also need to introduce the chemical potential term. One can see this as a modification of the Hamiltonian or of the action. To introduce it properly, we follow the following recipe. Let us first go back to real time and let us couple the theory to an external gauge field $A_\mu(x)$. The action becomes

$$S_M[\phi] = -\int_{t_i}^{t_f} dt \int d^3x \ \left\{ (\partial_\mu + iA_\mu)\phi^* (\partial^\mu - iA_\mu)\phi + m^2 \phi^* \phi + \frac{\lambda}{2}(\phi^*\phi)^2 \right\}. \tag{15.19}$$

The global U(1) symmetry $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$ has now been extended to a local symmetry where the external gauge field is transformed as well:

$$\phi \to e^{i\alpha(x)}\phi, \qquad \phi^* \to e^{-i\alpha(x)}\phi^*$$
$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \tag{15.20}$$

The conserved current on the microscopic or classical level then follows from

$$\frac{\delta S_M}{\delta A_\mu(x)} = J^\mu(x) = -i\phi^*(x)\partial^\mu \phi(x) + i\phi(x)\partial^\mu \phi^*(x) \tag{15.21}$$

The conserved particle number is

$$N = \int d^3x \ J^0(x). \tag{15.22}$$

If we take the chemical potential to be the time component of an external gauge field, it will automatically couple to the conserved number density. One may check that signs and factors of i indeed come out correctly. After analytic continuation to Euclidean time, one obtains

$$\frac{\partial}{\partial t} - iA_0 \to \frac{\partial}{\partial(-i\tau)} - iA_0 = i\left(\frac{\partial}{\partial\tau} - A_0\right) \to i\left(\frac{\partial}{\partial\tau} - \mu\right) \tag{15.23}$$

and the Euclidean action becomes

$$S_E[\phi] = \int_0^{1/T} d\tau \int d^3x \ \left\{ \left(\frac{\partial}{\partial\tau} + \mu\right)\phi^* \left(\frac{\partial}{\partial\tau} - \mu\right)\phi + \nabla\phi^*\nabla\phi + m^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \right\}$$

or, after partial integration,

$$S_E[\phi] = \int_0^{1/T} d\tau \int d^3x \ \left\{ \phi^*\left[-\left(\frac{\partial}{\partial\tau} - \mu\right)^2 - \nabla^2 + m^2 \right]\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \right\}. \tag{15.24}$$

In summary, we can write the grand canonical partition function for a complex scalar field as

$$Z(T,\mu,V) = e^{-\Omega(T,\mu,V)/T} = \int \mathcal{D}\phi \ e^{-S_E[\phi]} \tag{15.25}$$

with the action in equation (15.24) and fields that are periodic in Euclidean or Matsubara time

$$\phi(\tau = 0, \vec{x}) = \phi(\tau = 1/T, \vec{x}). \qquad (15.26)$$

The formalism can also be extended to fermionic (Grassmann) fields, where a careful consideration leads to anti-periodic boundary conditions

$$\psi(\tau = 0, \vec{x}) = -\psi(\tau = 1/T, \vec{x}). \qquad (15.27)$$

From here on, the formalism can be developed similar as in vacuum.

For concreteness, let us consider free complex scalars with mass m_B and chemical potential μ_B as well as free Dirac fermions with mass M and chemical potential μ_F. The Euclidean action is

$$\begin{aligned} S_E[\phi] = \int_0^{1/T} d\tau \int d^3x \ \bigg\{ & \phi^* \left[-\left(\frac{\partial}{\partial \tau} - \mu \right)^2 - \nabla^2 + m^2 \right] \phi \\ & + \bar{\psi} \left[i\gamma^0 \left(\frac{\partial}{\partial \tau} - \mu_F \right) + \gamma^j \partial_j + iM \right] \psi \bigg\} \end{aligned} \qquad (15.28)$$

We introduce now source fields J for bosons and η for fermions and write the partition function

$$\begin{aligned} Z[\bar{\eta}, \eta, J^*, J] &= e^{W[\bar{\eta}, \eta, J^*, J]} \\ &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi^*\mathcal{D}\phi \ \exp \left[-S_E[\bar{\psi}, \psi, \phi^*, \phi] + \int_{\tau, \vec{x}} \left\{ \bar{\eta}\psi + \bar{\psi}\eta + J^*\phi + \phi^*J \right\} \right] \end{aligned} \qquad (15.29)$$

The functional $W[\bar{\eta}, \eta, J^*, J]$ is the Schwinger functional in thermal equilibrium and the generating functional of connected correlation functions.

Furthermore, one may introduce the one-particle irreducible effective action as a Legendre transform

$$\Gamma[\bar{\Psi}, \Psi, \varphi^*, \varphi] = \int_{\tau, \vec{x}} \left\{ \bar{\eta}\Psi + \bar{\Psi}\eta + J^*\varphi + \varphi^*J \right\} - W[\bar{\eta}, \eta, J^*, J] \qquad (15.30)$$

On the right hand side, the sources $\bar{\eta}, \eta$ and J^*, J are evaluated at the extremum. One finds the integral representation

$$\begin{aligned} e^{-\Gamma[\bar{\Psi}, \Psi, \varphi^*, \varphi]} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi^*\mathcal{D}\phi \ \exp \Big[-S_E[\bar{\psi}, \psi, \phi^*, \phi] \\ &\quad + \int_{\tau, \vec{x}} \left\{ \bar{\eta}(\psi - \Psi) + (\bar{\psi} - \bar{\Psi})\eta + J^*(\phi - \varphi) + (\phi^* - \varphi^*)J \right\} \Big] \\ &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi^*\mathcal{D}\phi \ \exp \Big[-S_E[\bar{\Psi} + \bar{\psi}, \Psi + \psi, \varphi^* + \phi^*, \varphi + \phi] \\ &\quad + \int_{\tau, \vec{x}} \left\{ \bar{\eta}\psi + \bar{\psi}\eta + J^*\phi + \phi^*J \right\} \Big] \end{aligned} \qquad (15.31)$$

In the second equation, we have shifted the functional integral. This representation makes particularly transparent that the effective action Γ corresponds to an action where the effect of fluctuations in the quantum fields is taken into account. Let us now perform the functional integral. For a free field theory, this is a Gaussian integral and can be done analytically.

More generally, one could perform a steepest descent approximation, leading to a loop expansion. As a preparatory step, we need to develop an appropriate Fourier transform. We write the fields as

$$\begin{aligned} \phi(\tau, \vec{x}) &= T \sum_n \int \frac{d^3p}{(2\pi)^3} e^{-iw_n\tau + i\vec{p}\vec{x}} \phi(iw_n, \vec{p}), \\ \psi(\tau, \vec{x}) &= T \sum_n \int \frac{d^3p}{(2\pi)^3} e^{-iw_n\tau + i\vec{p}\vec{x}} \phi(iw_n, \vec{p}). \end{aligned} \qquad (15.32)$$

As a consequence of the periodic / anti-periodic boundary conditions, the frequencies are discrete. One has

$$w_n = \begin{cases} 2\pi n T, & \text{Bosons}, \\ 2\pi \left(n + \frac{1}{2}\right) T, & \text{Fermions}. \end{cases} \tag{15.33}$$

These are known as *Matsubara frequencies*.

Exercise 15.3. Show that this allows to write the action in momentum space as

$$S_E = T \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} \left\{ \phi^* \left[(w_n - i\mu_B)^2 + (\vec{p})^2 + m^2 \right] \phi + \bar{\psi} \left[i\gamma^0 (w_n - i\mu_F) - \vec{\gamma}\vec{p} + iM \right] \psi \right\}$$
$$=: \phi_i^+ p_{ij}^{(B)} \phi_j + \bar{\psi}_i p_{ij}^{(F)} \psi_j$$

The second line introduces an abstract index notation with

$$i = (n, \vec{p}), \qquad j = (n', \vec{p}'), \qquad \delta_{ij} = \frac{1}{T}\delta_{nn'}(2\pi)^3\delta^{(3)}(\vec{p} - \vec{p}'), \qquad \sum_i = T\sum_n \int \frac{d^3p}{(2\pi)^3} \tag{15.34}$$

and

$$p_{ij}^{(B)} = \left[(w_n - i\mu_B)^2 + (\vec{p})^2 + m^2 \right] \delta_{ij}$$
$$p_{ij}^{(F)} = \left[i\gamma^0 (w_n - i\mu_F) - \vec{\gamma}\vec{p} + iM \right] \delta_{ij}. \tag{15.35}$$

One obtains

$$e^{-\Gamma} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi^*\mathcal{D}\phi \ \exp\left[-(\bar{\Psi} + \bar{\psi})_i p_{ij}^{(F)} (\Psi + \psi)_j - (\varphi^* + \phi^*)_i p_{ij}^{(B)} (\varphi + \phi)_j \right.$$
$$\left. + \bar{\eta}_i\psi_i + \bar{\psi}_i\eta_i + J_i^*\phi_i + \phi_i^* J_i \right]$$
$$= \exp\left[-\bar{\Psi}_i p_{ij}^{(F)} \Psi_j - \varphi_i^* p_{ij}^{(B)} \varphi_j \right] \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi^*\mathcal{D}\phi \ \exp\left[-\bar{\psi}_i p_{ij}^{(F)}\psi_j - \phi_i^* p_{ij}^{(B)}\phi_j \right. \tag{15.36}$$
$$\left. + (\bar{\eta}_j - \bar{\Psi}_i p_{ij}^{(F)})\psi_j + \bar{\psi}_i(\eta_i - p_{ij}^{(F)}\Psi_j) + (J_j^* - \varphi_i^* p_{ij}^{(B)})\phi_j + \phi_i^*(J_i - p_{ij}^{(B)}\varphi_j) \right]$$

The first term is actually the microscopic action S_E evaluated at the field expectation values Ψ and φ. The terms linear in the fluctuating fields ψ, ϕ in the last line vanish actually at the point where the expectation values solve the field equations, like for example

$$\frac{\delta}{\delta\bar{\Psi}_i}\Gamma = \eta_i \tag{15.37}$$

These are the equations that determine the sources on the right hand side of a Legendre transform. We thus find

$$\Gamma[\bar{\Psi}, \Psi, \varphi^*, \varphi] = S_E[\bar{\Psi}, \Psi, \varphi^*, \varphi] - \ln\left(\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi^*\mathcal{D}\phi \ \exp\left[-\bar{\psi}_i p_{ij}^{(F)}\psi_j - \phi_i^* p_{ij}^{(B)}\phi_j \right] \right) \tag{15.38}$$

Or, recalling the Gaussian integrals for bosonic and fermionic (Grassmann) variables,

$$\Gamma[\bar{\Psi}, \Psi, \varphi^*, \varphi] = S_E[\bar{\Psi}, \Psi, \varphi^*, \varphi] + \ln\det(p^{(B)}) - \ln\det(p^{(F)}) \tag{15.39}$$

The effective action is given by the microscopic action plus a constant term. Because the ln-terms are independent of the field expectation values Ψ and φ, we could drop them, if we were only interested

in field correlation functions. However, in general this constant depends on the temperature T and the chemical potentials μ_B and μ_F, so we need to evaluate it if we are interested in thermodynamics. In the following, we discuss this evaluation. First note the useful relation

$$\ln \det M = \operatorname{tr} \ln M \qquad \text{for some matrix } M. \tag{15.40}$$

(It follows easily in the case where $M = \operatorname{diag}(m_1, m_2, \cdots)$ and also holds in general.) In our case, taking the trace involves a sum over the Matsubara frequencies and an integral over the momenta. Also note that, if one comprehends $p_{ij}^{(B)}$ as a matrix, one has

$$\begin{aligned}
p_{ij}^{(B)} = S_E^{(2)}(n, n', \vec{p}, \vec{p}') &= \left(\frac{T}{(2\pi)^3} \frac{\delta}{\delta \phi^*(iw_n, \vec{p})} \right) \left(\frac{T}{(2\pi)^3} \frac{\delta}{\delta \phi(iw_{n'}, \vec{p}')} \right) S \\
&= \frac{1}{T} \delta_{nn'} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \left[(w_n - \mu_B)^2 + (\vec{p})^2 + m^2 \right]
\end{aligned} \tag{15.41}$$

and similar for $p_{ij}^{(F)}$ (which also has a Dirac-matrix structure). When taking the trace, we need to set $n = n'$ and $\vec{p} = \vec{p}'$, sum over n and integrate over $\vec{p}$. This gives in particular a term

$$\frac{(2\pi)^3}{T} \delta^{(3)}(0) = \frac{1}{T} V \tag{15.42}$$

with spatial volume V. Thus, we obtain the effective action

$$\begin{aligned}
\Gamma = \int_0^{1/T} d\tau \int d^3x \; \Big\{ &\phi^* \left[-\partial_\tau^2 + \mu_B \partial_\tau - \nabla^2 + m^2 \right] \phi + U(\varphi^* \varphi) \\
&+ \bar{\psi} \left[-\gamma^0 \left(\frac{\partial}{\partial \tau} - \mu_F \right) + i\gamma^j \partial_j + iM \right] \psi \Big\}
\end{aligned} \tag{15.43}$$

with effective action

$$U(\varphi^* \varphi) = (m^2 - \mu_B)^2 \varphi^* \varphi + \Delta U \tag{15.44}$$

and the contribution from fluctuations

$$\Delta U = T \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \left\{ \ln \left[\left(w_n^{(B)} - i\mu_B \right)^2 + (\vec{p})^2 + m^2 \right] - \operatorname{tr} \ln \left[i\gamma^0 \left(w_n^{(F)} - i\mu_F \right) - \vec{\gamma} \vec{p} + iM \right] \right\} \tag{15.45}$$

The remaining trace in the second term is for the Dirac matrices. In order to calculate this further, we need to find a method to sum over the Matsubara frequencies. To this end, we will first use partial integration with respect to the spatial momentum, assuming that there are vanishing boundary values for $\vec{p} \to \infty$.

$$\begin{aligned}
\Delta U &= -T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{3} \left(p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} + p_3 \frac{\partial}{\partial p_3} \right) \{ \ln[\cdots] - \ln[\cdots] \} \\
&= -T \sum_n \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\frac{2}{3}(\vec{p})^2}{(w_n^{(B)} - i\mu_B)^2 + (\vec{p})^2 + m^2} \right. \\
&\qquad \left. - \frac{1}{3} \operatorname{tr} \left(- \left[i\gamma^0 \left(w_n^{(F)} - i\mu_F \right) - \vec{\gamma} \vec{p} + iM \right]^{-1} \vec{\gamma} \vec{p} \right) \right\}
\end{aligned} \tag{15.46}$$

And we have

$$\left[i\gamma^0 \left(w_n^{(F)} - i\mu_F \right) - \vec{\gamma} \vec{p} + iM \right]^{-1} = \frac{i\gamma^0 \left(w_n^{(F)} - i\mu_F \right) - \vec{\gamma} \vec{p} - iM}{(w_n^{(F)} - i\mu_F)^2 + (\vec{p})^2 + M^2} \tag{15.47}$$
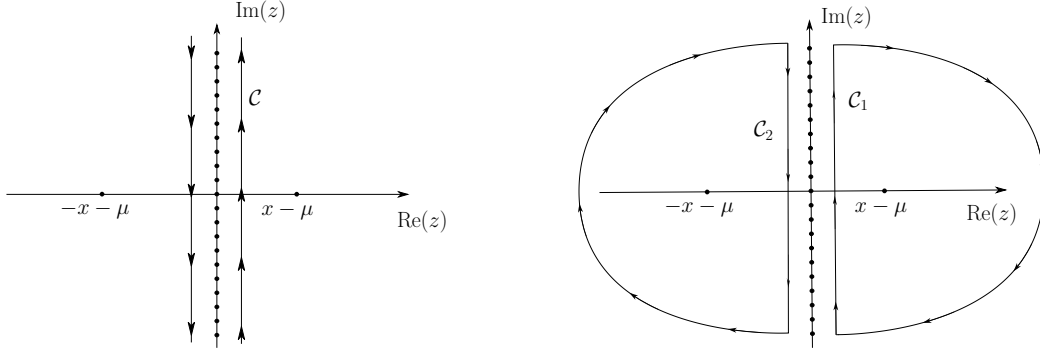
Figure 3. Different integration contours for J

and therefore

$$\Delta U = -T \sum_n \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{\frac{2}{3}(\vec{p})^2}{(w_n^{(B)} - i\mu_B)^2 + (\vec{p})^2 + m^2} - \frac{\frac{4}{3}(\vec{p})^2}{\left(w_n^{(F)} - i\mu_F\right)^2 + (\vec{p})^2 + M^2} \right\}$$ (15.48)

We used here $\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$.

In order to find a way to sum over n, we first consider a contour integral

$$J = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{1}{(-iz - i\mu)^2 + x^2} \left[\frac{1}{2} + n_B(z)\right]$$ (15.49)

Here $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$ while x, μ are parameters. As can be seen on the left hand side of figure (3), the integration contour $\mathcal{C}$ goes downwards slightly left of the imaginary z-axis and up again slightly to the right of it. We use here the Bose distribution function

$$n_B(z) = \left[e^{z/T} - 1\right]^{-1}$$ (15.50)

which has poles at $z = i2\pi n T$ with residue T. In contrast, the prefactor of the square bracket has poles at

$$z = \pm x - \mu.$$ (15.51)

We assume $|x| > |\mu|$, so that those poles are away from the imaginary z-axis. The contour can be closed at $z = \pm i\infty$ and we find from the residue theorem

$$J = T \sum_{-\infty}^{\infty} \frac{1}{(2\pi n T - i\mu)^2 + x^2}$$ (15.52)

This is precisely the infinite sum we need to calculate for the Boson contribution. On the other side, wen can also close the contour somewhat differently without changing the result of the integral as can be seen on the right hand side of figure (3).
We now get 2 contributions, one contour $\mathcal{C}_1$ that closes on the right and one contour $\mathcal{C}_2$ that closes on the left. From

$$\frac{1}{(-iz - i\mu)^2 + x^2} = \left(-\frac{1}{z - x + \mu} + \frac{1}{z + x + \mu}\right) \frac{1}{2x}$$ (15.53)

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one can read off the residues. Taking into account that the contour has now a clockwise orientation and that

$$\left[\frac{1}{2} + n_B(z)\right] = -\left[\frac{1}{2} + n_B(-z)\right] \tag{15.54}$$

gives the following alternative representation for our integral

$$\frac{1}{2x}\left[\frac{1}{2} + n_B(x - \mu) + \frac{1}{2} + n_B(x + \mu)\right] = J \overset{(15.52)}{=} T \sum_{-\infty}^{\infty} \frac{1}{(2\pi n T - i\mu)^2 + x^2} \tag{15.55}$$

This identity allows us to calculate the Matsubara sums for bosons. In a rather similar way, one can derive

$$\frac{1}{2x}\left[-\frac{1}{2} + n_F(x - \mu) - \frac{1}{2} + n_B(x + \mu)\right] = -T \sum_{-\infty}^{\infty} \frac{1}{\left(2\pi \left(n + \frac{1}{2}\right) T - i\mu\right)^2 + x^2} \tag{15.56}$$

with the Fermi distribution function

$$n_F(z) = \left[e^{z/T} + 1\right]^{-1} \tag{15.57}$$

These results actually have an interesting physical interpretation. The first term $1/2$ gives the contribution from bosonic "particle-like" quantum fluctuations, similar to the "quantum occupation $1/2$" of the simple harmonic oscillator, while the subsequent $n_B(z - \mu)$ gives the contribution of thermal fluctuations for "particle-like" excitations. The second $1/2$ comes from quantum fluctuations of anti-particle excitations and the corresponding thermal fluctuations are given by $n_B(x + \mu)$. Thus, anti-particles have opposite chemical potential.

The fermionic terms can be understood similarly - but now quantum fluctuations contribute $-1/2$! Moreover, the statistics is different as is apparent from $n_B \to n_F$.

Combining all terms, we find

$$\Delta U = -\int \frac{d^3p}{(2\pi)^3} \left\{ \frac{(\vec{p})^2}{3E_p^{(B)}} \left[\frac{1}{2} + n_B(E_p^{(B)} - \mu) + \frac{1}{2} + n_B(E_p^{(B)} + \mu)\right] \right. \\ \left. + \frac{2(\vec{p})^2}{3E_p^{(F)}} \left[-\frac{1}{2} + n_F(E_p^{(F)} - \mu) - \frac{1}{2} + n_F(E_p^{(F)} + \mu)\right] \right\} \tag{15.58}$$

with $E_p^{(B)} = \sqrt{(\vec{p})^2 + m^2}$ and $E_p^{(F)} = \sqrt{(\vec{p})^2 + M^2}$.

Note that there is an infinite contribution from quantum vacuum fluctuations, which however is independent of the temperature and the chemical potential. It would incidentally cancel if there were exactly as many complex bosons as complex fermions (here we have two complex fermions for spin $1/2$ degeneracy).

Subtracting the temperature independent term and recalling that pressure corresponds to $-U$, we find the thermodynamic pressure

$$p = -U + \text{const} = (\mu^2 - m^2)\varphi^*\varphi + \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{(\vec{p})^2}{3E_p^{(B)}} \left[\frac{1}{2} + n_B(E_p^{(B)} - \mu) + \frac{1}{2} + n_B(E_p^{(B)} + \mu)\right] \right. \\ \left. + \frac{2(\vec{p})^2}{3E_p^{(F)}} \left[-\frac{1}{2} + n_F(E_p^{(F)} - \mu) - \frac{1}{2} + n_F(E_p^{(F)} + \mu)\right] \right\} \tag{15.59}$$

And this is indeed the result that one would also obtain from the methods of quantum statistical mechanics. In particular, for photons one will thus obtain the Planck formula for blackbody radiation!

To understand the implications of what we have derived, let us investigate some limiting situations. First, calculate the boson density (for $\varphi^*\varphi = 0$ and $E := E_p^{(B)}$)

$$
\begin{aligned}
n &= \frac{\partial}{\partial\mu}p = \int \frac{d^3p}{(2\pi)^3} \frac{(\vec{p})^2}{3E} \frac{\partial}{\partial\mu} \left(n_B(E-\mu) + n_B(E+\mu)\right) \\
&= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{3}p^j \frac{\partial}{\partial p^j}E\right) \left(-\frac{\partial}{\partial E}n_B(E-\mu) + \frac{\partial}{\partial E}n_B(E+\mu)\right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{3}p^j \frac{\partial}{\partial p^j} \left(n_B(E-\mu) - n_B(E+\mu)\right) \\
&= \int \frac{d^3p}{(2\pi)^3} \left(n_B(E-\mu) - n_B(E+\mu)\right)
\end{aligned}
\tag{15.60}
$$

We see that the conserved particle number is the difference between occupation numbers for particles and anti-particles. Similarly for fermions.

Now consider the limit $T \to 0$, first for fermions. The occupation number becomes

$$
\lim_{T\to 0} n_F(E-\mu) = \lim_{T\to 0} \left[\exp\left(\frac{\sqrt{(\vec{p})^2 + M^2} - \mu}{T}\right) + 1\right]^{-1} = \theta(\mu - \sqrt{(\vec{p})^2 + M^2})
\tag{15.61}
$$

Similarly, $n_F(E+\mu) \to \theta(-\mu - \sqrt{(\vec{p})^2 + M^2})$. For $|\mu| < M$, there are no fermions at all. For $|\mu| > M$, a Fermi sea of particles develops where all states with $(\vec{p})^2 + M^2 < \mu^2$ are occupied once. Similarly, for $\mu < -M$, such a Fermi sea of anti-particles develops.

For bosons, the limit looks differently. The occupation number

$$
n_B(E-\mu) = \left[\exp\left(\frac{\sqrt{(\vec{p})^2 + M^2} - \mu}{T}\right) - 1\right]^{-1}
\tag{15.62}
$$

vanishes also when $|\mu| < m$ but would diverge for $\mu^2 = m^2$ at $\vec{p} = 0$. This is related to Bose-Einstein condensation. To see what happens, let us investigate the "classical" contribution to the effective potential in the interacting theory.

$$
U_{\rm cl} = -p = (m^2 - \mu^2)\varphi^*\varphi + \frac{1}{2}\lambda(\varphi^*\varphi)^2
\tag{15.63}
$$

For $m^2 > \mu^2$, the minimum of the effective potential is at $\rho := \varphi^*\varphi = \frac{m^2 - \mu^2}{\lambda} > 0$. The field has a non-vanishing, homogeneous expectation value. The particle number density has a contribution

$$
n = \frac{\partial p}{\partial\mu} = 2\mu\rho
\tag{15.64}
$$

which is positive for $\mu > 0$ and negative (anti-particles) for $\mu < 0$.

We should note here that in non-relativistic physics, one works with a chemical potential μ_{NR} related to the one used here through

$$
\mu = m + \mu_{NR}
\tag{15.65}
$$

Furthermore, the fields are normalised somewhat differently.

Finally, let us discuss the case of massless particles at vanishing chemical potential. This is a good approximation for many quantum fields in the early universe but also for quarks and gluons in relativistic heavy ion collisions at the Large Hadron Collider (LHC) at CERN.

Exercise 15.4. Show that upon performing the momentum integrals, one finds

$$p(T) = \frac{\pi^2}{90} \left(N_B + \frac{7}{8} N_F \right) T^4 \tag{15.66}$$

This formula actually holds more generally for N_B real bosonic massless fields where in our case for a complex scalar field $N_B = 2$. Similarly, N_F counts the number of real massless fermionic fields. In our case, we have 2 spin states and Dirac fermions are complex (in contrast to Majorana fermions), so $N_F = 4$.

It is very easy to derive the entropy density

$$s = \frac{\partial p}{\partial T} = 4 \frac{\pi^2}{90} \left(N_B + \frac{7}{8} N_F \right) T^3 \tag{15.67}$$

The energy density is given by

$$\epsilon = -p + sT = 3 \frac{\pi^2}{90} \left(N_B + \frac{7}{8} N_F \right) T^4 = 3p \tag{15.68}$$

In particular, the relation $p = \epsilon/3$ is a consequence of conformal symmetry (no mass scale except T) and shows that the velocity of sound is (in units of c)

$$c_s = \sqrt{\frac{\partial p}{\partial \epsilon}} = \frac{1}{\sqrt{3}}. \tag{15.69}$$

16 Functional renormalization

We have seen that loop integrals often contain ultraviolet divergencies if the UV cutoff is moved to infinity, or infrared divergencies if the volume is extended to infinity.

Quantities dominated by infrared fluctuations become predictable in terms of a few "renormalised couplings". This idea is the central point why quantum field theory has predictive power. Rather than dealing with this idea in a technical fashion, we will develop the *concepts* that explain why "technical miracles" (as the cancellation of the divergencies) occur in perturbation theory. This is done by introducing functional renormalization as developed by Wilson, Wegner, Symanzik and Kadanoff.

The main idea is to relate the microphysical laws embodied by the classical action S to the macrophysical laws that can be extracted from the effective action Γ. This is done in a continuous way by the effective average action Γ_k, which describes the laws at a length scale $\propto k^{-1}$. This effective average action interpolates smoothly between the classical action (at $k = \Lambda$ or $k \to \infty$) and the effective action (at $k = 0$). In this way, the effect of fluctuations is included stepwise. The way in which Γ_k depends on k is described by a so-called "flow-equation" or "renormalization group equation".

The effective average action includes the effects of all fluctuations with momenta q larger than k, i.e. $q^2 \geq k^2$, but does not include those with momenta smaller than k, i.e. $q^2 \leq k^2$. The small momentum fluctuations are "cut off" by an infrared cutoff function $R_k(q^2)$.
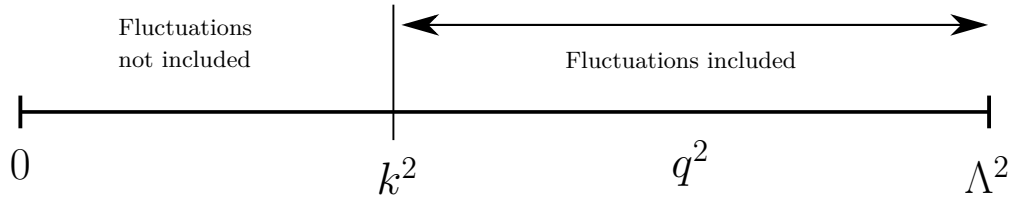
Consider to nearby scales k_1 and $k_2 < k_1$. The difference between Γ_{k_2} and Γ_{k_1} consists of the fluctuations in a finite momentum range $k_2^2 < q^2 < k_1^2$.
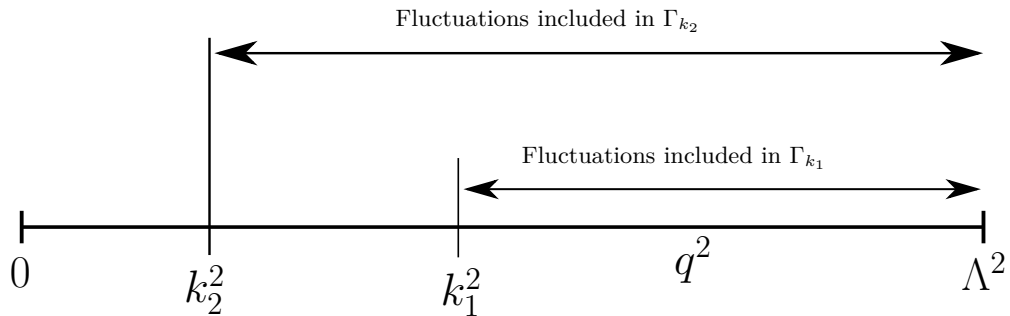
This is the reason why the flow equation (which describes how Γ_k *changes*) is dominated by this momentum range and both ultraviolet finite and infrared finite.

The momenta with $q^2 >> k_1^2$ and $q^2 << k_2^2$ simply do not matter for $\Gamma_{k_2} - \Gamma_{k_1}$. The ultraviolet

and infrared divergencies in perturbation theory have to do with properties of the *solutions* of the flow equations.

16.1 Effective average action

IR-cutoff

In order to implement these ideas, we add to the action an infrared cutoff piece

$$\Delta_k S = \frac{1}{2} \int_q \chi_a^*(q) R_k(q^2) \chi^a(q). \tag{16.1}$$

R_k is called regulator. This adds to the classical inverse propagator a cutoff piece

$$(S + \Delta_k S)^{(2)} = q^2 + R_k(q^2) + \cdots = P_k(q^2) + \cdots \tag{16.2}$$

An example for a regulator is

$$R_k(q^2) = (k^2 - q^2)\theta(k^2 - q^2), \tag{16.3}$$

resulting in

$$P_k(q^2) = \begin{cases} q^2, & \text{for } q^2 > k^2 \\ k^2, & \text{for } q^2 < k^2 \end{cases} \tag{16.4}$$

For $k > 0$, the momentum integrals in a loop expansion

$$\int_q \frac{1}{(P_k(q^2) + m^2)^n} \tag{16.5}$$

have no IR-divergencies, even for $m^2 = 0$.

Due to the dependence on k, we have a whole family of models with classical actions $S + \Delta_k S$. For $k > 0$, it is regularized in the infrared, a finite volume is no longer necessary. The ultraviolet divergencies in perturbation theory at this stage, since $P_k(q^2) = q^2$ for large momenta.

Generating functionals in presence of R_k

For $k > 0$, we can repeat all definitions for generating functionals. The Schwinger-functional becomes k-dependent,

$$W_k[J] = \ln \int \mathcal{D}\chi \, \exp\left\{-S - \Delta_k S + \int J \cdot \chi\right\}. \tag{16.6}$$

Only the action, not the construction, is modified. The same holds for the Legendre transform

$$\tilde{\Gamma}_k[\varphi] = -W_k + \int J \cdot \varphi. \tag{16.7}$$

The relation between φ and J depends on k,

$$\frac{\partial W_k}{\partial J(x)} = \varphi(x), \qquad \frac{\partial \tilde{\Gamma}_k}{\partial \varphi(x)} = J(x), \tag{16.8}$$

since W_k and $\tilde{\Gamma}_k$ depend on k.

For the effective average action Γ_k, we subtract from the effective action $\tilde{\Gamma}_k$ the IR-cutoff piece, now in terms of φ,

$$\Gamma_k = \tilde{\Gamma}_k - \Delta S_k = \tilde{\Gamma}_k - \frac{1}{2} \int_q \varphi_a^*(q) R_k(q^2) \varphi^a(q). \tag{16.9}$$

Matrix notation

Taking χ_a and φ_a as generalized vectors, the IR-cutoff is a matrix

$$\Delta S_k = \frac{1}{2} \int_{x,y} \varphi_a(x) \mathcal{R}_k^{ab}(x,y) \varphi_b(x) =: \frac{1}{2} \varphi^T \mathcal{R} \varphi. \tag{16.10}$$

Here $\mathcal{R}_k^{ab}(x,y)$ is the Fourier transform of

$$\mathcal{R}_k^{ab}(q,q') = R_k(q^2) \delta^{ab} \delta(q-q'), \tag{16.11}$$

i.e.

$$\mathcal{R}_k^{ab}(x,y) = \int_q \int_{q'} e^{iqx} \mathcal{R}_k^{ab}(q,q') e^{-iq'y} = \int_q e^{iq(x-y)} R_k(q^2) \delta^{ab}. \tag{16.12}$$

In the matrix notation, one thus has

$$\frac{\partial \Gamma_k}{\partial \varphi} = J - \mathcal{R}_k \cdot \varphi. \tag{16.13}$$

Background field identity

Inserting these definitions, one obtains

$$\exp\left(-\Gamma_k[\varphi]\right) = \int \mathcal{D}\chi' \, \exp\left(-S[\varphi+\chi'] + \frac{\partial \Gamma_k}{\partial \varphi} \chi' - \frac{1}{2}(\chi')^T \mathcal{R}_k \chi\right) \tag{16.14}$$

Comparing with $k=0$, where $\mathcal{R}_k = 0$, one sees that the IR-cutoff only acts on the fluctuations $\chi' = \chi - \varphi$.

Limits

(i) Assume $\lim_{k\to\infty} R_k(q) \to \infty$. Then one can show that

$$\lim_{k\to\infty} \Gamma_k[\varphi] \to S[\varphi]. \tag{16.15}$$

The corrections in the saddle point approximation vanish since the quadratic cutoff term diverges and the Gaussian integral becomes exact. The determinant from the one-loop correction is a field-independent constant. All fluctuations are cut off.

(ii) Assume $\lim_{k\to 0} R_k(q) \to 0$. Then

$$\lim_{k\to 0} \Gamma_k[\varphi] \to \Gamma[\varphi]. \tag{16.16}$$

Here, there is no IR-cutoff, all fluctuations are included in the quantum effective action.

> The flowing action $\Gamma_k[\varphi]$ interpolates between the microscopic action $S[\varphi]$ and the macroscopic quantum effective action $\Gamma[\varphi]$.

Some remarks:

I) In practice, there is a finite microscopic (UV) scale Λ. Instead of taking the limit $k \to \infty$, one sets $k \to \Lambda$. Thus, Γ_Λ can be associated with the microscopic action (though in principle, the first step is to compute Γ_Λ from S).

II) Symmetries of Γ_k:
All symmetries of $S + \Delta S$ (in absence of anomalies) or of Γ_Λ and ΔS are also symmetries of Γ_k. Sometimes the IR-cutoff can violate symmetries (e.g. gauge interactions).

III) Effective laws:

Γ_k encodes the effective laws at the momentum scale k, i.e. at the length scale k^{-1}. Thus, the flow to lower k can intuitively be understood as "zooming out" with a microscope that enables to adjust to variable resolutions. When fluctuations $q^2 < k^2$ are not yet included, Γ_k describes a situation analogous to an experiment with a finite probe size $\propto k^{-1}$. Therefore Γ_k is called the "flowing action".

IV) To take into account fluctuations only down to a certain momentum k can also be interpreted as averaging of fields, taking into account all interactions within a range k^{-1}. Therefore, one talks about the "effective average action".

16.2   Exact flow equation

We will derive the exact flow equation

$$\boxed{\partial_k \Gamma_k = \frac{1}{2}\mathrm{tr}\left((\Gamma^{(2)} + \mathcal{R}_k)^{-1} \partial_k \mathcal{R}_k \right)} \tag{16.17}$$

in several steps.

i)
$$\partial_k \tilde{\Gamma}_k[\varphi] = -\partial_k W_k[J] \tag{16.18}$$

Proof:

$$\partial_k \tilde{\Gamma}_{k|\varphi} = -\partial_k W_{k|J} - \int_x \frac{\delta W_k}{\delta J_a(x)} \partial_k J_a(x)_{|\varphi} + \int_x \varphi_a(x)\partial_k J_a(x)_{|\varphi} \tag{16.19}$$

ii)

$$\begin{aligned}
-\partial_k W_k[J] &= -\partial_k \ln \int \mathcal{D}\chi \, \exp\left(-S - \Delta_k S + J\cdot\chi\right) \\
&= -\frac{1}{Z}\int \mathcal{D}\chi \, \exp\left(-S - \Delta_k S + J\cdot\chi\right)\left(-\partial_k \Delta_k S\right) \\
&= \partial_k \langle \Delta_k S \rangle,
\end{aligned} \tag{16.20}$$

(because only $\Delta_k S$ depends on k).

iii) The formula for the propagator is derived as follows.

$$G_{ab}(x,y) = \langle \chi_a(x)\chi_b(y) \rangle - \langle \chi_a(x) \rangle \langle \chi_b(y) \rangle, \tag{16.21}$$

$$\langle \chi_a(x)\chi_b(y) \rangle = G_{ab}(x,y) + \varphi_a(x)\varphi_b(y), \qquad G_{ab}(x,y) = G_{ba}(y,x) \tag{16.22}$$

$$\partial_k \tilde{\Gamma}_k = \frac{1}{2}\int_{x,y} \partial_k \mathcal{R}_k^{ab}(x,y) G_{ab}(x,y) + \frac{1}{2}\int_{x,y} \varphi_a(x)\partial_k \mathcal{R}_k^{ab}(x,y)\varphi_b(y) \tag{16.23}$$

$$\Rightarrow \partial_k \Gamma_k = \frac{1}{2}\int_{x,y} G_{ab}(x,y)\partial_k \mathcal{R}_k^{ba}(y,x) = \frac{1}{2}\mathrm{tr}\left\{ G \cdot \partial_k \mathcal{R}_k \right\}. \tag{16.24}$$

Here

$$G_{ab}(x,y) = \frac{\partial^2 W_k}{\partial J_a \partial J_b} \tag{16.25}$$

is the propagator matrix in the presence of the IR-cutoff k. It depends on sources or fields.

iv) Propagator and inverse second functional derivative of Γ_k:
   We employ the general matrix identity for Legendre transforms

$$G(\Gamma_k + \mathcal{R}_k) = G\tilde{\Gamma}_k^{(2)} = 1. \tag{16.26}$$

This yields the final form of a closed flow equation:

$$\partial_k \Gamma_k = \frac{1}{2}\text{tr}\left((\Gamma^{(2)} + \mathcal{R}_k)^{-1}\partial_k \mathcal{R}_k\right) \tag{16.27}$$

and concludes the proof.

The dimensionless form is obtained by multiplying with k and by defining $\partial_t := k\partial_k$ (here t has nothing to do with time). Then

$$\boxed{\partial_t \Gamma_k = \frac{1}{2}\text{tr}\left((\Gamma^{(2)} + \mathcal{R}_k)^{-1}\partial_t \mathcal{R}_k\right).} \tag{16.28}$$

It is a functional differential equation and both Γ_k and $\Gamma_k^{(2)}$ are functionals of the macroscopic fields $\varphi_a = \varphi_a(x)$.

Properties of the flow equation

1. The flow equation is exact, we have made no approximations. All non-perturbative effects are included, e.g. topological defects, etc.

2. The particular form of the matrix $\mathcal{R}_k$ is not important. Only the quadratic form

$$\Delta_k S = \frac{1}{2}\chi^T \mathcal{R}_k \chi \tag{16.29}$$

matters. This allows generalisations to a wide range of situations where $\mathcal{R}_k$ is not necessarily interpreted as a momentum cutoff.

3. Finite momentum integrals. In momentum space, one has

$$\partial_t \Gamma_k = \frac{1}{2}\sum_a \int_q (\Gamma^{(2)} + \mathcal{R}_k)_{aa}^{-1}(q,q)\partial_t R_k(q^2). \tag{16.30}$$
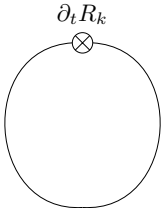
Note that for translation invariant configurations, one has $\Gamma_k^{(2)} + \mathcal{R}_k \propto \delta(q - q')$, $\delta(0) \Rightarrow \Omega_d$. The momentum integral is finite due to the presence of $\mathcal{R}_k$ in $(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$. It is also UV-finite due to the factor $\partial_t R_k$. Only momenta $q^2 \approx k^2$ contribute to the momentum integral.

4. For the example $R_k = (k^2 - q^2)\theta(k^2 - q^2)$, one has

$$\partial_t R_k = \begin{cases} 2k, & q^2 < k^2 \\ 0, & q^2 > k^2 \end{cases} \tag{16.31}$$

(The factor $(k^2 - q^2)\delta(k^2 - q^2)$ does not contribute if a suitable smooth limit is taken for all definitions of R_k.) In this case, the momentum integral is only over a range $q^2 < k^2$. In this range, one has $P_k(q^2) = k^2$.

5. Feynman graph.

$$\partial_t \Gamma_k = \qquad \overset{\partial_t R_k}{\underset{\bigotimes}{\bigcirc}} \tag{16.32}$$

This is an *exact* and *field dependent* propagator.

Renormalization group improved one-loop equation

One has the formal expression

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{tr} \left(\tilde{\partial}_t \ln(\Gamma_k^{(2)} + \mathcal{R}_k) \right), \qquad \tilde{\partial}_t := \partial_t \mathcal{R}_k \frac{\partial}{\partial \mathcal{R}_k}. \tag{16.33}$$

Recalling the one-loop formula

$$\Gamma^{(1)} = \frac{1}{2} \mathrm{tr} \ln S^{(2)}, \tag{16.34}$$

one sees the close correspondence of Feynman graphs. One can first formally derive the Feynman graphs, add $\mathcal{R}_k$ in the inverse propagator, perform $\tilde{\partial}_t$ and then integrate. One has, however, more vertices, since Γ_k has richer structure than S. Functional derivatives can be taken on bare scales. For finite quantities, $\int_q \tilde{\partial}_t = \tilde{\partial}_t \int_q$. This procedure has two important advantages compared to perturbation theory:

1. The momentum integrals in the loop expansion are finite,

2. no higher loops are needed. (The higher loops effects can be reproduced by an iterative solution of the flow equation.)

16.3   Derivative expansion

The flow equation is a functional differential equation. Except for a few particular cases (leading order large N expansion, few non-relativistic particles), it cannot be solved exactly.

Approximate solutions are constructed by *truncation*. A truncation is an ansatz for the general form of the effective average action in terms of a few free parameters or free functions. One computes the flow of these parameters or functions by inverting the ansatz on the r.h.s. of the flow equation for the computation of $\Gamma^{(2)}$.

For the *derivative expansion*, one expands $\Gamma_k[\varphi]$ in terms of its derivatives. For example, for a theory with $SO(N)$ symmetry, this yields

$$\Gamma_k = \int_x U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \varphi_a(x) \partial^\mu \varphi^a(x) + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial^\mu \rho + \mathcal{O}(\partial^4) \tag{16.35}$$

where $\rho = \frac{1}{2} \varphi_a \varphi^a$. The first order derivative expansion neglects terms with four or more derivatives. In this order, one has three functions: U, Z, Y.

If we simplify further: $Y_k = 0$, Z_k independent of ρ, then this is called "leading (order) potential approximation".

16.4   Flow of effective potential

We want to compute the flow equation for the effective potential $U_k(\rho)$. For this purpose, we evaluate Γ_k for φ independent of x. One needs to evaluate $\Gamma_k^{(2)}$ for constant φ. In momentum space, it reads

$$(\Gamma_k^{(2)})_{ab}(q, q') = \left(Z q^2 \delta_{ab} + \frac{\partial^2 U}{\partial \varphi_a \partial \varphi_b} \right) \delta(q - q'). \tag{16.36}$$

One infers, similar to section 14,

$$\boxed{\partial_t U = \frac{1}{2} \int_q \partial_t R_k (Z q^2 + R_k + U' + 2\rho U'')^{-1} + \frac{N-1}{2} \int_q \partial_t R_k (Z q^2 + R_k + U')^{-1}} \tag{16.37}$$

This can be compared to the one-loop approximation

$$U_{1l} = \frac{1}{2} \int_q \ln(Zq^2 + V' + 2\rho V'') + \frac{N-1}{2} \int_q \ln(Zq^2 + V'). \tag{16.38}$$

Replace $V \to U$, add R_k by $Zq^2 \to Zq^2 + R_k$ and take the $\tilde{\partial}_t$-derivative. There are no UV-divergencies in $\partial_t U_k$. We can choose the cutoff function

$$R_k(q^2) = Z(k^2 - q^2)\theta(k^2 - q^2). \tag{16.39}$$

The anomalous dimension is defined by

$$\eta := -\partial_t \ln Z. \tag{16.40}$$

It is typically very small and we can neglect the term proportional to η in $\partial_t R_k$. The result is

$$\partial_t U = \frac{1}{2} \int_{q^2 < k^2} \frac{2Z_k k^2}{Z_k k^2 + U' + 2\rho U''} + \frac{N-1}{2} \int_{q^2 < k^2} \frac{2Z_k k^2}{Z_k k^2 + U'}. \tag{16.41}$$

We define the renormalized dimensionless mass terms w_1 for the radial mode and w_2 for the Goldstone modes.

$$w_1 := \frac{U' + 2\rho U''}{Zk^2}, \qquad w_2 := \frac{U'}{Zk^2} \tag{16.42}$$

The momentum integrals are trivial

$$\int_{q^2 < k^2} = \alpha_d k^d, \qquad \alpha_d = \frac{4}{d} v_d$$

$$\text{e.g.} \qquad \alpha_2 = \frac{1}{4\pi}, \qquad \alpha_3 = \frac{1}{6\pi^2}, \qquad \alpha_4 = \frac{1}{32\pi^2} \tag{16.43}$$

We arrive at a very simple flow equation for the effective potential

$$\boxed{\partial_t U = \alpha_d k^d \left\{ \frac{1}{1 + w_1} + \frac{N-1}{1 + w_2} \right\}} \tag{16.44}$$

Since w_1 and w_2 involve ρ-derivatives of U, this is a differential equation for a single function U of the two variables k and ρ. For a given η or $\eta = 0$, it is closed.

The solution of this flow equation produces almost all characteristic features of the $O(N)$-scalar models in arbitrary dimension d. Some examples are provided below.

d=4

1. Spontaneous symmetry breaking and vacuum phase transition,

2. Renormalizable couplings (two),

3. Predictivity of all other couplings in terms of the renormalizable couplings,

4. Triviality for $\Lambda \to \infty$.

d=3

1. Critical behaviour at phase transition,

2. Critical exponents,

3. Wilson-Fisher fixed point

d=2

1. Absence of spontaneous symmetry breaking for continuous symmetries ($N > 2$), Mermin-Wagner theorem,

2. Kosterlitz-Thouless phase transition for $N = 2$ needs computation of η,

3. Non-perturbative mass generation for non-linear σ-models,

4. Essential scaling and jump in superfluid density in $d = 2$, $N = 2$ needs flow of $Z(\rho)$ and $Y(\rho)$.

16.5 Flowing couplings in four dimensions

Scale dependent minimum

If the effective average potential $U_k(\rho)$ has a minimum at $\rho_0(k)$, the condition for the minimum is for all k:

$$U'_k(\rho_0(k)) = 0. \tag{16.45}$$

The flow equation for $U'(\rho)$ at fixed ρ is obtained by taking a ρ-derivative of the flow equation for the potential,

$$
\begin{aligned}
\partial_t U'_k(\rho) &= \frac{k^4}{32\pi^2} \frac{\partial}{\partial \rho} \left\{ \frac{1}{1+w_1} + \frac{N-1}{1+w_2} \right\} \\
&= -\frac{k^4}{32\pi^2} \left\{ -\frac{1}{(1+w_1)^2} \frac{\partial w_1}{\partial \rho} + \frac{N-1}{(1+w_2)^2} \frac{\partial w_2}{\partial \rho} \right\}.
\end{aligned}
\tag{16.46}
$$

For simplicity, we take $Z = 1$, $(\eta = 0)$,

$$
\begin{aligned}
w_1 &= \frac{U' + 2\rho U''}{k^2}, & w_2 &= \frac{U'}{k^2} \\
\frac{\partial w_1}{\partial \rho} &= \frac{3U'' + 2\rho U'''}{k^2}, & \frac{\partial w_2}{\partial \rho} &= \frac{U''}{k^2}.
\end{aligned}
\tag{16.47}
$$

One infers

$$\partial_t U'_k(\rho) = -\frac{k^2}{32\pi^2} \left\{ -\frac{3U'' + 2\rho U'''}{(1+w_1)^2} + \frac{(N-1)U''}{(1+w_2)^2} \right\}. \tag{16.48}$$

For $\rho = \rho_0$, one has $U'(\rho_0) = 0$, $w_2 = 0$, $w_1 = 2\rho_0 U''(\rho_0)$. We define $\lambda := U''(\rho_0)$, $\nu := U'''(\rho_0)$ such that

$$\partial_t U'_k(\rho_0) = -\frac{k^2}{32\pi^2} \left\{ \frac{3\lambda + 2\rho_0\nu}{(1+2\rho_0\lambda)^2} + (N-1)\lambda \right\}. \tag{16.49}$$

For a fixed location ρ_0, the derivation $U'_k(\rho_0)$ does not remain zero. The location of the minimum therefore depends on k, according to

$$\partial_t U'_k(\rho_0) + \partial_t U''_k(\rho_0) \frac{\partial \rho}{\partial t} = 0, \qquad \text{or} \qquad \frac{\partial \rho_0}{\partial t} = -\frac{1}{\lambda} \partial_t U'_k(\rho_0). \tag{16.50}$$

The location of the minimum moves according to

$$\frac{\partial \rho_0}{\partial t} = \frac{k^2}{32\pi^2} \left\{ \frac{3 + 2\rho_0\nu/\lambda}{(1+2\rho_0\lambda)^2} + (N-1) \right\}. \tag{16.51}$$

As k is lowered, ρ_0 becomes smaller. Depending on the initial value at $k = \Lambda$, it may reach zero at some $k > 0$ or not. For small λ, we will see that $\nu \propto \lambda^3$. To lowest order in λ, the flow equation for ρ_0 simplifies:

$$\frac{\partial \rho_0}{\partial t} = \frac{k^2}{32\pi^2}(N + 2). \tag{16.52}$$

This has the simple solution

$$\rho_0(k) = \frac{k^2}{64\pi^2}(N + 2) + c_\Lambda, \tag{16.53}$$

with integration constant c_Λ, or

$$\rho_0 = \rho_\Lambda - \frac{\Lambda^2 - k^2}{64\pi^2}(N + 2), \qquad \text{with } \rho_\Lambda = \rho_0(k = \Lambda). \tag{16.54}$$

Different ρ_Λ label different "flow trajectories".

Phase transition

There is a critical value $\rho_{\Lambda,\mathrm{cr}}$ for which $\rho_0(k = 0) = 0$, namely

$$\rho_{\Lambda,\mathrm{cr}} = \frac{\Lambda^2}{64\pi^2}(N + 2). \tag{16.55}$$

For $\rho_\Lambda > \rho_{\Lambda,\mathrm{cr}}$, one has $\rho_0(k = 0) > 0$. This corresponds to the phase with spontaneous symmetry breaking (SSB). On the other hand, for $\rho_\Lambda < \rho_{\Lambda,\mathrm{cr}}$, one finds $\rho_0(k_t) = 0$ for $k_t > 0$. For $k < k_t$, the minimum is located at $\rho = 0$. The flow of U is then better described by the flow of $m_0^2 := U'(0)$. It increases with decreasing k. At $k = 0$, one finds $m_0^2 > 0$. Then the model is in the symmetric phase (SYM).

Quadratic divergence

For a given macroscopic or "remormalized" $\rho_{0,R} = \rho_0(k = 0)$, one finds for the microscopic or "bare" ρ_Λ,

$$\rho_\Lambda = \rho_{0,R} + \frac{\Lambda^2}{64\pi^2}(N + 2). \tag{16.56}$$

For $\Lambda \to \infty$, this diverges quadratically. The divergence arises from the relation between bare and renormalized parameters which in turn arises due to the flow that is generated by fluctuations.

Running quartic coupling

For the flow of $\lambda = U_k''(\rho_0)$, one needs

$$\partial_t \lambda = \partial_t U_k''(\rho_0) + U_k'''(\rho_0)\frac{\partial \rho_0}{\partial t} \tag{16.57}$$

where $\partial_t U_k''(\rho)$ is obtained from $\partial_t U_k'(\rho)$ by taking a further ρ-derivative. Thus,

$$\begin{aligned}
\partial_t U_k''(\rho) = &\frac{1}{16\pi^2}\left\{ \frac{(3U'' + 2\rho U''')^2}{(1 + w_1)^3} + (N - 1)\frac{(U'')^2}{(1 + w_2)^3} \right\} \\
&- \frac{k^2}{32\pi^2}\left\{ \frac{3U''' + 2\rho U''''}{(1 + w_1)^2} + (N - 1)\frac{U'''}{(1 + w_2)^2} \right\}.
\end{aligned} \tag{16.58}$$

Subsequently, we neglect $U'''(\rho_0)$ and $U''''(\rho_0)$ because they are of higher order in small λ (see below). This results in

$$\partial_t \lambda = \frac{1}{16\pi^2}\left\{ \frac{9\lambda^2}{(1 + 2\lambda\rho_0/k^2)^2} + (N - 1)\lambda^2 \right\}. \tag{16.59}$$

The leading order in a perturbative expansion in λ yields the one-loop β-function,

$$\beta_\lambda := \partial_t \lambda = \frac{\lambda^2}{16\pi^2}(N+8). \tag{16.60}$$

This is a typical "renormalization group equation" for a dimensionless coupling, that can also be found by perturbative renormalization. The β-function involves only λ at a given scale k, not the bare coupling $\lambda_\Lambda = \lambda(k = \Lambda)$. It does not involve k explicitly.

Feynman diagrams

$$\partial_t U = \quad \text{} \tag{16.61}$$

A ρ-derivative inserts external legs attached to a vertex, ,

$$\partial_t U' = \quad \text{} \tag{16.62}$$

and similarly,

$$\partial_t U'' = \quad \text{} \tag{16.63}$$

This is the perturbative Feynman diagram with the insertion of $\partial_t R_k$. Renormalized vertices replace the bare vertices. The $\partial_t R_k$ insertion removes divergencies from the momentum integral. The flow equation has the important property that only the couplings at a given scale k appear, not the bare couplings. (The computation of) a change of Γ_k only involves Γ_k!

Running coupling

The solution of the flow equation is easily found by

$$\frac{\mathrm{d}\lambda}{\lambda^2} = -\mathrm{d}\left(\frac{1}{\lambda}\right) = \frac{N+8}{16\pi^2}\mathrm{d}\ln k$$

$$\Rightarrow \quad \frac{1}{\lambda(k)} - \frac{1}{\lambda_\Lambda} = \frac{N+8}{16\pi^2}(\ln\Lambda - \ln k)$$

$$\Leftrightarrow \quad \frac{1}{\lambda(k)} = \frac{1}{\lambda_\Lambda} + \frac{N+8}{16\pi^2}\ln\frac{\Lambda}{k} \tag{16.64}$$

$$\Leftrightarrow \quad \lambda(k) = \frac{\lambda_\Lambda}{1 + \frac{(N+8)\lambda_\Lambda}{16\pi^2}\ln\frac{\Lambda}{k}}.$$

As k decreases, $\lambda(k)$ decreases.

Triviality

For any fixed Λ and $\lambda_\Lambda > 0$, one finds for $k \to 0$ that $\lambda(k \to 0) = 0$.
The interaction vanishes in this limit and one ends up with a free theory. This is called "triviality".
One can use the flow equation in order to show triviality without the assumption of small λ.

External momenta

Consider the momentum-dependent four-parent vertex

$$\lambda_k(p_1, p_2, p_3, p_4) = \frac{\partial^4 \Gamma_k}{\partial\varphi(p_1)\partial\varphi(p_2)\partial\varphi^*(p_3)\partial\varphi^*(p_4)} = \quad (16.65)$$

(We omit the internal indices, e.g. $N = 1$.) In lowest order, the flow equation is given by a one-loop diagram.

$$\partial_t \lambda = \quad (16.66)$$

It involves the renormalized vertices $\lambda_k(p_1, p_2, q, q')$ and $\lambda_k(q, q', p_3, p_4)$. Momentum conservation at the vertices implies

$$q + q' = p_1 + p_2 = p_3 + p_4, \qquad (q')^2 = (p_1 + p_2 - q)^2. \quad (16.67)$$

For $(p_1 + p_2)^2 = \mu^2$ and $k^2 << \mu^2$, only momenta $q^2 < k^2$ or momenta $(q')^2 \approx \mu^2$ contribute to the flow. This replaces in one of the propagators $1/(k^2 + m^2)$ by $1/(\mu^2 + m^2)$, leading to a suppression $\propto k^2/\mu^2$. The flow effectively stops for $k^2 < \mu^2$. One can associate

$$\lambda(\mu) \approx \lambda_{k^2 = \mu^2}(0), \quad (16.68)$$

where the l.h.s. represents the non-vanishing momenta, $k = 0$ and the r.h.s. the vanishing momenta, $k^2 = \mu^2$. Taking $\lambda_R(\mu) := \lambda_{k=0}(\mu)$, one has

$$\lambda_R(\mu) = \frac{\lambda_\Lambda}{1 + \frac{(N+8)\lambda_\Lambda}{16\pi^2} \ln \frac{\Lambda}{\mu}}. \quad (16.69)$$

The flow equation describes now the dependence of the vertex on the scale of the external momenta. It is equivalent to the perturbative renormalization group equation.

Landau pole and "incomplete theories"

For the standard model, the Fermi scale φ_0 contributes an effective infrared cutoff. The renormalized coupling at $k = \varphi_0$ is measured by the observation of the mass of the Higgs boson

$$m_H^2 = 2\lambda(\varphi_0)\varphi_0^2. \quad (16.70)$$

We can use the flow equation in order to compute λ at shorter distance scales $\lambda(k > \varphi_0)$:

$$\frac{1}{\lambda(k)} - \frac{1}{\lambda(\varphi_0)} = -\frac{N+8}{16\pi^2} \ln \frac{k}{\varphi_0}$$

$$\Rightarrow \lambda(k) = \frac{\lambda(\varphi_0)}{1 - \frac{(N+8)\lambda(\varphi_0)}{16\pi^2} \ln \frac{k}{\varphi_0}}. \quad (16.71)$$

We observe that $\lambda(k)$ diverges at a Landau pole k_L, i.e. $\lambda(k \to k_L) \to \infty$. One concludes that the $O(N)$-model with non-zero renormalized coupling $\lambda_R(\mu)$ can not be continued to infinitely short scales. It is an "incomplete theory". One finds similarly that the standard model is an incomplete theory. Some new physics is necessary at very short length scales in order to make the standard model a well-defined QFT. The Landau pole appears far beyond the Planck scale for gravity. The completion of the standard model could therefore be provided by quantum gravity, changing the flow of couplings for $k > M_p$, where $M_p \approx 10^{18}$GeV.

Predictivity

Let us compute the flow equation for $\nu = U'''(\rho_0)$. We neglect $U''''(\rho)$ and higher ρ-derivatives.

$$\partial_t U_k'''(\rho) = -\frac{3}{16\pi^2 k^2} \left\{ \frac{(3U'' + 2\rho U''')^3}{(1 + w_1)^4} + (N - 1)\frac{(U'')^3}{(1 + w_2)^4} \right\} \\ - \frac{1}{16\pi^2} \left\{ \frac{5U'''(3U'' + 2\rho U'')}{(1 + w_1)^3} + (N - 1)\frac{U'''U''}{(1 + w_2)^3} \right\} . \tag{16.72}$$

With $U'' = \lambda$, $U''' \propto \lambda^3$, the leading term in an expansion in small λ is

$$\partial_t \nu = -\frac{3(N + 26)\lambda^3}{16\pi^2 k^2}. \tag{16.73}$$

For the dimensionless ratio $\tilde{\nu} = \nu k^2$, one has

$$\partial_t \tilde{\nu} = 2\tilde{\nu} - \frac{3(N + 26)\lambda^3}{16\pi^2} = \beta_{\tilde{\nu}}. \tag{16.74}$$

The function $\beta_{\tilde{\nu}}$ has a zero for

$$\tilde{\nu}_* = \frac{3(N + 26)\lambda^3}{32\pi^2 k^2}. \tag{16.75}$$

Indicating the flow for decreasing k by arrows, one obtains
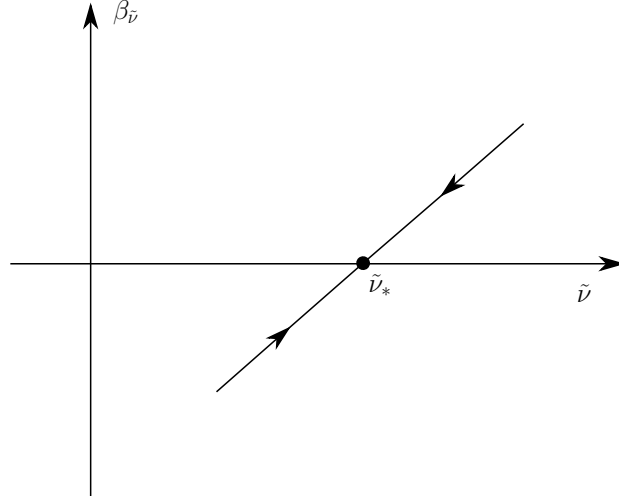
The solution of the flow equation attracts $\tilde{\nu}$ to the partial-IR fixed point $\tilde{v}_*$. For a given k, say $k = \varphi_0$, this predicts

$$\nu(k) = \frac{3(N + 26)\lambda(k)^3 k^2}{32\pi^2}. \tag{16.76}$$

More precisely, one has

$$\partial_t \left(\frac{\tilde{\nu}}{\lambda^3}\right) = 2 \left(\frac{\tilde{\nu}}{\lambda^3}\right) - \frac{3(N+26)}{16\pi^2} - \frac{\tilde{\nu}}{\lambda^4}\partial_t \lambda$$

$$= \left(2 - \underbrace{\frac{N+8}{16\pi^2}\lambda}_{\text{can be neglected}}\right)\frac{\tilde{\nu}}{\lambda^3} - \frac{3(N+26)}{16\pi^2}. \tag{16.77}$$

The effects of the running of λ can be neglected and one finds for $x := \tilde{\nu}/\lambda^3$,

$$\partial_t x = 2(x - x_*), \qquad x_* := \frac{\tilde{\nu}_*}{\lambda^3}. \tag{16.78}$$

The solution is a power law behaviour

$$x - x_* = c_0 \frac{k^2}{\Lambda^2}. \tag{16.79}$$

This implies

$$\tilde{\nu} - \tilde{\nu}_* = c_0 \lambda^3 \frac{k^2}{\Lambda^2}, \qquad \text{or} \qquad \nu = \frac{\tilde{\nu}_*}{k^2} + \frac{c_0 \lambda^3}{\Lambda^2}. \tag{16.80}$$

For $k^2 \ll \Lambda^2$, the initial value (a bare coupling) $\nu(\Lambda)$ that is specified by c_0 plays no role. The flow "looses the memory about its microphysics". This happens to all couplings except for the two renormalizable couplings λ_R and $\rho_{0,R}$. All other couplings can be produced in terms of λ_R and $\rho_{0,R}$!

16.6 Flow of the effective potential for an N-component scalar theory in arbitrary dimension

The scaling form of the flow equation eliminates the explicit dependence on the scale k and on Z_k. Let us introduce $u_k = U_k/k^d$, as well as renormalized dimensionless fields $\tilde{\rho} := Z_k k^{2-d}\rho$. Then

$$\kappa := Z_k k^{2-d}\rho_0,$$
$$u' = \frac{\partial u}{\partial \tilde{\rho}} = k^{-d}\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial \tilde{\rho}} = \frac{1}{Z_k k^2}U' = w_2,$$
$$u'' = \frac{1}{Z_k k^2}\frac{k^{d-2}}{Z_k}U'' \tag{16.81}$$
$$\tilde{\rho}u'' = \frac{1}{Z_k k^2}\rho U'' \Rightarrow w_1 = u' + 2\tilde{\rho}u''.$$

$$\partial_t u_{|\rho} = \alpha_d \left(\frac{N-1}{1+u'} + \frac{1}{1+u'+2\tilde{\rho}u''}\right) - du \tag{16.82}$$

We want to calculate the k-dependence at a fixed $\tilde{\rho}$:

$$\partial_t u_{|\tilde{\rho}} = \partial_t u_{|\rho} + \frac{\partial u}{\partial \rho}\partial_t \rho_{|\tilde{\rho}}$$
$$= \partial_t u_{|\rho} + \underbrace{u' Z_k k^{2-d}}_{\partial u/\partial \rho}\underbrace{(d-2+\eta)\rho}_{\partial_t \rho_{|\tilde{\rho}}}. \tag{16.83}$$

Thus

$$\boxed{\partial_t u_{|\tilde{\rho}} = -du + (d-2+\eta)\tilde{\rho}u' + \alpha_d \left(\frac{N-1}{1+u'} + \frac{1}{1+u'+2\tilde{\rho}u''}\right).} \tag{16.84}$$

In the scaling form, there is no k and no Z_k. The scaling solution can be computed as follows.

$$\partial_t u_{|\tilde{\rho}} = 0 \tag{16.85}$$

A simultaneous fixed point for $\kappa, u''(k), u'''(k)$, etc. A fixed point for infinitely many couplings, no scale present, all dimensionful couplings scale with appropriate powers of k. An example for an approximate scaling solution is

$$u_* = \frac{1}{2}\lambda_*(\tilde{\rho} - \kappa_*)^2 \tag{16.86}$$

(i) fixed κ_*:

$$\rho_0 = \frac{k^{d-2}\kappa_*}{Z_k}, \qquad d = 3, \ \rho_0 \propto k. \tag{16.87}$$

(ii) fixed λ_*:

$$\bar{\lambda} = U_k''(\rho_0) = Z_k^2 k^{4-d}\lambda_*, \qquad d = 3, \ \bar{\lambda} \propto k. \tag{16.88}$$

Solution for IR-divergence for $\bar{\lambda}$ and $d = 3$!

The fixed point for λ_* in $d = 3$ is a Wilson-Fisher fixed point. There is no fixed point, nor any scaling solution in $d = 4$ (triviality).

17 Gauge theories

17.1 Gauge groups and generators

Gauge theories are models with a *local* symmetry.
For the example of complex fermions or scalars ψ, one has

$$\psi(x) \rightarrow U(x)\psi(x). \tag{17.1}$$

An important example are the strong interactions which are described by the gauge group $SU(3)$. Here $\psi(x)$ are quarks as u, d, s, c, b, t that are each in a color-triplet, e.g. ψ is a complex three component Grassmann variable,

$$\begin{aligned}
\psi &= \psi_i(x), & i, j &= 1, \cdots, 3 : \text{color index} \\
U &= U_{ij}(x), & \psi_i'(x) &= U_{ij}(x)\psi_j(x).
\end{aligned} \tag{17.2}$$

The transformation matrices $U(x)$ form the group $SU(3)$ of special unitary transformation in three complex dimensions,

$$U^\dagger U = 1, \qquad \det U = 1. \tag{17.3}$$

The group structure is obvious, with the unit matrix being the 1-element, $U^{-1} = U^\dagger$, $U_1 U_2 = U_3$, $U_3^\dagger = (U_1 U_2)^\dagger = U_2^\dagger U_1^\dagger$, $U_3^\dagger U_3 = U_2^\dagger U_1^\dagger U_1 U_2 = 1$, $\det U_3 = \det U_1 \det U_2 = 1$, $U_1(U_2 U_3) = (U_1 U_2)U_3$.

Similarly, the weak interactions involve an $SU(2)$-gauge symmetry, for which left-handed leptons and quarks are doublets, i.e. two-component complex Grassmann fields

$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L, \qquad \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L, \qquad \text{etc.,} \tag{17.4}$$

where

$$\psi_L := \frac{1 + \gamma_5}{2}\psi. \tag{17.5}$$

In this case, U is a complex 2x2-matrix. For the standard model, one has an additional abelian $U(1)$-symmetry under which left- and right-handed fermions transform with different charges.
We consider an $SU(n)$-symmetry with fermions in the fundamental n-component representation. We can write

$$U(x) = \exp\left(i\alpha^z(x)t_z\right) \tag{17.6}$$

where t_z are the generators of the gauge group in the fundamental representation.
The generators are hermitian, traceless nxn matrices,

$$t_z^\dagger = t_z, \qquad \text{tr } t_z = 0. \tag{17.7}$$

This implies $U^\dagger = \exp\left(-i\alpha^z(x)t_z\right)$. For $SU(2)$, one has $z = 1, \cdots, 3$,

$$t_z = \frac{1}{2}\tau_z. \tag{17.8}$$

For $SU(3)$, there are eight generators, $z = 1, \cdots, 8$,

$$t_z = \frac{1}{2}\lambda_z, \tag{17.9}$$

and λ_z are the eight "Gell-Mann matrices", to be given explicitly later. The normalization is

$$\text{tr}(t_z t_y) = \frac{1}{2}\delta_{zy}. \tag{17.10}$$

It is sufficient to consider infinitesimal gauge transformations, α^z infinitesimal,

$$
\begin{aligned}
U &\approx 1 + i\alpha^z(x)t_z, \\
\psi &\to \psi + i\alpha^z(x)t_z\psi =: \psi + \delta\psi.
\end{aligned} \tag{17.11}
$$

Finite gauge transformations can then be obtained by consecutive infinitesimal transformations.

17.2 Gauge fields and covariant derivatives

Partial derivatives do not transform homogeneously under local gauge transformations

$$\delta\partial_\mu\psi = \partial_\mu\delta\psi = i\alpha^z t_z\partial_\mu\psi + i\partial_\mu\alpha^z t_z\psi. \tag{17.12}$$

Similarly to the local $U(1)$-symmetry of electromagnetism, one introduces gauge fields and defines a covariant derivative. For the $SU(2)$ gauge symmetry of the weak interaction, these are three W-bosons, for quantum chromodynamics (QCD), there are eight gluons.
We denote the gauge fields by A_μ^z. There is one field for each generator t_z (e.g. $z = 1, \cdots, 3$ for $SU(2)$ or $z = 1, \cdots, 8$ for $SU(3)$).
The covariant derivative is defined as

$$D_\mu\psi := (\partial_\mu - igA_\mu^z t_z)\psi, \tag{17.13}$$

with g the gauge coupling. We want a transformation of the gauge fields such that the covariant derivative transforms homogeneously,

$$\delta(D_\mu\psi) = \delta(\partial_\mu\psi) - ig\delta(A_\mu^z)t_z\psi - igA_\mu^z t_z\delta\psi \stackrel{(!)}{=} i\alpha^z t_z(D_\mu\psi). \tag{17.14}$$

This requires the relation

$$i\partial_\mu\alpha^z t_z\psi - ig\delta A_\mu^z t_z\psi + gA_\mu^z t_z\alpha^y t_y\psi = i\alpha^z t_z(-igA_\mu^y t_y\psi) = g\alpha^z A_\mu^y t_z t_y\psi. \tag{17.15}$$

In order to proceed, we employ the property that the commutator of two generators is again a linear combination of generators,

$$[t_y, t_z] = if_{yz}^w t_w. \tag{17.16}$$

The coefficients f_{yz}^w are the so-called "structure constants" of $SU(n)$. For example, for $SU(2)$, one has

$$\left[\frac{1}{2}\tau_y, \frac{1}{2}\tau_z\right] = \frac{1}{4}[\tau_y, \tau_z] = \frac{1}{4}2i\varepsilon_{yz}^w\tau_w = i\varepsilon_{yz}^w\frac{1}{2}\tau_w, \tag{17.17}$$

and the structure constants are thus $f_{yz}^w = \varepsilon_{yz}^w$.

There is a general proof that every traceless, hermitian matrix can be written as a linear combination of the generators with real coefficients. One employs that $-i[t_y, t_z]$ is hermitian:

$$(-i[t_y, t_z])^\dagger = i[t_y, t_z]^\dagger = i[t_z^\dagger, t_y^\dagger] = i[t_z, t_y] = -i[t_y, t_z]. \tag{17.18}$$

The gauge transformation of the gauge fields can be expressed in terms of these structure constants,

$$\delta A^z_\mu = \frac{1}{g}\partial_\mu \alpha^z + f^z_{uw} A^u_\mu \alpha^w. \tag{17.19}$$

Inserting this expression into the l.h.s. of equation (17.15), one finds

$$
\begin{aligned}
i\partial_\mu \alpha^z t_z \psi &- ig\left(\frac{1}{g}\partial_\mu \alpha^z + f^z_{uw} A^u_\mu \alpha^w\right) t_z \psi + g A^z_\mu t_z \alpha^y t_y \psi \\
&= -igf^z_{uw} A^u_\mu \alpha^w t_z \psi + g A^z_\mu t_z \alpha^y t_y \psi \\
&= -g A^u_\mu \alpha^w [t_u, t_w]\psi + g A^u_\mu \alpha^w t_u t_w \psi \\
&= g A^u_\mu \alpha^w t_w t_u \psi \\
&= g \alpha^z A^y_\mu t_z t_y \psi,
\end{aligned}
\tag{17.20}
$$

which is exactly the r.h.s. of eq. (17.15). This proves that requiring δA^z_μ to transform as in eq. (17.19) leads to the homogeneous transformation property of the covariant derivative,

$$\delta(D_\mu \psi) = i\alpha^z t_z (D_\mu \psi). \tag{17.21}$$

17.3 Gauge covariant action

With

$$\delta\bar\psi = -i\alpha^z \bar\psi t_z, \tag{17.22}$$

it is now easy to construct a gauge invariant kinetic form for the fermions,

$$S_\psi = \int_x i\bar\psi \gamma^\mu D_\mu \psi = \int_x i\bar\psi \gamma^\mu \partial_\mu \psi + g A^z_\mu \bar\psi \gamma^\mu t_z \psi. \tag{17.23}$$

It generates a vertex

$$\tag{17.24}$$

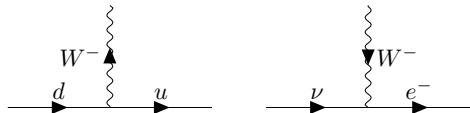similar to the vertex $e\gamma^\mu$ for photons. An example is

$$\tag{17.25}$$

The fermion species can be changed in the vertex due to $(t_z)i\gamma$!
An example is the decay of a neutron,

$$n \to p^+ + e^- + \bar\nu, \tag{17.26}$$

$n : udd$ and $p : uud$, the charge of u is $2/3$ and of d is $-1/3$.

$$\tag{17.27}$$

The four point function has a tree contribution

(17.28)

For small momenta, the W-propagator can be approximated by $m_W^{-2}\delta_{\mu\nu}$. This leads to the pointlike four-fermion interaction

$$\propto (g\bar{u}_L\gamma^\mu d_L)\frac{1}{m_W^2}(g\bar{e}_L\gamma_\mu\nu_L) \tag{17.29}$$

With $m_W^2 \propto g^2\varphi_0^2$, the strength of the interaction is φ_0^{-2}. This is a crucial difference between non-abelian and abelian gauge theories. For an abelian gauge theory, the photon has no self-interaction since it is neutral. As a consequence, Maxwell's equations are linear.

In contrast, for non-abelian gauge theories such as Yang-Mills theories, there is a self-interaction between gluons. The gluons carry color charge, not only the quarks. The field equations thus become non-linear. The necessity for self-interaction is clear for the $W^\pm$-bosons. They love to interact with the photon.

We next need a gauge covariant kinetic form which generalizes the Maxwell action for the photon. It is given by

$$S_F = \frac{1}{4}\int_x F_z^{\mu\nu}F_{\mu\nu}^z. \tag{17.30}$$

Here we define the non-abelian field strength as

$$F_{\mu\nu}^z = \partial_\mu A_\nu^z - \partial_\nu A_\mu^z + gf_{uw}^z A_\mu^u A_\nu^w. \tag{17.31}$$

The action S_F contains terms with three and with four gluon fields, as expressed in the vertices

(17.32)

Similarly one has interactions

(17.33)

We still need to prove the gauge-invariance of the action S_F. One could insert the definition of δA_μ^z and employ properties of products of structure functions. A more elegant way introduces matrix valued gauge fields

$$A_\mu(x) := gA_\mu^z(x)t_z. \tag{17.34}$$

In this formulation, $A_\mu(x)$ is an nxn-matrix, just as t_z. The covariant derivative reads

$$D_\mu\psi = (\partial_\mu - iA_\mu)\psi. \tag{17.35}$$

Likewise, we introduce a matrix-valued transformation parameter

$$\alpha(x) := \alpha^z(x)t_z. \tag{17.36}$$

In this language, the gauge transformation is

$$\delta\psi = i\alpha\psi, \qquad \delta A_\mu = \partial_\mu\alpha - i[A_\mu, \alpha]. \tag{17.37}$$

This can be verified as follows,

$$
\begin{aligned}
\delta A_\mu &= g\delta A_\mu^z t_z \\
&= g\left(\frac{1}{g}\partial_\mu\alpha^z + f_{uw}^z A_\mu^u \alpha^w\right)t_z \\
&= \partial_\mu\alpha^z t_z - ig[t_u, t_w]A_\mu^u \alpha^w \\
&= \partial_\mu\alpha - i[gA_\mu^u t_u, \alpha^w t_w] \\
&= \partial_\mu\alpha - i[A_\mu, \alpha].
\end{aligned} \tag{17.38}
$$

The field strength

$$F_{\mu\nu} = gF_{\mu\nu}^z t_z \tag{17.39}$$

is then given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \tag{17.40}$$

It transforms as

$$\delta F_{\mu\nu} = i[\alpha, F_{\mu\nu}]. \tag{17.41}$$

The last two relations follow directly upon insertion of the definitions. The action

$$S_F = \int_x \frac{1}{2g^2}\text{tr}\left(F^{\mu\nu}F_{\mu\nu}\right) = \int_x \frac{1}{2}\text{tr}\left(t_z t_y\right)F_{\mu\nu}^z F^{y\ \mu\nu} \tag{17.42}$$

is invariant,

$$
\begin{aligned}
\delta S_F &= \int_x \frac{1}{g^2}\text{tr}\left(F^{\mu\nu}\delta F_{\mu\nu}\right) \\
&= \frac{1}{g^2}\int_x \text{tr}\left(F^{\mu\nu}[\alpha, F_{\mu\nu}]\right) \\
&= \frac{1}{g^2}\int_x \text{tr}\left(F^{\mu\nu}\alpha F_{\mu\nu} - F^{\mu\nu}F_{\mu\nu}\alpha\right) \\
&= 0
\end{aligned} \tag{17.43}
$$

17.4 Inverse propagator of gauge fields

For the computation of the inverse propagator of the gauge fields in vacuum ($A_\mu^z = 0$), the interaction terms in S_F can be neglected. Each gauge field has the same inverse propagator as the photon,

$$(S^2)_{zy}^{\mu\nu}(q, q') = \frac{\partial^2 S^2}{\partial A_\mu^z(q)\partial A_\nu^y(-q')} \propto \delta_{zy}\delta(q - q')\tilde{P}^{\mu\nu}(q^2). \tag{17.44}$$

We need

$$S_F = \frac{1}{4} \int_q F_{\mu\nu}^z(q) F_z^{\mu\nu}(-q) \tag{17.45}$$

with

$$F_{\mu\nu}^z(q) = iq_\mu A_\nu(q) - iq_\nu A_\mu(q)$$
$$F_z^{\mu\nu}(-q) = (F_z^{\mu\nu}(q))^* = -iq^\mu A^\nu(-q) + iq^\nu A^\mu(-q) \tag{17.46}$$

one has

$$\begin{aligned}
S_F &= \frac{1}{4} \int_q \{q^\mu A^\nu(-q) - q^\nu A^\mu(-q)\}\{q_\mu A_\nu(q) - q_\nu A_\mu(q)\} \\
&= \frac{1}{2} \int_q \{q^2 A^\mu(-q) A_\mu(q) - q_\mu q^\nu A^\mu(-q) A_\nu(q)\} \\
&= \frac{1}{2} \int_q A_\mu(-q) \left(q^2 \eta^{\mu\nu} - q^\mu q^\nu\right) A_\nu(q).
\end{aligned} \tag{17.47}$$

Now we split $A_\nu(q)$ into transversal and longitudinal parts

$$\begin{aligned}
A_\nu^L(q) &= \frac{q_\nu q^\rho}{q^2} A_\rho(q) = (P_L)_\nu{}^\rho A_\rho(q), \\
A_\nu^T(q) &= A_\nu(q) - A_\nu^L(q) = (P_T)_\nu{}^\rho A_\rho(q).
\end{aligned} \tag{17.48}$$

The projectors obey the relationships

$$P_L^2 = P_L, \qquad P_T^2 = P_T, \qquad P_L + P_T = 1, \qquad P_L P_T = 0 = P_T P_L. \tag{17.49}$$

Proof.

$$(P_L^2)_\mu{}^\rho = (P_L)_\mu{}^\nu (P_L)_\nu{}^\rho = \frac{q_\mu q^\nu}{q^2} \frac{q_\nu q^\rho}{q^2} = \frac{q_\mu q^\rho}{q^2} = (P_L)_\mu{}^\rho, \tag{17.50}$$

$$P_T = 1 - P_L \qquad \Rightarrow$$

$$P_T P_L = P_L - P_L = 0, \qquad (P_T)^2 = (1 - P_L)^2 = 1 - 2P_L + P_L^2 = 1 - P_L = P_T. \tag{17.51}$$

∎

Decoupling of longitudinal gauge fields

In quadratic order, the action depends only on the transversal gauge fields, not on the longitudinal ones.

$$S_{F,L} = \frac{1}{2} \int_q A_\mu(-q) q^2 \eta^{\mu\nu} (P_T)_\nu{}^\rho A_\rho(q) = \frac{1}{2} \int_q A_\mu(-q) q^2 (A^T)^\mu(q) \tag{17.52}$$

This follows from

$$q^2 \eta^{\mu\nu} (P_T)_\nu{}^\rho = q^2 \frac{1}{q^2} (\eta^{\mu\rho} - q^\mu q^\rho). \tag{17.53}$$

The second functional derivative reads

$$(S^{(2)})^{\mu\nu} = q^2 \eta^{\mu\rho} (P_T)_\rho{}^\nu(q) \delta(q - q') \tag{17.54}$$

or

$$(S^{(2)})_\mu{}^\nu = q^2 (P_T)_\mu{}^\nu(q) \delta(q - q'). \tag{17.55}$$

Every propagator has only eigenvalues $\lambda = 1$ or $\lambda = 0$. This follows from $P^2 = P$, $\lambda^2 = \lambda$. The propagator P_T has three eigenvalues 1 and one eigenvalue 0. The 0-eigenvalue corresponds to the longitudinal gauge field.

As a consequence, $S^{(2)}$ is *not invertible*. The propagator for the gauge field A_μ is not defined!

Gauge degrees of freedom

This problem is a direct consequence of the local gauge symmetry. There is a direction in field space on which a gauge invariant action does not depend. Those are the directions into which gauge transformations change a given field. The gauge transformation of the field $A_\mu = 0$ is precisely the longitudinal gauge field. With

$$\delta A_\mu^z = \frac{1}{g}\partial_\mu \alpha^z, \qquad \delta A_\mu^z(q) = \frac{i}{g}q_\mu \alpha^z(q), \tag{17.56}$$

one has

$$(P_L)_\mu{}^\nu \delta A_\nu^z(q) = \delta A_\mu^z(q), \tag{17.57}$$

according to

$$(P_L)_\mu{}^\nu q_\nu = \frac{q_\mu q^\nu}{q^2}q_\nu = q_\mu. \tag{17.58}$$

Redundant description

Gauge theories are redundant descriptions. Tehy involve fields that do not appear in the action. This generalizes for $A_\mu(x)$: There is always a particular direction in field space into which an infinitesimal gauge transformation changes a given $A_\mu(x)$.

For non-abelian gauge theories, the construction is not linear and no "global gauge invariant field" exists. The reason why one uses gauge theories rather than a formulation involving only physical fields is *locality*. Projections are non-local, and any action formulated in terms of gauge invariant dields would not be local. Local formulations are very usful for causality, unitarity, renormalizability, derivative expansions, etc.
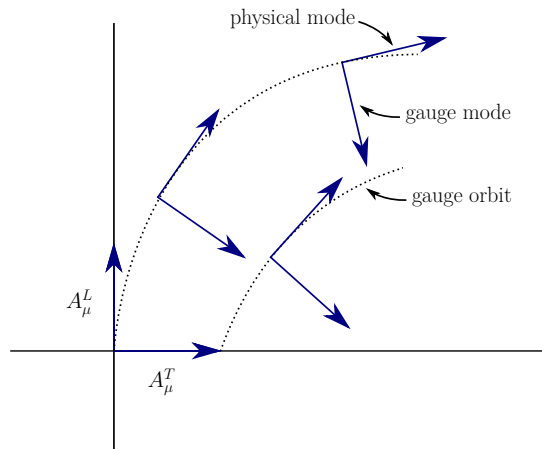
17.5 Functional integral for gauge theories

Problem with gauge modes

Let us try to proceed with the standard definition

$$Z_0 = \int \mathcal{D}A \, \exp\left(-S[A] + \int A_\mu^z(x)j_z^\mu(x)\right). \tag{17.59}$$

The functional integral contains an integral over the gauge modes that are the extension of the longitudinal gauge bosons to the whole space of A_μ.

The action does not depend on the gauge modes. As a consequence, the functional integral diverges. This could be cured by dividing by a *j-independent* diverging constant. Furthermore, perturbation theory cannot be used. Hence, $S^{(2)}$ is not invertible, we cannot proceed by a saddle point expansion.

17.6   Gauge fixing

(i) **Abelian gauge theory**

Gauge fixing term

For an abelian theory, we can define a global physical field A_μ^T and a global gauge degree of freedom A_μ^L. Since the action does not depend on A_μ^L, we may simply "take out" the longitudinal photon from the integration by invoking a δ-function:

$$Z_1 = \int \mathcal{D}A \ \delta(A_\mu^L(x)) \exp\left(-S[A] + \int j^\mu A_\mu\right). \tag{17.60}$$

This replaces also $A_\mu \to A_\mu^T$ in the source term. Thus Z_1 only depends on the "transversal sources" or "physical sources"

$$(j^T)^\mu = j^\nu (P_T)_\nu{}^\mu. \tag{17.61}$$

This follows from

$$\int j^\nu A_\nu^T = \int j^\nu (P_T)_\nu{}^\mu A_\mu \qquad \text{and} \qquad P_T^2 = 1. \tag{17.62}$$

The physical sources are conserved,

$$\partial_\mu (j^T)^\mu = 0, \tag{17.63}$$

corresponding to all conserved currents in electromagnetism.
We can replace the δ-function by a Gaussian,

$$\delta(A_\mu^L) = \lim_{\alpha \to 0} \exp\left(-\frac{1}{2\alpha} \int_x (\partial^\mu A_\mu^L)^2\right), \tag{17.64}$$

and define the "gauge fixing term" as an addition to the gauge invariant action S,

$$Z_2 = \int \mathcal{D}A \ \exp\left(-S - S_{gf} + \int (j^T)^\mu A_\mu^T\right) \tag{17.65}$$

with

$$S_{gf} = \frac{1}{2\alpha} \int_x (\partial^\mu A_\mu^L)^2. \tag{17.66}$$

The difference to the δ-function is only a source-independent constant. We take $\alpha \to 0$ at the end. This is called "Landau gauge fixing".

Propagator with gauge fixing

The gauge fixing term provides an inverse propagator for the longitudinal photon. From

$$S_{gf} = \frac{1}{2\alpha} \int A_\nu(-q) q^\nu q^\mu A_\mu(q), \tag{17.67}$$

we infer

$$(S_{gf}^{(2)})^{\mu\nu} = \frac{1}{\alpha}q^{\mu}q^{\nu} = \frac{1}{\alpha}q^2(P_L)^{\mu\nu}\delta(q-q'). \tag{17.68}$$

The classical inverse propagator with gauge fixing is invertible

$$S^{(2)} \propto \eta^{\mu\nu}q^2 + \left(\frac{1}{\alpha} - 1\right)q^{\mu}q^{\nu}. \tag{17.69}$$

Now perturbation theory can be defined.

Functional integral for the effective action

$$\exp\left(-\Gamma[A]\right) = \int \mathcal{D}a' \, \exp\left(-S[A+a'] - S_{gf}[a'] + \int \frac{\partial\Gamma}{\partial A}a'\right), \tag{17.70}$$

$$S_{gf} = \frac{1}{2\alpha}\int_x (\partial^{\mu}a'_{\mu})^2, \quad \alpha \to 0.$$

We employ the gauge fixing here for the fluctuation a'_{μ}.

(ii) **Non-abelian gauge theories**

Gauge condition
The physical modes ("transversal modes") obey

$$D^{\mu}[A]a'_{\mu} = 0. \tag{17.71}$$

A gauge fixing which puts the gauge modes to zero for a given macroscopic field A^z_{μ} and $(a')^z_{\mu} = (A')^z_{\mu} - A^z_{\mu}$ amounts for every z to

$$G^z = 0, \tag{17.72}$$

with

$$G^z = (D^{\mu}[A])^z{}_w(a')^w_{\mu} = \partial^{\mu}(a')^z_{\mu} - gA^{y\mu}f_y{}^z{}_w(a')^w_{\mu}. \tag{17.73}$$

Here D^{μ} is the covariant derivative in the adjoint representation. In general, the covariant derivative involves generators, and those depend on the representation. In principle, one could also use other gauge connections, i.e. other choices for G^z.

"Insertion of one"
We use the general identity

$$\int d\alpha \, \delta(G(\alpha)) \det \frac{\partial G}{\partial \alpha} = 1, \tag{17.74}$$

with

$$\det \frac{\partial G}{\partial \alpha} = \det\left(\frac{\partial G^z}{\partial \alpha^y}\right), \tag{17.75}$$

involving the matrix

$$\mathcal{N}^z{}_y = \frac{\partial G^z}{\partial \alpha^y}. \tag{17.76}$$

It follows from

$$\prod_z \int dG^z \; \delta(G^z) = 1 := \int dG \; \delta(G) \tag{17.77}$$

by a change of the integration variable $G = G(\alpha)$. The determinant corresponds to the Jacobian of the variable transformation.

We insert this at every x into the functional integral

$$\exp\left(-\Gamma[A]\right) = \int \mathcal{D}a' \; \exp\left(-S[A + a'] + \int \frac{\partial\Gamma}{\partial A}a'\right) \times$$
$$\times \int \mathcal{D}\alpha \; \delta(G(A, a'(\alpha)) \det \frac{\partial G(A, a'(\alpha))}{\partial\alpha}, \tag{17.78}$$

with

$$\int \mathcal{D}\alpha = \prod_x \int d\alpha(x). \tag{17.79}$$

We choose the dependence of G on α such that it arises only via the dependence of $(a')_\mu^z$ on α. We take α^y to be the parameter of a gauge transformation, with $(a')_\mu^z(\alpha)$ the gauge transform of $(a')_\mu^z$ obeying

$$(a')_\mu^z(\alpha + \delta\alpha) = (A')_\mu^z(\alpha) + \delta(A')_\mu^z(\alpha) - A_\mu^z. \tag{17.80}$$

The macroscopic field A is kept fixed.

Faddeev-Popov determinant

The new element is the Faddeev-Popov determinant

$$\tilde{M} = \det \frac{\partial G}{\partial\alpha}, \tag{17.81}$$

with

$$G^z = (D^\mu(A))^z_{\;w}(a')_\mu^w. \tag{17.82}$$

We compute

$$\frac{\partial G^z}{\partial\alpha^y} = \frac{\partial G^z}{\partial(a')_\mu^w}\frac{\partial(a')_\mu^w}{\partial\alpha^y} = (D^\mu[A])^z_{\;w}\frac{\partial(a')_\mu^w}{\partial\alpha^y}. \tag{17.83}$$

We employ the infinitesimal gauge transformation

$$(A')_\mu^w \to (A')_\mu^w + \frac{1}{g}\left(\partial_\mu\alpha^w + g f_{uy}^{\;\;\;w}(A')_\mu^u\alpha^y\right) = (A')_\mu^w + \frac{1}{g}(D_\mu[A'])^w_{\;y}\alpha^y \tag{17.84}$$

in order to obtain, with $A' = A'(\alpha)$,

$$\tilde{M} = \det\left[\frac{1}{g}(D^\mu[A])^z_{\;w}(D_\mu[A'])^w_{\;y}\right]. \tag{17.85}$$

We observe that $\tilde{M}$ does not depend on α but depends on A and A' or a'. For an abelian gauge theory, $\tilde{M}$ is independent of the gauge fields. It produces only an overall factor that we have omitted. Similarly, the factor $\frac{1}{g}$ can be dropped.

Variable transformation in functional integral

We exchange the order of integrations,

$$\int \mathcal{D}\alpha' \int \mathcal{D}\alpha \to \int \mathcal{D}\alpha \int \mathcal{D}\alpha'. \tag{17.86}$$

As a next step, we make a variable transformation in the functional integral over the gauge fields

$$A' \to \tilde{A}' := A'(\alpha), \qquad q' \to \tilde{q}' := a'(\alpha). \tag{17.87}$$

We employ

$$\int \mathcal{D}a' = \int \mathcal{D}\tilde{a}', \qquad S[\tilde{A}'_\mu] = S[A'_\mu], \tag{17.88}$$

which express the gauge invariance of the functional measure and the action. Also the source term is invariant. We arrive at

$$\exp\left(-\Gamma[A]\right) = \int \mathcal{D}\alpha \int \mathcal{D}\tilde{a}' \,\exp\left(-S[A+\tilde{a}'] + \int \frac{\partial\Gamma}{\partial A}\tilde{a}'\right) \times$$
$$\times\, \delta(G(A,\tilde{a}'(\alpha))\tilde{M}[A,\tilde{a}']. \tag{17.89}$$

The integral factorizes. The factor $\int \mathcal{D}\alpha$ is independent of the fields and only yields a constant factor, corresponding to the product of group volumes at every x. It can be omitted. Writing the δ-function again as a Gaussian, one gets the final result

$$\boxed{\exp\left(-\Gamma[A]\right) = \int \mathcal{D}a' \,\exp\left(-S[A+a'] - S_{gf}[A,a'] + \int \frac{\partial\Gamma}{\partial A}a'\right) \tilde{M}[A,\tilde{a}'].} \tag{17.90}$$

Here S_{gf} is the gauge fixing term (for $\alpha \to 0$)

$$S_{gf} = \frac{1}{2\alpha} \int_x (D^\mu[A]a'_\mu)^z (D^\nu[A]a'_\nu)_z. \tag{17.91}$$

If Γ is gauge invariant, the r.h.s. is gauge invariant.

Ghosts

For practical calculations with

$$\tilde{M} = \det(\tilde{\mathcal{N}}) = \det(\tilde{\mathcal{N}}^z_{\;y}), \tag{17.92}$$

one uses the identities for Grassmann variables,

$$\int d\bar{c}\, dc \,\exp\left(-\bar{c}_z \tilde{\mathcal{N}}^z_{\;y} c^y\right) = \det \tilde{\mathcal{N}}^z_{\;y}. \tag{17.93}$$

One adds to the functional integral an integral over "ghost fields" $\bar{c}$ and c, and adds a "ghost action"

$$S_{gh} = \int_x \bar{c}_z (D^\mu[A])^z_{\;w} (D_\mu[A+a'])^w_{\;y} c^y. \tag{17.94}$$

This produces a ghost propagator and a ghost-gauge boson vertex.

$$\tag{17.95}$$

Ghosts are Grassmann variables that belong to the adjoint representation of the gauge group. They cannot be observed as particles.

17.7 Lie groups

Below is an outline of the role of Lie groups in QFT. The reader interested in mathematical definitions of Lie groups, Lie algebras and representations is recommended to have a look at the corresponding literature.

For example, for an overview of Lie groups in a physical context, have a look at the lecture notes of Stefan Floerchinger's course "Lectures on symmetries and particle physics" (https://www.thphys.uni-heidelberg.de/∽floerchinger/teaching/SymmetriesAndParticlePhysicsFloerchinger.pdf) or at chapter two of the book "A modern Introduction to QFT" by Michele Maggiore. Introductions in the mathematical literature can for instance be found in chapter 14 of the book "An Introduction to Manifolds" by Loring W. Tu or in chapter 7 of the book "Introduction to Smooth Manifolds" by John M. Lee.

A *Lie group* is a continuous group where the elements can be smoothly connected to the unit element. They have the interesting property that they can be fully characterized by the transformations that are infinitesimally close to the unit element which form a *Lie algebra*.

Finite group transformations can be composed of many small ones with the exponential map

$$U = \lim_{N \to \infty} \left(\mathbb{1} + \frac{i\alpha^z T_z}{N} \right)^N = \exp\left[i\alpha^z T_z \right]. \tag{17.96}$$

To combine two transformations, one needs the Baker-Campbell-Hausdorff formula

$$e^x e^y = e^{Z(x,y)} \tag{17.97}$$

with

$$Z(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]] + \cdots \tag{17.98}$$

This shows that it is enough to know how to calculate commutators between the Lie algebra generators T_z.

The commutator between generators for a given Lie group (such as $SU(N)$ or $SO(N)$) is of the form

$$[T_y, T_z] = if_{yz}{}^w T_w, \tag{17.99}$$

where the structure constants $f_{yz}{}^w$ characterize the Lie algebra and therefore indirectly the Lie group. The generators T_z can be realized through different *representations*. Typically there are matrices such that the commutation relation (17.99) is fulfilled.

An example is the *fundamental representation*

$$(T_z^{(F)})^i{}_j = (t_z)^i{}_j. \tag{17.100}$$

For $SU(N)$, the generators in the fundamental representation t_z are hermitian and traceless $N \times N$ matrices, e.g. Pauli matrices for $SU(2)$ and Gell-Mann matrices for $SU(3)$. When the generators are hermitian (which is the case for compact Lie groups), the structure constants are real, i.e. $f_{yz}{}^w \in \mathbb{R}$.

The generators also satisfy the so-called Jacobi identity

$$[T_x, [T_y, T_z]] + [T_y, [T_z, T_x]] + [T_z, [T_x, T_y]] = 0, \tag{17.101}$$

which can easily be seen by writing this out explicitly. For the structure constants, this implies

$$f_{xy}{}^v f_{yz}{}^u + f_{yu}{}^v f_{zx}{}^u + f_{zu}{}^v f_{xy}{}^u = 0. \tag{17.102}$$

From the Jacobi identity, one can see that the structure constants can actually be used to construct another representation, the so-called *adjoint representation.* Here one sets the matrices to

$$(T_z^{(A)})^v{}_u = i f_{zu}{}^v. \tag{17.103}$$

Indeed, one has now

$$[T_x^{(A)}, T_y^{(A)}] = i f_{xy}{}^w T_w^{(A)}. \tag{17.104}$$

The adjoint representation is for $SU(3)$ given by 8×8 matrices.

The fundamental and the adjoint representation are the most important representations of a *Lie algebra* and we will need them in the following. However, there are many more and they all induce corresponding representations of the *Lie group* through the exponential map.

Different representations of covariant derivatives

We have seen that the generators of the Lie algebra exist in different representations and so do the covariant derivatives that can be constructed out of them,

$$D_\mu = \partial_\mu - i A_\mu^z T_z. \tag{17.105}$$

In fact, the appropriate generator for a covariant derivative depends on what object the derivative is acting on. For a field in some representation R, we must use

$$D_\mu^{(R)} \psi^{(R)} = \left(\partial_\mu - i A_\mu^z T_z^{(R)} \right) \psi^{(R)}. \tag{17.106}$$

For example, if the field is in the fundamental representation as for quarks, we ned to use $T_z^{(F)} = t_z$ and for neutral fields, one has the trivial representation, $T_z^{(o)} = 0$, so that a covariant derivative becomes an ordinary derivative.

Representation theory makes sure that the covariant derivative fulfills the Leibniz rule

$$D_\mu(AB) = (D_\mu A)B + A(D_\mu B), \tag{17.107}$$

even though A and B may be in different representations. This is in particular useful for partial integration.

Commutator of covariant derivatives

Consider the covariant derivative in some representation of the Lie algebra

$$D_\mu = \partial_\mu - i A_\mu^z T_z^{(A)} = \partial_\mu - i A_\mu. \tag{17.108}$$

Let us calculate the commutator of the two covariant derivatives,

$$\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu - i A_\mu, \partial_\nu - i A_\nu] \\
&= -i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]) \\
&\overset{(17.40)}{=} -i F_{\mu\nu} = -i F_{\mu\nu}^z T_z^{(R)}.
\end{aligned} \tag{17.109}$$

The commutator gives the fields strength in the same representation.

Three numbers are useful to characterize a given representation R. The first is its *dimension*

$D_{(F)} = N$ for the fundamental representation or $D_{(A)} = N^2 - 1$ for the adjoint representation. The second is the *index* $T_{(R)}$ which is defined through

$$\mathrm{tr}\left\{T_u^{(R)}T_v^{(R)}\right\} = T_{(R)}\delta_{uv}. \tag{17.110}$$

For the standard normalization of generators in the fundamental representation, e.g. $t_z = \sigma_z/2$ for $SU(2)$ and $t_z = \lambda_z/2$ for $SU(3)$, one has $T_{(F)} = 1/2$. For the adjoint representation, one can show that $T_{(A)} = N$.

The sum $T_z^{(R)}T_z^{(R)}$ commutes with all generators and must be a number times the identity matrix. The number is the *quadratic Casimir* $C_{(R)}$. It is easy to show from eq. (17.110) that

$$C_{(R)} = \frac{T_{(R)}}{D_{(R)}}D_{(A)}. \tag{17.111}$$

In particular, for the fundamental representation of $SU(N)$, one has $C_{(F)} = \frac{N^2-1}{2N}$ and for the adjoint representation $C_{(A)} = T_{(A)} = N$.

Finite gauge transformations

Let us now discuss different gauge transformation behaviours in this context. For the fermion or quark field, we have the infinitesimal transformation

$$\psi \to (\mathbb{1} + i\alpha^z t_z)\,\psi, \tag{17.112}$$

or, as a finite transformation

$$\psi \to U\psi, \qquad U = \exp\left[i\alpha^z t_z\right]. \tag{17.113}$$

Quarks transform in the *fundamental representation* of the Lie group.

For the gauge bosons, we had the infinitesimal transformation

$$\begin{aligned} A_\mu^z &\to A_\mu^z + \partial_\mu\alpha^z - \alpha^w f_{wu}{}^z A_\mu^u \\ &= A_\mu^z + \partial_\mu\alpha^z + i\alpha^w (T_w^{(A)})^z{}_u A_\mu^u. \end{aligned} \tag{17.114}$$

We see that the gauge field transforms under the *adjoint representation*! This can also be written nicely in terms of matrix valued gauge fields and the gauge parameter field

$$\begin{aligned} A_\mu &= A_\mu^z t_z, \qquad \alpha = \alpha^z t_z, \\ A_\mu &\to A_\mu + \partial_\mu\alpha + i[\alpha, A_\mu]. \end{aligned} \tag{17.115}$$

In particular, one can recognize the infinitesimal form of the finite transformation,

$$A_\mu \to U A_\mu U^\dagger + iU\partial_\mu U^\dagger, \tag{17.116}$$

such that the covariant derivative (in the fundamental representation) transforms as

$$D_\mu = (\partial_\mu - iA_\mu) \to U D_\mu U^\dagger. \tag{17.117}$$

Action of QCD with gauge fixing

Let us now recall the functional integral for QCD with gauge fixing à la Faddeev-Popov. Without quarks, we had

$$e^{-\Gamma[A]} = \int \mathcal{D}a' \mathcal{D}\bar{c}\mathcal{D}c \, \exp\left(-S[A, a', \bar{c}, c] + \int \frac{\delta\Gamma}{\delta A} a'\right), \tag{17.118}$$

where

$$S[A, a', \bar{c}, c] = S_{YM}[A + a'] + S_{gf}[A, a'] + S_{gh}[A, a', \bar{c}, c]. \tag{17.119}$$

Here, we use the actual Yang-Mills action

$$S_{YM} = \int_x \frac{1}{2g^2} \text{tr}\,\{F^{\mu\nu}F_{\mu\nu}\},$$
$$\text{where } F_{\mu\nu} = \partial_\mu(A_\nu + a'_\nu) - \partial_\nu(A_\mu + a'_\mu) - i[A_\mu + a'_\mu, A_\nu + a'_\nu], \tag{17.120}$$

together with the gauge fixing term,

$$S_{gf} = \int_x \frac{1}{2g^2\xi}(D^\mu[A]a'_\mu)^z (D^\nu[A]a'_\nu)^z, \tag{17.121}$$

and the ghost action

$$S_{gh} = -\int_x \bar{c}^z (D^\mu[A])^z{}_w (D_\mu[A + a'])^w{}_y c^y. \tag{17.122}$$

Note that we are now using a normalization of the fields such that the coupling strength appears as a factor $1/(2g^2)$ in front of the field strength term in the action.

The covariant derivatives appearing in the gauge fixing term and the ghost action are in the adjoint representation, e.g.

$$(D_\mu[A])^z{}_w = \delta^z_w \partial_\mu + A^y_\mu f_{yw}{}^z = \delta^z_w \partial_\mu - iA^y_\mu (T^{(A)}_y)^z{}_w. \tag{17.123}$$

Gauge invariance of action

Although we have fixed the gauge by the Faddeev-Popov method, there actually still is a gauge symmetry remaining. Under this symmetry, the gauge field expectation value transforms as usual,

$$A_\mu \to UA_\mu U^\dagger + iU\partial_\mu U^\dagger, \tag{17.124}$$

while the fluctuating part of the gauge field transforms as

$$a'_\mu \to Ua'_\mu U^\dagger, \tag{17.125}$$

and similarly, the ghost fields transform as

$$c = c^z t_z \to UcU^\dagger,$$
$$\bar{c} = \bar{c}^z t_z \to U\bar{c}U^\dagger. \tag{17.126}$$

These are in fact the transformation laws for matter fields in the adjoint representation of the gauge group. Of course we can easily add more matter fields, e.g. fermions in the fundamental representations for quarks.

> **Exercise 17.1.** Assuming that the functional integral measure is invariant, show that the above transformation of background fields and fluctuating fields leaves the action Γ invariant.

Effective action with quantum corrections

Let us now attempt to calculate the quantum effective action $\Gamma[A]$. Because of the invariance under gauge transformations on A_μ, only gauge invariant terms can appear in it. The leading term is expected to be of the form

$$\Gamma = \int_x \frac{1}{2g^2} \mathrm{tr}\, \{F_{\mu\nu} F^{\mu\nu}\}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \tag{17.127}$$

This is the form of the microscopic action but the coupling g may differ from the microscopic coupling by renormalization group running.

We will now perform a one-loop calculation, based on

$$\Gamma[A] = S[A] + \frac{1}{2}\mathrm{STr}\left\{\ln\left[S^{(2)}[A]\right]\right\}. \tag{17.128}$$

The operation STr corresponds to a trace over all indices of the fields, including momentum and frequency. The S stands for "super" and should remind us that we used to add a minus sign for fermionic degrees of freedom.

In order to calculate the quantum correction to $1/g^2$, we need to determine the propagator for the fluctuating fields, a'_μ, $\bar{c}$ and c in the presence of a background field A_μ. Eventually, we will expand the left hand side and the right hand side of eq. (17.128) to quadratic order in A_μ in order to identify the coefficient of the term $\mathrm{tr}\, \{F_{\mu\nu} F^{\mu\nu}\}$.

Yang-Mills term

Let us start with the Yang-Mills term in the action

$$S_{YM} = \int_x \frac{1}{2g^2} \mathrm{tr}\, \{F_{\mu\nu} F^{\mu\nu}\}. \tag{17.129}$$

The field strength for background plus fluctuation fields can be decompsed as

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu(A_\nu + a'_\nu) - \partial_\nu(A_\mu + a'_\mu) - i[A_\mu + a'_\mu, A_\nu + a'_\nu] \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] + \partial_\mu a'_\nu - \partial_\nu a'_\mu - i[A_\mu, a'_\nu] - i[a'_\mu, A_\nu] - i[a'_\mu, a'_\nu] \\
&= \bar{F}_{\mu\nu} + D_\mu a'_\nu - D_\nu a'_\mu - i[a'_\mu, a'_\nu],
\end{aligned} \tag{17.130}$$

where D_μ is the covariant derivative in the adjoint representation.

To calculate $S^{(2)}$, we specifically need the term quadratic in a'. One has

$$S_{YM} = \cdots + \int_x \frac{1}{2g^2} \mathrm{tr}\, \left\{(D_\mu a'_\nu - D_\nu a'_\mu)(D^\mu(a')^\nu - D^\nu(a')^\mu) - 2iF^{\mu\nu}[a_\mu, a_\nu]\right\} \tag{17.131}$$

Using $\mathrm{tr}(t_z t_w) = \delta_{zw}/2$, this can be rewritten as

$$S_{YM} = \cdots + \int_x \frac{1}{2g^2} \left\{(a'_\mu)^z \left[-g^{\mu\nu}(D_\alpha D^\alpha)_{zw} + (D^\nu D^\mu)_{zw}\right] (a'_\nu)^w + f_{uv}{}^w F^{\mu\nu}_w a^u_\mu a^v_\nu\right\}. \tag{17.132}$$

Adding the gauge fixing term leads to

$$\begin{aligned}
S_{YM} + S_{gf} = \cdots + \int_x \frac{1}{2g^2} \Bigg\{(a'_\mu)^z \Bigg[-g^{\mu\nu}(D_\alpha D^\alpha)_{zw} &+ \left(1 - \frac{1}{\xi}\right) D^\mu D^\nu + \\
&+ (\underbrace{[D^\nu, D^\mu]}_{=iF^{\mu\nu}} + iF^{\mu\nu})_{zw}\Bigg](a'_\nu)^w\Bigg\}.
\end{aligned} \tag{17.133}$$

The field strength is here in the adjoint representation as appropriate. In a similar, we can write the quadratic part of the ghost action as follows.

$$S_{gh} = \cdots + \int_x \bar{c}^z (-D_\alpha D^\alpha)_{zw} c^w. \tag{17.134}$$

This form of the quadratic action allows to directly read off the (inverse) propagators for $A_\mu = 0$. For the gluon, we have

$$g^2 \frac{g_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2}}{p^2} \tag{17.135}$$

and for the ghosts

$$\frac{1}{p^2}. \tag{17.136}$$

In the following, it will be beneficial to work in Feynman gauge, i.e. $\xi = 1$.

The A_μ-dependent terms can be treated as a perturbation. We will expand in them. We write the inverse propagator for gluons as

$$
\begin{aligned}
[-g^{\mu\nu}(D_\alpha D^\alpha)_{zw} + 2i(F^{\mu\nu})_{zw}] &= p_{0,2w}^{\mu\nu} + (\mathcal{F}_A^{\mu\nu})_{zw} + (\mathcal{F}_{AA}^{\mu\nu})_{zw} + (\mathcal{F}_J^{\mu\nu})_{zw} \\
&= [g^{\mu\nu}(-\partial_\alpha\partial^\alpha)]\,\delta_{zw} + ig^{\mu\nu}\left[(\partial_\alpha A^\alpha)_{zw} + 2(A_\alpha)_{zw}\partial^\alpha\right] + g^{\mu\nu}(A_\alpha A^\alpha)_{zw} + F_{zw}^{\alpha\beta}(J_{\alpha\beta})^{\mu\nu},
\end{aligned} \tag{17.137}
$$

with

$$(J_{\alpha\beta})^{\mu\nu} = i(\delta_a^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu) \tag{17.138}$$

the generator of Lorentz transformations in the vector representation. Here, the first term simply is the inverse propagator for free gluons, while the remaining terms correspond to vertices that couple them to the external gauge field.

In a similar way, one can expand the ghost inverse propagator

$$
\begin{aligned}
[-D_\mu D^\mu]_{zw} &= (p_0)_{zw} + (\mathcal{F}_A)_{zw} + (\mathcal{F}_{AA})_{zw} \\
&= [-\partial_\mu\partial^\mu]\delta_{zw} + i[(\partial_\mu A^\mu)_{zw} + 2(A_\mu)_{zw}\partial^\mu] + (A_\mu A^\mu)_{zw}.
\end{aligned} \tag{17.139}
$$

In order to obtain the contribution proportional to $\mathrm{tr}\{F_{\mu\nu}F^{\mu\nu}\}$ in the effective action, we expand both sides in the field A_μ up to quartic order. Because $S^{(2)}$ contains terms up to quadratic order in A_μ, we can write

$$\frac{1}{2}\mathrm{STr}\left\{\ln S^{(2)}\right\} = \frac{1}{2}\mathrm{STr}\left\{\ln(p_0 + \mathcal{F}_1 + \mathcal{F}_2 + \cdots)\right\} \tag{17.140}$$

where $\mathcal{F}_1$ is linear in A and $\mathcal{F}_2$ quadratic in A. Expanding this further gives

$$
\begin{aligned}
\frac{1}{2}\mathrm{STr}\left\{\ln S^{(2)}\right\} = {}& \frac{1}{2}\mathrm{STr}\left\{\ln p_0\right\} + \frac{1}{2}\mathrm{STr}\left\{\ln\left(1 + p_0^{-1}\mathcal{F}_1 + p_0^{-1}\mathcal{F}_2\right)\right\} \\
= {}& \frac{1}{2}\mathrm{STr}\left\{\ln p_0\right\} + \frac{1}{2}\mathrm{STr}\left\{p_0^{-1}\mathcal{F}_1 + p_0^{-1}\mathcal{F}_2\right\} \\
& - \frac{1}{4}\mathrm{STr}\left\{p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_1 + 2p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_2 + p_0^{-1}\mathcal{F}_2 p_0^{-1}\mathcal{F}_2\right\} \\
& + \frac{1}{8}\mathrm{STr}\left\{p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_1 + 3p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_2 + \cdots\right\} \\
& - \frac{1}{8}\mathrm{STr}\left\{p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_1 p_0^{-1}\mathcal{F}_1 + \cdots\right\}
\end{aligned} \tag{17.141}
$$

The terms on the left hand side have a diagrammatic interpretation. The first term is independent of the background field and contributes only to the constant part of the effective action. The subsequent terms are loopp expressions with external background field insertions.

$$\frac{1}{2}\; \bigcirc \qquad \frac{1}{2}\; \bigcirc \qquad -\frac{1}{4}\; \bigcirc \qquad -\frac{1}{2}\; \bigcirc$$

$$-\frac{1}{4}\; \bigcirc \qquad \frac{1}{6}\; \bigcirc \qquad \frac{1}{2}\; \bigcirc \qquad -\frac{1}{8}\; \bigcirc$$

$$\tag{17.142}$$

Specifically, this series contains all terms with up to four external fields while higher orders have been suppressed.

The effective action $\Gamma[A]$ contains not only the term proportional to $\mathrm{tr}\,\{F_{\mu\nu}F^{\mu\nu}\}$ but also other structures. In order to project to the coefficient of this term, we could either compose the terms $\propto A^2$, the terms $\propto A^3$ or the terms $\propto A^4$. We will follow the last strategy ($\propto A^4$) first and then discuss the first strategy ($\propto A^2$).

Note that

$$\int_x \frac{1}{2g^2}\mathrm{tr}\,\{F_{\mu\nu}F^{\mu\nu}\}= \cdots + \int_x \frac{1}{4g^2} A_\mu^u A_\nu^v A^{w\mu} A^{x\nu} f_{uv}{}^y f_{wx}{}^y. \tag{17.143}$$

Since the right hand side contains no derivatives, we can evaluate it in momentum space at vanishing momentum, or, in other words, for homogeneous fields. In fact, we can rewrite the above as

$$\frac{1}{4g^2}\int_{p_1\ldots p_4} A_\mu^u(p_1) A_\nu^v(p_2) A^{w\mu}(p_3) A^{x\nu}(p_4)(2\pi)^4\delta^{(4)}(p_1 + p_2 + p_3 + p_4) f_{uv}{}^y f_{wx}{}^y. \tag{17.144}$$

To project to this structure, we will evaluate $\Gamma^{(4)}$ at vanishing momenta, $p_1 = p_2 = p_3 = p_4 = 0$. For vanishing momenta, we can read off the coupling of the external fields from the expansion of the inverse propagator in momentum space, e.g. for the gluons

$$\tag{17.145}$$

$$- 2A_\mu^z p^\mu T_z^{(A)} g^{\alpha\beta}$$

We find from this (we denote the gluon lines by ww, ghost lines by - - - - - and external background field insterions by ww),

$$-\frac{1}{8}\; \bigcirc \quad = \quad -\frac{1}{8}\; \bigcirc \quad + \quad \frac{1}{8}\; \bigcirc$$

$$= -\frac{1}{8} A_{\mu_1}^{z_1}\cdots A_{\mu_4}^{z_4}\int_p \mathrm{tr}\left\{ 2p^{\mu_1} g^{\alpha_1\beta_1} T_{z_1}^{(A)}\frac{g_{\beta_1\alpha_2}}{p^2}\cdots 2p^{\mu_4} g^{\alpha_4\beta_4} T_{z_4}^{(A)}\frac{g_{\beta_4\alpha_1}}{p^2}\right\}$$

$$+ \frac{2}{8} A_{\mu_1}^{z_1}\cdots A_{\mu_4}^{z_4}\int_p \mathrm{tr}\left\{ 2p^{\mu_1} T_{z_1}^{(A)}\frac{1}{p^2}\cdots 2p^{\mu_4} T_{z_4}^{(A)}\frac{1}{p^2}\right\}$$

$$\tag{17.146}$$

The second loop is from ghosts and has an additional minus sign for Grassmann fields and a factor 2 for complex fields. However, it is a scalar. The generators $T_z^{(A)}$ are all in the adjoint representation. The trace over Lorentz indices in the first diagram gives a factor 4.

Combining the diagrams gives

$$-\frac{(4-2)}{8} A_{\mu_1}^{z_1} \cdots A_{\mu_4}^{z_4} \operatorname{tr}\left\{T_{z_1}^{(A)} \cdots T_{z_4}^{(A)}\right\} \int_p \left\{\frac{2p^{\mu_1}}{p^2} \cdots \frac{a p^{\mu_4}}{p^2}\right\} \tag{17.147}$$
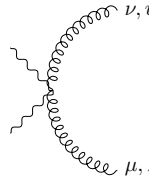
The momentum integral is of the structure

$$\int \frac{d^4p}{(2\pi)^4} \, p^{\mu_1} p^{\mu_2} p^{\mu_3} p^{\mu_4} f(p^2)$$
$$= \frac{1}{24}\left[g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right] \int \frac{d^4p}{(2\pi)^4} \, p^4 f(p^2) \tag{17.148}$$

which leads to

$$-\frac{1}{6}\operatorname{tr}\left\{2A_\mu A^\mu A_\nu A^\nu + A_\mu A_\nu A^\mu A^\nu\right\} \int_p \frac{1}{p^4}. \tag{17.149}$$

For the remaining diagrams, we also need the vertex

$$g^{\mu\nu}[A_\lambda A^\lambda] + (F^{\alpha\beta})_{zw}(J_{\alpha\beta})^{\mu\nu} \tag{17.150}$$

and similar for the ghost loop but without the last term $\propto J$. We then find

$$= \frac{1}{2}\operatorname{tr}\left\{A_\lambda A^\lambda A_\mu A^\mu\right\} \cdot (4-2) \cdot \int_p \left\{\frac{2p^\mu}{p^2}\frac{2p^\nu}{p^2}\frac{1}{p^2}\right\}. \tag{17.151}$$

there is no contribution from the term $\propto J$ because it is anti-symmetric in the Lorentz indices. Using

$$\int \frac{d^4p}{(2\pi)^4} p^\mu p^\nu f(p^2) = \frac{1}{4}g^{\mu\nu}\int \frac{d^4p}{(2\pi)^4}p^2 f(p^2) \tag{17.152}$$

leads to

$$\operatorname{tr}\left\{A_\mu A^\mu A_\nu A^\nu\right\} \int_p \frac{1}{p^4}. \tag{17.153}$$

Finally, we have

$$= -\frac{1}{4}\operatorname{tr}\left\{A_\mu A^\mu A_\nu A^\nu\right\}(4-2) \cdot \int_p \frac{1}{p^4} - \frac{1}{4}\operatorname{tr}\left\{F^{\mu\nu}F^{\rho\sigma}\right\}(J_{\mu\nu})^{\alpha\beta}(J_{\rho\sigma})_{\beta\alpha}\int_p \frac{1}{p^4}. \tag{17.154}$$

We can calculate explicitly

$$(J_{\mu\nu})^{\alpha\beta}(J_{\rho\sigma})_{\beta\alpha} = 2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{17.155}$$

This allows to combine the terms to

$$-\frac{1}{2}\mathrm{tr}\left\{A_\mu A^\mu A_\nu A^\nu\right\} \cdot \int_p \frac{1}{p^4} - \mathrm{tr}\left\{F^{\mu\nu}F_{\mu\nu}\right\}\int_p \frac{1}{p^4}. \tag{17.156}$$

Combining now all three types of diagrams leads to

$$\left(\frac{1}{6}\mathrm{tr}\left\{A_\mu A^\mu A_\nu A^\nu\right\} - \frac{1}{6}\mathrm{tr}\left\{A_\mu A_\nu A^\mu A^\nu\right\} - \mathrm{tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}\right)\int_p \frac{1}{p^4}. \tag{17.157}$$

Recall that for constant A_μ, one has

$$F_{\mu\nu} = -i[A_\mu, A_\nu] \tag{17.158}$$

such that

$$\mathrm{tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} = -2\mathrm{tr}\left\{A_\mu A_\nu A^\mu A^\nu\right\} + 2\mathrm{tr}\left\{A_\mu A^\mu A_\nu A^\nu\right\}. \tag{17.159}$$

We can therefore combine everything to (recall that all fields are here in the adjoint representation)

$$\begin{aligned}
&-\frac{11}{12}\mathrm{tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}\int_p \frac{1}{p^4}\\
&= -\frac{11}{12}F_{\mu\nu}^u F^{v\ \mu\nu}\mathrm{tr}\left\{T_u^{(A)}T_v^{(A)}\right\}\int_p \frac{1}{p^4}\\
&= -\frac{11}{12}T_{(A)}F_{\mu\nu}^z F^{z\ \mu\nu}\int_p \frac{1}{p^4}.
\end{aligned} \tag{17.160}$$

For $SU(N)$, one has $T_{(A)} = N$. We thus find for the effective action

$$\Gamma = \int_x \left(\frac{1}{4g^2} - \frac{11N}{12}\int_p \frac{1}{p^4}\right)F_{\mu\nu}^z F^{z\ \mu\nu}. \tag{17.161}$$

Finally, we note that

$$\int_p \frac{1}{p^4} = \frac{1}{(2\pi)^4}2\pi^2\int_0^\infty \frac{dp}{p} \to \frac{1}{(4\pi)^2}2\int_\mu^\Lambda \frac{dp}{p} = \frac{2}{(4\pi)^2}\ln\left(\frac{\Lambda}{\mu}\right) \tag{17.162}$$

We have introduced a UV cutoff Λ and and IR cutoff μ.

Running coupling constant

In summary, we find the effective coupling constant g with quantum corrections (one loop) to obey

$$\frac{1}{g^2} = \frac{1}{\bar{g}^2} - \frac{11N}{3}\frac{1}{(4\pi)^2}2\ln\left(\frac{\Lambda}{\mu}\right) \tag{17.163}$$

where $\bar{g}$ is the bare coupling constant.

In particular, this depdends on the infrared regulator scale μ. In fact, if we would have done the calculation at non-vanishing external momenta, a corresponding scale would have appeared instead of μ naturally.

We find for the renormalization group action

$$\mu\frac{\partial}{\partial\mu}g = \beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N\right). \tag{17.164}$$

This is in fact the renormalization group equation for the coupling constant of $SU(N)$ gauge theory without fermions at one loop. More generally, for a group with quadratic Casimir $C_{(A)} = T_{(A)}$ and n_f Dirac fermions in a representation with index $T_{(DF)}$ as well as n_s complex scalars in a representation with index $T_{(CS)}$, we find

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left[\frac{11}{3} C_{(A)} - \frac{4}{3} n_f T_{(DF)} - \frac{1}{3} n_s T_{(CS)} \right] . \tag{17.165}$$

Specifically for QCD, one has $C_{(A)} = N = 3$, and at high energies where all quarks can be counted massless, $n_f = 6$, $T_{(DF)} = 1/2$. The beta function is then

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left[11 - 4 \right] . \tag{17.166}$$

Asymptotic freedom

The interesting property of non-abelian gauge theories is that for small enough n_f, the beta function is negative, $\beta(g) < 0$. This implies that the coupling becomes weaker and weaker when one goes to higher and higher energies (the UV scale).
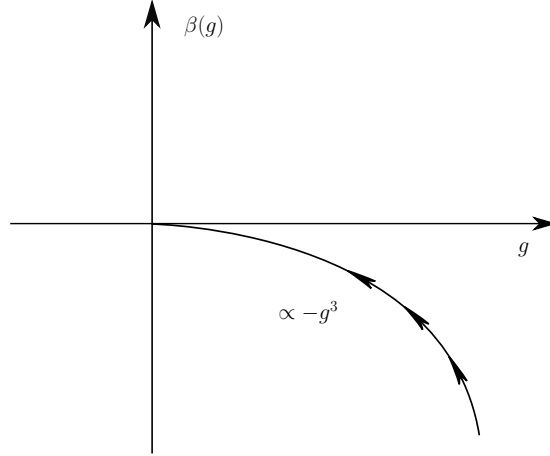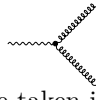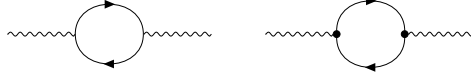
The coupling constant flows into a fixed point at $g = 0$. This property is called *asymptotic freedom*. At asymptotically large momenta, the theory becomes free.

Alternative calculation through self-energy

While we have calculated the contribution to $F^z_{\mu\nu} F^{z\ \mu\nu}$ through the term $\propto A^4$, one could as well have considered the quadratic term $\propto A^2$ including the appropriate derivative structure. In fact, as a result of gauge invariance, the coefficient must be the same. It is actually quite instructive to do this calculation as well. We have the following contributions

$$\tag{17.167}$$

The vertex describes the coupling through the magnetic moment term $\propto J$. Fermions (quarks) are taken into account. There are also corresponding diagrams.

$$\tag{17.168}$$

Here we calculate the contributions from gluons and ghosts. The first diagram gives

$$-\frac{1}{4}\int_k A_\mu^z(-k)A_\nu^w(k)\int_p \text{tr}\left\{\frac{g_{\rho\sigma}}{p^2}(2p+k)^\mu g^{\sigma\kappa}T_z\frac{g_{\kappa\lambda}}{(p+k)^2}(2p+k)^\nu g^{\lambda\rho}T_w\right\}. \tag{17.169}$$

The $-1/4$ arises from the expansion of the logarithm and $\frac{1}{2}\text{STr}(\cdots)$. The ghost contribution is similar but with an additional sign due to fermionic fields, and a factor 2 because they are complex.

$$\frac{1}{2}\int_k A_\mu^z(-k)A_\nu^w(k)\int_p \text{tr}\left\{\frac{1}{p^2}(2p+k)^\mu T_z\frac{1}{(p+k)^2}(2p+k)^\nu T_w\right\}. \tag{17.170}$$

We can now use the definition of the index

$$\text{tr}\left\{T_z T_w\right\}= \delta_{zw}T_{(A)} \tag{17.171}$$

and $g_{\rho\sigma}g^{\sigma\kappa}g_{\kappa\lambda}g^{\lambda\rho} = \delta_\rho^\rho = d$ and combine the terms

$$-(d-2)\frac{1}{4}T_{(A)}\int_k A_\mu^z(-k)A_\nu^w(k)\int_p\left\{\frac{1}{p^2}(2p+k)^\mu\frac{1}{(p+k)^2}(2p+k)^\nu\right\}. \tag{17.172}$$

In a similar way, also the second diagram can be evaluated and combined with the corresponding ghost loop. This yields

$$\frac{1}{2}(d-2)T_{(A)}\int_k A_\mu^z(-k)A_\nu^z(k)\int_p\left\{\frac{g^{\mu\nu}}{p^2}\right\}. \tag{17.173}$$

Finally, for the diagram $\propto J^2$, one obtains

$$-\frac{1}{4}T_{(A)}\int_k A_\mu^z(-k)A_\nu^w(k)\int_p\left\{\frac{g_{\alpha\beta}}{p^2}(-2ik_\rho J^{\mu\rho})^{\beta\gamma}\frac{g_{\gamma\delta}}{(p+k)^2}(2ik_\sigma J^{\nu\sigma})^{\delta\alpha}\right\} \tag{17.174}$$

With the explicit expression for the generator of Lorentz boosts $(J_{\alpha\beta})$ in the spin 1 representation, one finds

$$\begin{aligned}\text{tr}\left\{J^{\mu\rho}J^{\nu\sigma}\right\} &= (J^{\mu\rho})^\kappa_{\;\beta}(J^{\nu\sigma})^\beta_{\;\alpha}\\ &= (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\rho\nu})\cdot 2.\end{aligned} \tag{17.175}$$

For other representations, the prefactor would be different to 2, e.g. 1 for spin 1/2 and 0 for spin 0. This allows to simplify this term to

$$-2T_{(A)}\int_k A_\mu^z(-k)A_\nu^z(k)\int_p\left\{\frac{1}{p^2}\frac{1}{(p+k)^2}(k^2g^{\mu\nu}-k^\mu k^\nu)\right\}. \tag{17.176}$$

At this point, we can combine the different contributions which leads to

$$T_{(A)}\int_k A_\mu^z(-k)A_\nu^z(k)\int_p\frac{1}{p^2(p+k)^2}\left\{-\frac{(d-2)}{4}(2p+k)^\mu(2p+k)^\nu+\right.$$
$$\left.+\frac{(d-2)}{2}(p+k)^2g^{\mu\nu}-2k^2g^{\mu\nu}+2k^\mu k^\nu\right\}. \tag{17.177}$$

To calculate this further, we introduce Feynman parameters

$$\frac{1}{AB} = \int_0^1 d\alpha_1 \ d\alpha_2 \ \frac{1}{[\alpha_1 A + \alpha_2 B]^2} \delta(\alpha_1 + \alpha_2 - 1) \tag{17.178}$$

and write the integral over p as

$$\int_0^1 d\alpha_1 \ d\alpha_2 \ \delta(\alpha_1 + \alpha_2 - 1) \int_p \frac{1}{[(p + \alpha_2 k)^2 + (\alpha_2 - \alpha_2^2)k^2]^2} \left\{\cdots\right\}. \tag{17.179}$$

We can now shift the integral $p + \alpha_2 k \to p$ and perform the integral over α_1. The integral over p is then symmetric around $p = 0$ so that odd terms can be dropped. We are left with

$$\begin{aligned}
T_{(A)} \int_k A_\mu^z(-k) A_\nu^z(k) \int_0^1 d\alpha \int_p &\frac{1}{[p^2 + (\alpha - \alpha^2)k^2]^2} \Bigg\{(d-2)\left(\frac{1}{2}p^2 g^{\mu\nu} - p^\mu p^\nu\right) + \\
&+ \left[2 - \frac{(d-2)}{4}(1-2\alpha)^2\right] k^\mu k^\nu - \left[2 - \frac{(d-2)}{2}(1-\alpha)^2\right] k^2 g^{\mu\nu}\Bigg\}.
\end{aligned} \tag{17.180}$$

The puzzling feature of this result is that there is a k-independent piece that is acutally quadratically divergent. Indeed, we can replace $p^\mu p^\nu \to \frac{1}{d}g^{\mu\nu}p^2$ in the first line and find for $d = 4$ a contribution $\propto g^{\mu\nu} \int_p \frac{1}{p^2}$. This has the form of a mass term! So, naively, from this analysis, one might conclude that gluons shoudl develop a mass. However, one must be careful with this conclusion. First, a gluon mass term would actually destroy gauge invariance. Second, the argument above would actually not give a definite value but predict it to be infinite. Before one can say anything in such a situation, one must regularize the theory so that the quantum corrections become regular or finite. In the present situation, it is also important that the regularization conserves the gauge invariance of the theory. It turns out that if one uses a gauge invariant regularization instead of a simple sharp momentum cutoff, the quadratic term goes away and no gluon mass is generated.

Dimensional regularization

One regularization scheme that is often used for perturbative calculations and that has the advantage to preserve gauge invariance is dimensional regularization. The idea is to evaluate loop integrals not in $d = 4$ dimensions but to promote d to an arbitrary complex number and to take the limit $d \to 4$ only in the end.

Let us first derive an expression for the surface of the unit sphere Ω_d in d dimensions,

$$(\sqrt{\pi})^d = \left(\int_{-\infty}^\infty dx \ e^{-x^2}\right)^d = \int d\Omega_d \int_0^\infty dr \ r^{d-1} e^{-r^2} = \Omega_d \frac{1}{2}\Gamma\left(\frac{d}{2}\right), \tag{17.181}$$

so that the surface element is

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{17.182}$$

Here $\Gamma(x)$ is the Gamma function which is an analytic continuation of the factorial function. For integer arguments, it evaluates to

$$\Gamma(1) = 1, \qquad \Gamma(2) = 1, \qquad \Gamma(3) = 2, \qquad \Gamma(n+1) = n! \qquad \text{(for } n \geq 0\text{)}. \tag{17.183}$$

For half integer arguments, one has

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{n!2^{(2n)}}\sqrt{\pi} \qquad \text{for} \qquad n \geq 0. \tag{17.184}$$

At $x = 0$ and negative integers, there are simple poles

$$\Gamma(-u + x) = \frac{(-1)^n}{n!} \left[\frac{1}{x} - \gamma_E + \sum_{k=1}^{n} \frac{1}{k} + \sigma(x) \right] \tag{17.185}$$

and in particular

$$\Gamma(x) = \frac{1}{x} - \gamma_E + \sigma(x). \tag{17.186}$$

Here γ_E is the Euler-Mascheroni constant, $\gamma_E \approx 0.577$.
Special cases of the surface element are

$$\Omega_1 = 2, \qquad \Omega_2 = 2\pi, \qquad \Omega_3 = 4\pi, \qquad \Omega_4 = 2\pi^2. \tag{17.187}$$

Furthermore, the following relation is very useful:

$$\int_0^\infty dk \, \frac{k^a}{(k^2 + \Delta)^b} = \Delta^{\frac{a+1}{2} - b} \frac{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma(b)}. \tag{17.188}$$

Combining terms, this allows to write rather generally

$$\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^a}{(k^2 + \Delta)^b} = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma\left(a + \frac{d}{2}\right) \Gamma\left(b - a - \frac{d}{2}\right)}{\Gamma(b) \Gamma\left(\frac{d}{2}\right)}. \tag{17.189}$$

This formula is useful for many loop calculations. Because the Gamma function is analytic except for the simple poles on the non-positive integers, the integral above is actually regular for general complex d. As an example, consider $a = 0$, $b = 2$. Then

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Gamma(2)}. \tag{17.190}$$

The Gamma function has a pole at $d = 4$ corresponding to the logarithmic UV divergence of the integral. Moreover, there is a second divergence for $\Delta \to 0$ when $2 - d/2 > 0$. This is an infrared divergence.
Using $d = 4 - \epsilon$ and $\Gamma(2) = 1$, one obtains

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) \approx \frac{2}{\epsilon} - \gamma_E + \sigma(\epsilon), \tag{17.191}$$

as well as

$$\frac{1}{(4\pi)^2} \frac{1}{\Delta^{2-d/2}} = \frac{1}{(4\pi)^2} \left(\frac{4\pi}{\Delta}\right)^{\frac{\epsilon}{2}} = \frac{1}{(4\pi)^2} \left[1 + \frac{\epsilon}{2} \ln(4\pi) - \frac{\epsilon}{2} \ln(\Delta)\right] \tag{17.192}$$

leads to

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta)^2} = \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} - \ln(\Delta) + \ln(4\pi) - \gamma_E + \sigma(\epsilon)\right]. \tag{17.193}$$

Now, one may subtract the diverging terms by introducing counter terms to cancel the poles $\propto 1/\epsilon$ etc. in a concrete perturbative Feynman diagram calculation. Moreover, the term $\ln(\Delta)$ is like an infrared regulator we had introduced previously,

$$\int \frac{d^d k}{(2\pi)^4} \to \frac{1}{(4\pi)^2} 2 \int_\mu^\Lambda dk \, \frac{1}{k} = \frac{1}{(4\pi)^2} \left[\ln(\Lambda^2) - \ln(\mu^2)\right]. \tag{17.194}$$

This allows to directly read off the renormalization group equations from the integrals regularized through dimensional regularisation.

Note now that an additional factor k^2 in the numerator of the otherwise logarithmically divergent integral leads within dimensional regularization effectively to the replacements

$$
\begin{aligned}
\int_k \frac{k^2}{(k^2+\Delta)^2} &\to \Delta \frac{\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(2-\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)}\int_k \frac{1}{(k^2+\Delta)^2} \\
&= \Delta \frac{\frac{d}{2}}{1-\frac{d}{2}}\int_k \frac{1}{(k^2+\Delta)^2} \\
&= \Delta \frac{1}{\frac{2}{d}-1}\int_k \frac{1}{(k^2+\Delta)^2}
\end{aligned}
\tag{17.195}
$$

where we used $z\Gamma(z)=\Gamma(z+1)$.
With this we can go back to our calculation of the gluon energy quantum corrections. One finds that one can replace in eq. (17.180)

$$
\begin{aligned}
(d-2)\left(\frac{1}{2}p^2 g^{\mu\nu}-p^\mu p^\nu\right) &\to (d-2)\left(\frac{1}{2}-\frac{1}{d}\right)p^2 g^{\mu\nu} \\
&\to (\alpha-\alpha^2)k^2(d-2)\left(\frac{1}{2}-\frac{1}{d}\right)\frac{1}{\frac{2}{d}-1} \\
&= (\alpha-\alpha^2)k^2(d-2)\frac{1}{2}.
\end{aligned}
\tag{17.196}
$$

This way we get rid of the "superficially" quadratic divergence term. In a subsequent step, one can keep only the leading k-dependent terms and perform the integral over the Feynman parameter. The final result is a contribution to the effective action that has precisely the same form as we have calculated from the quartic terms. We emphasize again that a gauge invariant regularization scheme was crucial to make this work.

18 Wilson lines, lattice gauge theory and confinement

This section follows rather closely section 82 of [Mark Srednicki, Quantum field theory (2007)].

Wilson links

Let us consider pure Yang-Mills theory with the Euclidean action

$$S = \int d^4x \ \frac{1}{2g^2} \text{tr} \left\{ F^{\mu\nu} F_{\mu\nu} \right\} \tag{18.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{18.2}$$

and the gauge fields $A_\mu = A_\mu^z t_z$ are matrices in the adjoint representation of the gauge group $SU(N)$.

Let us take two spacetime points x^μ and $x^\mu + \varepsilon^\mu$ where ε^μ is infinitesimal. We define the *Wilson link* as

$$\begin{aligned} W(x+\varepsilon, x) &= \exp\left[i\varepsilon^\mu A_\mu(x)\right] \\ &= \mathbb{1} + i\varepsilon^\mu A_\mu(x) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{18.3}$$

Because A_μ is an $N \times N$-matrix, this is also the case for $W(x+\varepsilon, x)$. We now determine how the Wilson link transforms under gauge transformations. First note that infinitesimal gauge transformations (including fermions in the fundamendal representation),

$$\begin{aligned} \psi(x) &\to [\mathbb{1} + i\alpha(x)] \ \psi(x), \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu \alpha(x) - i \left[A_\mu(x), \alpha(x)\right], \end{aligned} \tag{18.4}$$

can be extended to the finite transformations

$$\begin{aligned} \psi(x) &\to U(x)\psi(x), \\ A_\mu(x) &\to U(x)A_\mu(x)U^\dagger(x) + iU(x)\partial_\mu U^\dagger(x), \end{aligned} \tag{18.5}$$

where $U(x) = [i\alpha(x)]$. The Wilson link (to order ε) transforms as

$$W(x+\varepsilon, x) \to \mathbb{1} + i\varepsilon^\mu U(x)A_\mu(x)U^\dagger(x) - \varepsilon^\mu U(x)\partial_\mu U^\dagger(x). \tag{18.6}$$

Because $U^\dagger(x)U(x) = \mathbb{1}$ and accordingly

$$U\partial_\mu U^\dagger + (\partial_\mu U)U^\dagger = 0, \tag{18.7}$$

one can write this as

$$W(x+\varepsilon, x) \to [(\mathbb{1} + \varepsilon^\mu \partial_\mu)U(x)] \ U^\dagger(x) + i\varepsilon^\mu U(x)A_\mu(x)U^\dagger(x). \tag{18.8}$$

Up to terms of order $\mathcal{O}(\varepsilon^2)$, we can replace $(\mathbb{1} + \varepsilon^\mu \partial_\mu)U(x)$ by $U(x+\varepsilon)$ and $\varepsilon^\mu U(x)$ by $\varepsilon^\mu U(x+\varepsilon)$. We then find the transformation

$$\begin{aligned} W(x+\varepsilon, x) &\to U(x+\varepsilon) \left[\mathbb{1} + i\varepsilon^\mu A_\mu(x)\right] U^\dagger(x) \\ &= U(x+\varepsilon)W(x+\varepsilon, x)U^\dagger(x). \end{aligned} \tag{18.9}$$

Wilson line

We can now consider a Wilson line as a chain of infinitesimal Wilson links. It goes along some path ξ, connecting two spacetime points x and $y = x + \varepsilon_1 + \cdots + \varepsilon_n$,

$$W_\xi(y,x) = W(y, y - \varepsilon_n)W(y - \varepsilon_n, y - \varepsilon_n - \varepsilon_{n-1}) \cdots W(x + \varepsilon_1 + \varepsilon_2, x + \varepsilon_1)W(x + \varepsilon_1, x). \quad (18.10)$$

The transformation behaviour under gauge transformations is rather simple,

$$W_\xi(y,x) \to U(y)W_\xi(y,x)U^\dagger(x). \quad (18.11)$$

For a Wilson link, one can write

$$W^\dagger(x + \varepsilon, x) = W(x, x + \varepsilon). \quad (18.12)$$

For a finite Wilson line, this extends to

$$W_\xi^\dagger(y,x) = W_{\bar\xi}(x,y), \quad (18.13)$$

where $\bar\xi$ denotes the reverse of the path ξ.

Wegner-Wilson loop

Consider now a closed path or oriented curve $\xi = \mathcal{C}$. The Wegner-Wilson loop is the trace of the Wilson line along the closed curve,

$$W_\mathcal{C} = \mathrm{Tr}\left\{W_\mathcal{C}(x,x)\right\}. \quad (18.14)$$

The trace goes over the $SU(N)$ matrix indices and the Wilson loop is accordingly not a matrix but a scalar.
From the transformation law of the Wilson lines, it follows that this is in fact gauge invariant,

$$W_\mathcal{C} \to W_\mathcal{C}. \quad (18.15)$$

Furthermore, the complex conjugate is

$$W_\mathcal{C}^* = W_{\bar{\mathcal{C}}}. \quad (18.16)$$

Wegner - Wilson loop for QED

Let us consider the expectation value of the Wegner-Wilson loop of abelian gauge theory (QED) in the Euclidean theory at vanishing temperature $T = 0$. One can write

$$\langle W_\mathcal{C} \rangle = \int \mathcal{D}A \, \exp\left[i \int_\mathcal{C} dx^\mu \, A_\mu\right] e^{-S}, \quad (18.17)$$

where

$$S = \int d^4x \left\{\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}\right\}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (18.18)$$

The line integral in the exponential resembles the current term

$$\exp\left[\int d^4x \left\{J^\mu(x)A_\mu(x)\right\}\right] \quad (18.19)$$

that we introduced to define the partition function. In fact, one can set

$$J^\mu(x) = i \int_{\mathcal{C}} d\tilde{x}^\mu \; \delta^{(4)}(x - \tilde{x}) \tag{18.20}$$

to make the expressions agree.

For the free abelian gauge field, the partition function is known and one has (in Euclidean space)

$$\left\langle \exp\left[\int_x J^\mu(x) A_\mu(x) \right] \right\rangle = \exp\left[\frac{g^2}{2} \int_{x,y} \{ J^\mu(x) \Delta_{\mu\nu}(x - y) J^\nu(y) \} \right]. \tag{18.21}$$

Accordingly, one finds for the Wegner-Wilson loop

$$\langle W_{\mathcal{C}} = \exp\left[-\frac{g^2}{2} \int_{\mathcal{C}} dx^\mu \int_{\mathcal{C}} dy^\nu \Delta_{\mu\nu}(x - y) \right] \tag{18.22}$$

with the Euclidean photon propagator

$$\begin{aligned}
\Delta_{\mu\nu}(x - y) &= \delta_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2} \\
&= \delta_{\mu\nu} \frac{4\pi}{(2\pi)^4} \int_0^\infty \frac{k^3 \, dk}{k^2} \underbrace{\int_0^\pi d\theta \; \sin^2\theta \; e^{ik|x-y|\cos\theta}}_{\pi J_1(k(x-y))/(k|x-y|)} \\
&= \delta_{\mu\nu} \frac{1}{4\pi^2(x-y)^2} \int_0^\infty du \; J_1(u) \\
&= \frac{\delta_{\mu\nu}}{4\pi^2(x-y)^2}.
\end{aligned} \tag{18.23}$$

We can now evaluate the double line integral for the Wegner-Wilson line. One can split this into an integral over relative distances on the loop and the average portion. Because the photon propagator depends only on the former, the result will be of the form

$$\langle W_{\mathcal{C}} \rangle = \exp\left[-\frac{g^2}{a} \tilde{c} P \right] \tag{18.24}$$

with $P = \int_{\mathcal{C}} dx$, the perimeter of the curve $\mathcal{C}$.

In fact, the integral has a short distance (UV) divergence, which is cut off at the distance a. The constant $\tilde{c}$ depends on details of the curve shape and on how precisely the cutoff is imposed. That the Wegner-Wilson loop scales exponentially with P is called "Perimeter law". It is typical for weakly-interacting, non-confined theories.

Interaction potential from Wegner-Wilson loop

One may obtain the interaction potential between two very heavy or static particles from the Wegner-Wilson loop. To this end, consider a rectangular closed path with length T in time and R in space direction such that $a << R << T$. When $\langle W_{\mathcal{C}} \rangle$ is computed, we are actually solving the functional integral in the presence of static, opposite charges with separation R. The Euclidean path integral will then be proportional to $\exp\left[-E(R)T\right]$, where $E(R)$ is the interaction energy of the two heavy particles.

Doing the calculation (exercise), one finds with $\alpha = g^2/(4\pi)$,

$$\langle W_{\mathcal{C}} \rangle = \exp\left[-\left(\text{const} - \frac{\alpha}{R} \right) T \right]. \tag{18.25}$$

The constant part is in fact divergend and corresponds to the self energy of a classical charged particle. From the R-dependece, one can read off the constant potential $V(R) = -\alpha/R$.

In a weak coupling expansion of a non-abelian gauge theory like QCD, one also finds the Coulomb potential. However this cannot be the full story. We now describe a strong coupling expansion for the Wegner-Wilson line.

Lattice regularization

Imagine that we formulate the gauge theory on a discrete spacetime lattice with lattice spacing a. We now consider a Wilson loop of the form
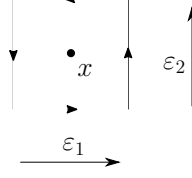
going around a point x. We define this loop to be the plaquette. Multiplying the Wilson links gives

$$W_{\text{plaq}} = \text{Tr}\left\{ e^{-iaA_2\left(x - \frac{\varepsilon_1}{2}\right)} e^{-iaA_1\left(x + \frac{\varepsilon_2}{2}\right)} e^{iaA_2\left(x + \frac{\varepsilon_1}{2}\right)} e^{iaA_1\left(x - \frac{\varepsilon_2}{2}\right)} \right\}.$$

$$\downarrow \qquad\qquad \leftarrow \qquad\qquad \uparrow \qquad\qquad \rightarrow$$

(18.26)

Assume now that the gauge fields are smooth and expand in a,

$$W_{\text{plaq}} = \text{Tr}\Big\{ e^{-iaA_2(x) + ia^2\partial_1 A_2(x)/2 + \dots}\ e^{-iaA_1(x) - ia^2\partial_2 A_1(x)/2 + \dots}$$
$$\cdot\, e^{iaA_2(x) + ia^2\partial_1 A_2(x)/2 + \dots}\ e^{iaA_1(x) - ia^2\partial_2 A_1(x)/2 + \dots} \Big\}.$$

(18.27)

With help of $e^A e^B = e^{A+B+[A,B]/2+\cdots}$, one can combine the exponentials in the first line and in the second line and then both lines together. The result is

$$W_{\text{plaq}} = \text{Tr}\left\{ e^{ia^2(\partial_1 A_2 - \partial_2 A_1 - i[A_1, A_2])} \right\} = \text{Tr}\left\{ e^{ia^2 F_{12}} \right\}.$$

(18.28)

The Wilson loop of the same plaquette in the opposite sense gives

$$W_{\overline{\text{plaq}}} = \text{Tr}\left\{ e^{-ia^2(\partial_1 A_2 - \partial_2 A_1 - i[A_1, A_2])} \right\}.$$

(18.29)

Adding them and expanding the exponentials gives

$$W_{\text{plaq}} + W_{\overline{\text{plaq}}} = 2N - a^4 \text{Tr}\left\{ F_{12} F^{12} \right\} + \cdots$$

(18.30)

Interestingly, this is precisely of the form we need for the action. We can take the lattice action of Yang-Mills theory to be

$$S = -\frac{1}{2g^2} \sum_{\text{plaq}} W_{\text{plaq}},$$

(18.31)

where the sum goes over the plaquettes around each lattice point including both orientations. Each plaquette is expressed as the product of four link matrices U. The functional integral can be written as an integral over these link matrices,

$$Z = \int \mathcal{D}U\ e^{-S},$$

(18.32)

where

$$\mathcal{D}U := \prod_{\text{links}} dU_{\text{links}}$$

(18.33)

and dU is the so-called Haar measure for a special unitary matrix. It has the properties (for $N \geq 3$)

$$\int dU \ U_{ij} = 0,$$
$$\int dU \ U_{ij} U_{kl} = 0, \tag{18.34}$$
$$\int dU \ U_{ij} U_{kl}^* = \frac{1}{N} \delta_{ik} \delta_{jl}.$$

Let us now consider a Wilson loop composed of a sequence of link variables,

$$\langle W_{\mathcal{C}} \rangle = \frac{1}{Z} \int \mathcal{D}U \ W_{\mathcal{C}} e^{-S}. \tag{18.35}$$

We will evaluate this in the strong coupling expansion in powers of $1/g^2$. To lowest order $e^{-S} \to 1$ and the result vanishes because $\int dU \ U_{ij} = 0$ for each link. Clearly, each link U in $W_{\mathcal{C}}$ must be balanced by a conjugate link U^* from the expansion of e^{-S}.

In fact, to get a non-zero result, we must fill the interior of the Wilson loop by opposite plaquettes from the action using a division like the following.
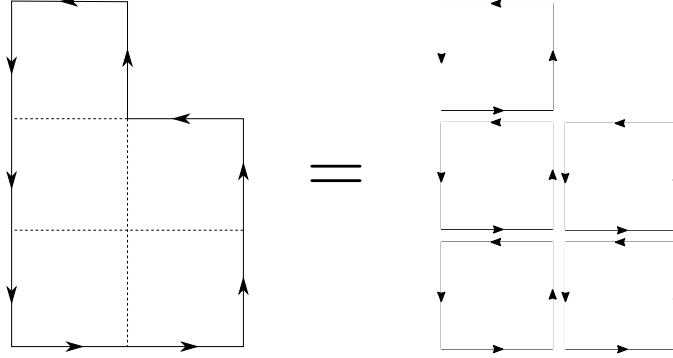
Each plaquette comes with a factor $1/g^2$ and the number of plaquettes needed is the area A/a^2. Accordingly,

$$\langle W_{\mathcal{C}} \rangle = \left(\frac{1}{g^2} \right)^{A/a^2} \propto e^{-\sigma A}, \tag{18.36}$$

where

$$\sigma = \frac{\ln(g^2)}{a^2} \tag{18.37}$$

is the string tension. This area law is a signal for confinement. The consideration of a static quark-antiquark pair now leads to

$$\langle W_{\mathcal{C}} \rangle \propto e^{-\sigma RT} \tag{18.38}$$

and thus

$$V(R) = \sigma R. \tag{18.39}$$

The energy becomes infinitely large when one tries to separate the quark and anti-quark. In reality, the string breaks when the energy is large enough to produce another quark-antiquark pair. Implementing lattice QCD numerically, one can go beyond the strong and weak coupling expansion.

19 Characterization of particle excitations

We discuss here how particle-type excitations in a quantum field theory, either fundamental or composite, can be characterized in terms of correlation functions and in terms of their transformation behaviour under space-time symmetries.

19.1 Källén-Lehmann spectral representation

We start from the spectral representation for the complex argument Greens function in vacuum

$$G_{ab}(p) = \int_0^\infty d\mu^2 \; \rho_{ab}(\mu^2) \frac{1}{p^2 + \mu^2} \tag{19.1}$$

with spectral density $\rho_{ab}(\mu^2)$. By evaluating this on different contours close to the real frequency axis, one can obtain various Greens functions,

$$\begin{aligned}
\Delta_{ab}^R(p) &= G_{ab}(w + i\varepsilon, \vec{p}), \\
\Delta_{ab}^A(p) &= G_{ab}(w - i\varepsilon, \vec{p}), \\
\Delta_{ab}^F(p) &= G_{ab}(w + i\varepsilon \; \mathrm{sign}(w), \vec{p}),
\end{aligned} \tag{19.2}$$

where

$$\Delta_{ab}^F(x - y) = i\langle T\{\phi_a(x)\phi_b(y)\}\rangle = \int_p \Delta_{ab}^F(p) e^{ip(x-y)} \tag{19.3}$$

is the Feynman propagator and

$$\Delta_{ab}^R(x - y) = i\theta(x^0 - y^0)\langle[\phi_a(x), \phi_b(y)]\rangle = \int_p \Delta_{ab}^R(p) e^{ip(x-y)} \tag{19.4}$$

is the retarded propagator and

$$\Delta_{ab}^A(x - y) = -i\theta(y^0 - x^0)\langle[\phi_a(x), \phi_b(y)]\rangle = \int_p \Delta_{ab}^R(p) e^{ip(x-y)} \tag{19.5}$$

is the adcanced propagator. One can also write

$$\Delta_{ab}(x - y) = \int_p \int_0^\infty d\mu^2 \; \rho_{ab}(\mu^2) \left[\frac{1}{p^0 \pm i\varepsilon + \sqrt{\vec{p}^2 + \mu^2}} + \frac{1}{-(p^0 \pm i\varepsilon) + \sqrt{\vec{p}^2 + \mu^2}} \right] \frac{e^{ip(x-y)}}{2\sqrt{\vec{p}^2 + \mu^2}}$$

where the combinations with $+i\varepsilon, +i\varepsilon$ gives the retarded, the one with $-i\varepsilon, -i\varepsilon$ the advanced and $-i\varepsilon, +i\varepsilon$ gives the Feynman propagator.

The Feynman propagator is used in most perturbative calculations while the retarded propagator corresponds to a Greens function with causal boundary conditions, that is non-zero only when $x^0 > y^0$.

Interestingly, the spectral representation holds independently of whether ϕ_a, ϕ_b are fundamental or composite fields. It also holds for fermionic (Grassmann) fields, although there the definition of Δ^R and Δ^A is based on anti-commutators instead of commutators.

The spectral function $\rho_{ab}(\mu^2)$ is typically real and positive, although negative values sometimes occur, e.g. for the gluon propagator in the region governed by confinement.

The spectral function $\rho_{ab}(\mu^2)$ has typically singular contributions in the form of Dirac distributions $\rho(\mu^2) \propto \delta(\mu^2 - m^2)$, as well as a continuum. We will now show that the former describes particle-type excitations.

19.2 Retarded propagator

Let us consider the retarded propagator describing the causal reaction at x to a small perturbation at y,

$$\Delta^R_{ab}(x-y) = \int_p \int_0^\infty d\mu^2 \, \rho_{ab}(\mu^2) \left[\frac{1}{p^0 + i\varepsilon + \sqrt{\vec{p}^2 + \mu^2}} + \frac{1}{-p^0 - i\varepsilon + \sqrt{\vec{p}^2 + \mu^2}} \right] \frac{e^{ip(x-y)}}{2\sqrt{\vec{p}^2 + \mu^2}}.$$

We will investigate this for large time separation $x^0 - y^0$. It is useful to perform the integral over frequencies p^0. The integration contour can be closed below and evaluated with the residue theorem.
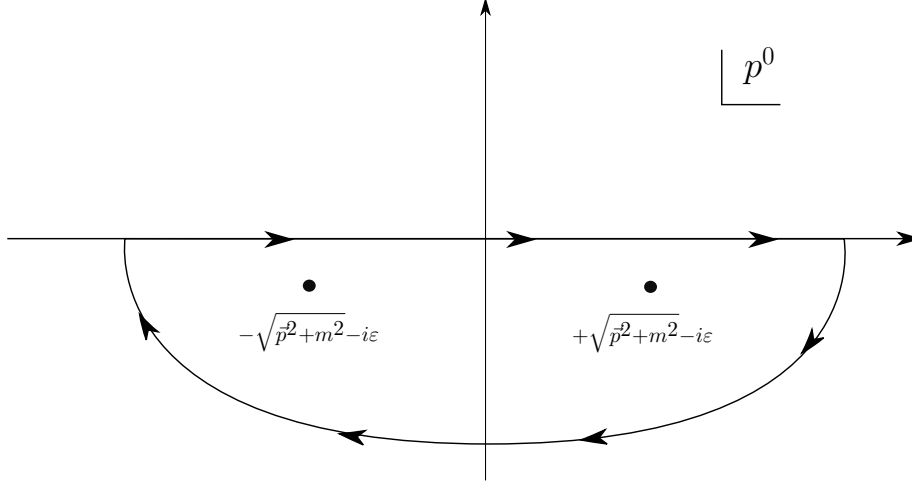
$$\Delta^R_{ab}(x-y) = \int_p \int_0^\infty d\mu^2 \, \rho_{ab}(\mu^2) \frac{1}{2\sqrt{\vec{p}^2 + \mu^2}} \left[-ie^{i\sqrt{\vec{p}^2 + \mu^2}(x^0 - y^0) + i\vec{p}(\vec{x} - \vec{y})} \right.$$
$$\left. + ie^{-i\sqrt{\vec{p}^2 + \mu^2}(x^0 - y^0) + i\vec{p}(\vec{x} - \vec{y})} \right]$$

There are two contributions, one with the negative frequency $-\sqrt{\vec{p}^2 + \mu^2}$, one with the positive frequency $\sqrt{\vec{p}^2 + \mu^2}$.

There is also an integral over the parameter μ^2 with weight given by the spectral density $\rho_{ab}(\mu^2)$. One may substitute variables and integrate over $E = \sqrt{\vec{p}^2 + \mu^2}$ instead of μ^2. Using $dE = d\mu^2/(2E)$, one finds

$$\Delta^R_{ab}(x-y) = \int_{\vec{p}} \int_0^\infty dE \, \rho_{ab}(E^2 - \vec{p}^2) \left[-ie^{iE(x^0 - y^0) + i\vec{p}(\vec{x} - \vec{y})} \right.$$
$$\left. + ie^{-iE(x^0 - y^0) + i\vec{p}(\vec{x} - \vec{y})} \right].$$

Let us now work out the consequences of a continuum contribution in $\rho_{ab}(E^2 - \vec{p}^2)$ (as opposed to a sharp δ-distribution contribution). For fixed $\vec{p}$, we assume that $\rho_{ab}(E^2 - \vec{p}^2)$ is approximately constant in an interval $(E - \Delta E/2, E + \Delta E/2)$. This allows to integrate and one finds for example

$$\int_{E-\Delta E/2}^{E+\Delta E/2} dE \, e^{-iE(x^0 - y^0)} = \left[\frac{e^{-iE(x^0 - y^0)}}{-i(x^0 - y^0)} \right]_{E-\Delta E/2}^{E+\Delta E/2} = e^{-iE(x^0 - y^0)} \frac{\sin\left(\frac{\Delta E}{2}(x^0 - y^0)\right)}{\frac{1}{2}(x^0 - y^0)}. \tag{19.6}$$

The important point here is that this decays for $x^0 - y^0 \to \infty$. This is in fact a general statement, the contributions from a continuum in the spectral function decay for large time separation $x^0 - y^0$. The only way to get a non-vanishing contribution is from a contribution

$$\rho_{ab}(\mu^2) \propto c_{ab}\delta(\mu^2 - m^2). \tag{19.7}$$

In that case, we obtain the retarded Greens function

$$\Delta_{ab}^R(x - y) = c_{ab}\int_{\vec{p}}\frac{1}{2\sqrt{\vec{p}^2 + m^2}}\Bigg[-ie^{i\sqrt{\vec{p}^2 + m^2}(x^0 - y^0) + i\vec{p}(\vec{x} - \vec{y})}$$
$$+ ie^{-i\sqrt{\vec{p}^2 + m^2}(x^0 - y^0) + i\vec{p}(\vec{x} - \vec{y})}\Bigg].$$

which looks indeed very much like the result for freely propagating particles.

19.3   Particles and plane waves

We have seen that stable particles essentially correspond to plane wave contributions to the two-point function or retarded propagator

$$\propto e^{-iE_p t + i\vec{p}\vec{x}} = e^{ip_\mu x^\mu} \tag{19.8}$$

where $E_p = \sqrt{\vec{p}^2 + m^2}$. These are in fact eigenstates of the energy-momentum operator

$$\hat{p}_\mu = -i\partial_\mu \tag{19.9}$$

with eigenvalue p_μ. Note that the latter is the generator for translations, e.g.

$$\phi(x + a) = e^{a^\mu \partial_\mu} f(x) = e^{ia^\mu \hat{p}_\mu} f(x). \tag{19.10}$$

19.4   Particles as representations of the Poincaré group

We have seen that stable particles correspond to Dirac distribution like contributions to the spectral function or poles in the (retarded) two-point correlation function. These in turn can be understood as being given by plane waves which are eigenstates with respect to translations. In fact, one can understand particle-type excitations as representations of the Poincaré group consisting of Lorentz transformations and translations. The Poincaré group is a Lie group and we now discuss its properties briefly.

19.5   Lorentz and Pioncaré group

We use here conventions where the metric in four dimensional Minkowski space is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1) .$$

Infinitesimal Lorentz transformations and rotations in Minkowski space are of the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu \tag{19.11}$$

with $\Lambda^\mu{}_\nu \in \mathbb{R}$ such that the metric $\eta_{\mu\nu}$ is invariant.

Exercise: *Show that this implies*

$$\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu} \; .$$

The spatial-spatial components describe rotations the three dimensional subspace and the spatial-temporal components Lorentz boost in Minkowski space or rotations around a particular three-dimensional direction in Euclidian space. Representations of the Lorentz group with

$$U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda)$$

can be written in infinitesimal form as

$$U(\Lambda) = \mathbb{1}_4 + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \; ,$$

where $M^{\mu\nu} = -M^{\nu\mu}$ are the generators of the Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}\right). \tag{19.12}$$

The fundamental representation (19.11) has the generators

$$(M_F^{\mu\nu})^\alpha{}_\beta = -i(\eta^{\mu\alpha}\delta_\beta^\nu - \eta^{\nu\alpha}\delta_\beta^\mu) \; .$$

It acts on the space of four-dimensional vectors x^α and the infinitesimal transformation in (19.11) induces the infinitesimal change

$$\delta x^\alpha = \frac{i}{2}\delta\omega_{\mu\nu}(M_F^{\mu\nu})^\alpha{}_\beta\, x^\beta \; .$$

One can decompose the generators into the spatial-spatial part

$$J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk}, \tag{19.13}$$

and a spatial-temporal part,

$$K_j = M^{j0}. \tag{19.14}$$

Equation (19.12) implies the commutation relations

$$[J_i, J_j] = i\,\varepsilon_{ijk}J_k,$$
$$[J_i, K_j] = i\,\varepsilon_{ijk}K_k,$$
$$[K_i, K_j] = -i\,\varepsilon_{ijk}J_k.$$

In the fundamental representation one has

$$(J_i^F)^j{}_k = -i\varepsilon_{ijk}$$

where j, k are spatial indices. All other components vanish, $(J_i^F)^0{}_0 = (J_i^F)^0{}_j = (J_i^F)^j{}_0 = 0$. Note that J_i^F is hermitian, $(J_i^F)^\dagger = J_i^F$. The generator K_j has the fundamental representation

$$(K_j^F)^0{}_m = -i\delta_{jm}, \qquad (K_j^F)^m{}_0 = -i\delta_{jm}$$

and all other components vanish, $(K_j^F)^0{}_0 = (K_j^F)^m{}_n = 0$. From these expression one finds that the conjugate of the fundamental representation of the Lorentz algebra has the generators

$$J_j^C = (J_j^F)^\dagger = J_j^F, \qquad K_j^C = (K_j^F)^\dagger = -K_j^F. \tag{19.15}$$

This implies that K_j^F is anti-hermitian,

$$(K_j^F)^\dagger = -K_j^F \; .$$

One can define the linear combinations of generators

$$N_j = \frac{1}{2}(J_j - iK_j), \qquad \tilde{N}_j = \frac{1}{2}(J_j + iK_j),$$

for which the commutation relations become

$$[N_i, N_j] = i\varepsilon_{ijk}N_k,$$
$$[\tilde{N}_i, \tilde{N}_j] = i\varepsilon_{ijk}\tilde{N}_k,$$
$$[N_i, \tilde{N}_j] = 0.$$

This shows that the representations of the Lorentz algebra can be decomposed into two representations of SU(2) with generators N_j and $\tilde{N}_j$, respectively. Note that N_j and $\tilde{N}_j$ are hermitian and linearly independent. Nevertheless, there is an interesting relation between the two: Consider the hermitian conjugate representation of the Lorentz group as related to the fundamental one by eq. (19.15). The representation of the generators $N_j, \tilde{N}_j$ is

$$N_j^C = \frac{1}{2}(J_j^C - iK_j^C) = \frac{1}{2}(J_j^F + iK_j^F) = \tilde{N}_j^F,$$
$$\tilde{N}_j^C = \frac{1}{2}(J_j^C + iK_j^C) = \frac{1}{2}(J_j^F - iK_j^F) = N_j^F.$$

This implies that the role of N_j and $\tilde{N}_j$ is interchanged in the hermitian conjugate representation. Representations of SU(2) are characterized by spin n of half integer or integer value. Accordingly, the representations of the Lorentz group can be classified as $(2n + 1, 2\tilde{n} + 1)$. For example

$$(1, 1) = \text{scalar or singlet},$$
$$(2, 1) = \text{left-handed spinor},$$
$$(1, 2) = \text{right-handed spinor},$$
$$(2, 2) = \text{vector}.$$

Poincaré group

Poincaré transformations consist of Lorentz transformations plus transformations of the form

$$X^\mu \to \Lambda^\mu{}_\nu x^\nu - b^\mu. \tag{19.16}$$

It is clear that these transformations form a group.

Exercise: *Show the Poincaré group indeed is a group with the composition law*

$$(\Lambda_2, b_2) \circ (\Lambda_1, b_1) = (\Lambda_2\Lambda_1, \Lambda_2 b_1 + b_2) .$$

As transformations of fields, translations are generated by the momentum operator

$$\hat{P}_\mu := -i\partial_\mu .$$

For example

$$\phi(x) \mapsto \phi'(x) = \phi(\Lambda^{-1}(x + b))$$
$$\approx \phi(x^\mu - \delta\omega^\mu{}_\nu x^\nu + b^\mu)$$
$$= (1 + \frac{i}{2}\delta\omega_{\mu\nu}\mathcal{M}^{\mu\nu} + ib^\mu P_\mu)\phi(x) .$$

One finds easily

$$[P_\mu, P_\nu] = 0 \tag{19.17}$$

and

$$\mathcal{M}^{\mu\nu}, P_\rho t \qquad = i(\delta_\rho{}^\mu P^\nu - \delta^\nu{}_\rho P^\mu) \tag{19.18}$$
$$[\mathcal{M}^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu)$$

which together with

$$[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}] = i(\eta^{\mu\rho}\mathcal{M}^{\nu\sigma} - \eta^{\mu\sigma}\mathcal{M}^{\nu\rho} - \eta^{\nu\rho}\mathcal{M}^{\mu\sigma} + \eta^{\nu\sigma}\mathcal{M}^{\mu\rho})$$

forms the Poincaré algebra. The commutator (19.17) tells that the different components of the energy-momentum operator can be diagonalized simultaneously, while (19.18) says that P^ρ transforms as a vector under Lorentz transformations.

Particles as representations One can understand particles as representations of the Poincaré algebra. Energy and momentum are the eigenvalues of

$$P_\mu = -i\partial_\mu \ ,$$

and the spin tells information about $\mathcal{M}^{\mu\nu}$. One Casimir operator is

$$P^2 = P_\mu P^\mu \ ,$$

which obviously commutes with $\mathcal{M}^{\mu\nu}$ and P_μ. Moreover,

$$P^2 \left| p \right\rangle = -m^2 \left| p \right\rangle$$

gives the particle mass. The other Casimir follows from the Pauli-Lubanski vector

$$W_\sigma = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{M}^{\mu\nu}P^\rho \ .$$

Massive particles

For massive particles, $m > 0$, we can go to their rest frame where P^ρ gives $p_*^\rho = (m, 0, 0, 0)$ and

$$W_0 = 0,$$
$$W_j = \frac{1}{2}\varepsilon_{jmn}\mathcal{M}^{mn} = mJ_j,$$

with spin operator J_j. The second Casimir of the Poincaré algebra is $W_\mu W^\mu$ and

$$\frac{1}{m^2}W_\mu W^\mu \left| p, j \right\rangle = \boldsymbol{J}^2 \left| p, j \right\rangle = j(j+1) \left| p, j \right\rangle$$

so the states are characterized by the labels p and j. The commutation relations

$$[W_\mu, W_\nu] = i\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma$$

reduce to the rotation algebra in the rest frame.

As in quantum mechanics, one can choose a basis where the total angular momentum $\vec{J}^2 = j(j+1)$ as well as the spin operator in a specific direction, e.g. J_z have definite eigenvalues $v_z = -j, \cdots, j$. In summary, massive particles are charaterized by a four momentum p_μ with $p_\mu p^\mu + m^2 = 0$ and the so-called *little group* consisting of rotations in the particle's rest frame which is characterized by the wave numbers or quantum numbers j, v_z.

Massless particles

For massless particles, $m^2 = 0$, we can choose a frame where $p^\mu = (E, 0, 0, E)$. The little group now in particular contains rotations around the axis of the momentum. Eigenstates are of the form $e^{ih\phi}$ where ϕ is the angle around the momentum axis and h is the helicity.

Bosonic fields must be invariant under full rotations so that $h = 0, \pm 1, \pm 2$ and so on.

Massless scalars would have helicity 0, massless vectors ± 1 and massless gravitons would have helicity ± 2. Fermionic fields must pick up a factor -1 under full rotations so that $h = \pm 1/2, \pm 3/2$ and so on.

The little group for massless particles has in fact two more parameters which are however associated to gauge transformations. We do not discuss this further here.

20 Quantum field dynamics

So far we have mainly been concerned with the description of correlation functions and field expectation values for certain specific states such as the Minkowski space vacuum or a finite temperature state. From the correlation functions in the standard Minkowski space vacuum, one can in particular calculate the scattering S-matrix or the spectrum of bound states of the theory. Calculations at finite temperature give access to the thermodynamic equation of state and transport properties such as for example shear and bulk velocity or electric conductivity.

In the present section, we shall ask how the quantum state of a theory can actually be specified in full generality and how it can evolve in time. This allows to address non-equilibrium physics as well, for example to describe the quantum state of the evolving universe.

Quantum field dynamics out of equilibrium can also be investigated experimentally, for example with ultracold atomic quantum gases or with high energy heavy ion collisions.

Another benefit of a complete dynamical description is that it allows to investigate quantum field theory from an information theoretic perspective. This is particularly interesting in the context of black holes, of cosmology and it is important to understand questions related to (local) thermalisation.

20.1 The density matrix

Recall that in quantum mechanics for N particles, one can specify an arbitrary pure state at some time t in terms of a Schrödinger wave function

$$\psi_t(\vec{x}_1, \cdots, \vec{x}_N), \tag{20.1}$$

or, more abstractly, in terms of a state in a Hilbert space,

$$|\psi_t\rangle. \tag{20.2}$$

A general mixed state needs to be described by a *density matrix* or a *density operator*. For a mixture of states $|\psi^{(j)}\rangle$ with probability p_j such that $\sum_j p_j = 1$, the density operator is given by

$$\rho_t = \sum_j p_j |\psi^{(j)}\rangle\langle\psi^{(j)}|. \tag{20.3}$$

From the density operator, one can calculate expectation values at time t as

$$\langle A(t)\rangle = \text{Tr}\,\{\rho_t A\} = \sum_j p_j \text{Tr}\left\{|\psi^{(j)}\rangle\langle\psi^{(j)}|A\right\} = \sum_j p_j \langle\psi^{(j)}|A|\psi^{(j)}\rangle. \tag{20.4}$$

For the concrete case of an N-particle state, one would have

$$\rho_t(\vec{x}_1, \cdots, \vec{x}_N; \vec{y}_1, \cdots, \vec{y}_N) = \sum_j p_j \psi^{(j)}(\vec{x}_1, \cdots, \vec{x}_N)\psi^{(j)}(\vec{y}_1, \cdots, \vec{y}_N). \tag{20.5}$$

An arbitrary operator can be written as

$$A(\vec{u}_1, \cdots, \vec{u}_N; \vec{v}_1, \cdots, \vec{v}_N) \tag{20.6}$$

in position space representation and the expectation value would be

$$\langle A(t)\rangle = \text{Tr}\,\{\rho_t A\} = \int_{\vec{x}_1, \ldots, \vec{x}_N} \int_{\vec{y}_1, \ldots, \vec{y}_N} \rho_t(\vec{x}_1, \cdots, \vec{x}_N; \vec{y}_1, \cdots, \vec{y}_N) A(\vec{y}_1, \cdots, \vec{y}_N; \vec{x}_1, \cdots, \vec{x}_N).$$

Let us now go to quantum field theory. Instead of the positions $\vec{x}_1, \cdots, \vec{x}_N$, the defrees of freedom are now the field variables $\phi(\vec{x})$ at some fixed time t, for all possible spatial positons $\vec{x}$. The spatial position $\vec{x}$ now plays the role of the index $n = 1, \cdots, N$ and labels the different degrees of freedom (quantum fields).

A pure state at some time t is now specified by a so-called *Schrödinger functional*

$$\psi_t[\phi] \tag{20.7}$$

and a mixed state in a similar way by a *density matrix functional*

$$\rho_t[\phi_+, \phi_-]. \tag{20.8}$$

The most general observable is also specified by a similar functional

$$A[\phi_1, \phi_2] \tag{20.9}$$

and an expectation value is given by

$$\langle A \rangle = \int \mathcal{D}\phi_+ \mathcal{D}\phi_- \ \rho_t[\phi_+, \phi_-]A[\phi_+, \phi_-]. \tag{20.10}$$

The functional integrals $\int \mathcal{D}\phi_+$ and $\int \mathcal{D}\phi_-$ are here over fields at constant time t but for all spatial positions $\vec{x}$.

20.2   States on Cauchy surfaces

In a relativistic quantum field theory, one can specify a state not only at a fixed time t but somewhat more generally on any so-called Cauchy surface Σ. This is a $(d - 1)$ dimensional submanifold of spacetime, a so-called hypersurface, with a normal vector that points in a time-like direction everywhere.
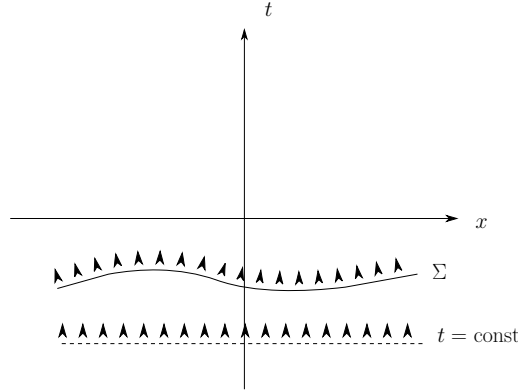
A hypersurface $t = $ const with normal vector $n^\mu = (1, 0, 0, 0)$ is then just a special case.

In the more general case, the density matrix on the hypersurface Σ is specified as a double functional of fields $\phi_+(x)$ and $\phi_-(x)$ where the coordinates are now on the hypersurface, that is $x \in \Sigma$, $\rho = \rho_\Sigma[\phi_+, \phi_-]$.

In this formulation, a generalization of time evolution would correspond to an evolution between neighbouring Cauchy surfaces, e.g. $\Sigma_1 \to \Sigma_2 \to \cdots \to \Sigma_N$.
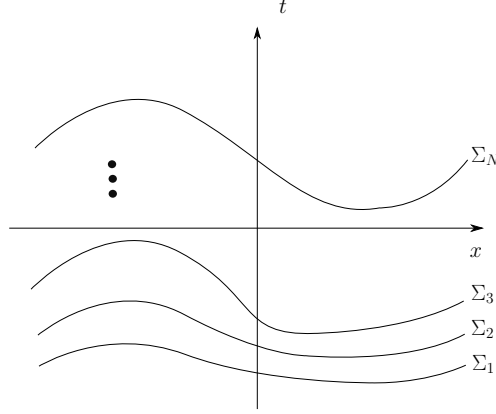
20.3 Evolution operators

Similar as in quantum mechanics, the evolution in time, or between Cauchy surfaces, is realized by unitary evolution operators. For N-body quantum mechanics, this would be an operator of the type

$$U_{t_2 \leftarrow t_1}(\vec{x}_1, \cdots, \vec{x}_N; \vec{y}_1, \cdots, \vec{y}_N) \tag{20.11}$$

such that

$$\psi_{t_2}(\vec{x}_1, \cdots, \vec{x}_N) = \int_{\vec{y}_1, \ldots, \vec{y}_N} U_{t_2 \leftarrow t_1}(\vec{x}_1, \cdots, \vec{x}_N; \vec{y}_1, \cdots, \vec{y}_N) \psi_{t_1}(\vec{y}_1, \cdots, \vec{y}_N). \tag{20.12}$$

The density matrix also needs the hermitian conjugate operator

$$U^\dagger_{t_1 \to t_2}(\vec{x}_1, \cdots, \vec{x}_N; \vec{y}_1, \cdots, \vec{y}_N) \tag{20.13}$$

so that the density matrix evolves as

$$\begin{aligned}
\rho_{t_2}(\vec{x}_1, \cdots, \vec{x}_N; \vec{y}_1, \cdots, \vec{y}_N) = \int_{\vec{u}_1, \cdots, \vec{u}_N} \int_{\vec{v}_1, \cdots, \vec{v}_N} U_{t_2 \leftarrow t_1}(\vec{x}_1, \cdots, \vec{x}_N; \vec{u}_1, \cdots, \vec{u}_N) \cdot \\
\cdot \rho_{t_1}(\vec{u}_1, \cdots, \vec{u}_N; \vec{v}_1, \cdots, \vec{v}_N) U^\dagger_{t_1 \to t_2}(\vec{v}_1, \cdots, \vec{v}_N; \vec{y}_1, \cdots, \vec{y}_N)
\end{aligned} \tag{20.14}$$

In a quantum field theory, one can specify in a similar way the unitary operator for evolution from one hypersurface to the next, e.g. $\Sigma_1 \to \Sigma_2$,

$$U_{\Sigma_2 \leftarrow \Sigma_1}[\phi_2, \phi_1] \tag{20.15}$$

such that the density matrix functional evolves as

$$\rho_{\Sigma_2}[\phi_{2+}, \phi_{2-}] = \int \mathcal{D}\phi_{1+} \int \mathcal{D}\phi_{1-} U_{\Sigma_2 \leftarrow \Sigma_1}[\phi_{2+}, \phi_{1+}] \rho_{\Sigma_1}[\phi_{1+}, \phi_{1-}] U^\dagger_{\Sigma_1 \to \Sigma_2}[\phi_{1-}, \phi_{2-}]. \tag{20.16}$$

To make these formal statements more concrete, let us specify how the time evolution operator would be written as a functional integral. One would have

$$U_{\Sigma_2 \leftarrow \Sigma_1}[\phi_2, \phi_1] = \frac{1}{Z} \int_{\phi_1, \phi_2} \mathcal{D}\phi \; e^{iS[\phi]} \tag{20.17}$$

where the functional integral now goes over fields $\phi(x)$ on spacetime, with the boundary conditions

$$\begin{aligned}
\phi(x) &= \phi_1(x) &&\text{on } \Sigma_1, \\
\phi(x) &= \phi_2(x) &&\text{on } \Sigma_2.
\end{aligned} \tag{20.18}$$

In a similar way, one would have

$$U^{\dagger}_{\Sigma_1 \leftarrow \Sigma_2}[\phi_1, \phi_2] = \frac{1}{Z} \int_{\phi_1, \phi_2} \mathcal{D}\phi \; e^{-iS^*[\phi]} \tag{20.19}$$

where the boundary conditions are as above but we now employ an action $S^*[\phi]$ where the complex conjugate symbol is a reminder to replace $i\varepsilon \to -i\varepsilon$ wherever it appears.

The evolution equation of the density matrix functional (20.16) is reminiscent of the corresponding equation in quantum mechanics

$$\rho_{t_2} = e^{-iH(t_2-t_1)} \rho_{t_1} e^{iH(t_2-t_1)}. \tag{20.20}$$

20.4 Schwinger-Keldysh double time path

Note that if one inserts (20.17) and (20.19) into equation (20.16), one obtains a functional integral expression with a forward and a backward path. This is because both, the "ket" and the "bra" of the density matrix need to be evolved in time.
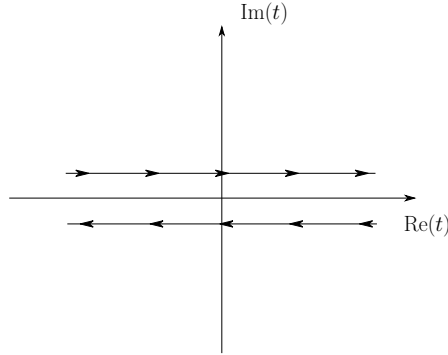
One can actually understand

$$\exp\left[-iS[\phi]\right] = \exp\left[-i \int_{t_1}^{t_2} dt \int d^3x \; \mathcal{L}\right] = \exp\left[+i \int_{t_2}^{t_1} dt \int d^3x \; \mathcal{L}\right] \tag{20.21}$$

as an evolution backwards in time. In this sense, there is one branch that evolves forward and one branch that evolves backward in time.

The infinitesimal it-terms can actually be accounted for by going to slightly negative imaginary times when evolving forward so that the contour in the complex time plane is slightly inclined downwards. The backward branch must also slightly go downwards when gping from large to small real times.

What we have derived here is known as the Schwinger-Keldysh double time path. Before discussing the evolution in time further, let us discuss the density matrix functional in somewhat more detail.

20.5 Density matrix of thermal states

Let us first discuss the density matrix of a thermal state. It is interesting by itself and allows to obtain the standard vacuum case in the limit $T \to 0$.

At fixed time t, the thermal density matrix is formally given by

$$\frac{1}{Z} e^{-H/T}, \qquad Z = \mathrm{Tr}\left\{e^{-H/T}\right\}. \tag{20.22}$$

In a field theory, the Hamiltonian can be written as

$$H = \int d^3x \; \mathcal{H}(x) \tag{20.23}$$

with Hamiltonian density $\mathcal{H}(x)$. The latter is in fact the time-time component of an energy-momentum tensor $T^{\mu\nu}(x)$.

We have already discussed that the partition function Z at finite temperature T can be realized as a functional integral in a Euclidean spacetime, or, equivalently, in Minkowski space but with imaginary time $t = -i\tau$ where τ is integrated from 0 to $\beta = 1/T$, or equivalently, t is integrated from 0 to $-i\beta$.

The density matrix functional is then given by

$$\rho[\phi_+, \phi_-] = \frac{1}{Z} \int_{\phi_+, \phi_-} \mathcal{D}\phi \; e^{-S[\phi]} \tag{20.24}$$

where

$$S[\phi] = \int_0^\beta d\tau \int d^3x \; \{-\mathcal{L}\} = -i \int_0^{-i\beta} dt \int d^3x \; \mathcal{L}. \tag{20.25}$$

We used that $\mathcal{L}_E = -\mathcal{L}$, with Minkowski space Lagrangian density $\mathcal{L}$. The boundary conditions for the functional integral are

$$
\begin{aligned}
\phi(\tau = 0, \vec{x}) &= \phi_+(x), \\
\phi(\tau = \beta, \vec{x}) &= \phi_-(x).
\end{aligned}
\tag{20.26}
$$

One easily confirms that this is now normalized correctly,

$$\mathrm{Tr}\rho = \int \mathcal{D}\phi \; \rho[\phi_+, \phi_+] = \frac{1}{Z} \int \mathcal{D}\phi_+ \int_{\phi_+, \phi_+} \mathcal{D}\phi \; e^{-S[\phi]} = \frac{1}{Z} \int \mathcal{D}\phi \; e^{-S[\phi]} = 1. \tag{20.27}$$

The standard Minkowski vacuum follows for $T \to 0$ or $\beta \to \infty$. The Euclidean time is now integrated in an infinite interval.

Let us now generalize this to an arbitrary Cauchy surface. The covariant generalization of equation (20.22) is

$$\frac{1}{Z} \exp\left[-\int d\Sigma_\mu \; \{\beta_\nu(x)T^{\mu\nu}(x) + \alpha(x)N^\mu(x)\}\right]. \tag{20.28}$$

Here we employ the hypersurface integral element

$$d\Sigma_\mu = d^3x\sqrt{g_3}n_\mu \tag{20.29}$$

where the integral element d^3x is on the hypersurface and $g_3 = \det(g_{ij})$ is the determinant of the metric on the hypersurface. Finally, n^μ is the normal vector normalized to $g_{\mu\nu}n^\mu n^\nu = -1$.

A hypersurface can be specified by a condition $\tilde{t}(x) = t_0$ where $\tilde{t}(x)$ is some "time function". In that case, one can write

$$d\Sigma_\mu = d^4x \; \sqrt{g}\delta(\tilde{t}(x) - t_0)\partial_\mu \tilde{t}(x) \tag{20.30}$$

where now $g = -\det(g_{\mu\nu})$ is the determinant of the four-dimensional metric.

Other elements in (20.28) are the combination of fluid velocity and temperature

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)}, \tag{20.31}$$

the energy-momentum tensor $T^{\mu\nu}(x)$ and a conserved number current $N^\mu(x)$. We have coupled the latter to a chemical potential μ and use

$$\alpha(x) = \frac{\mu(x)}{T(x)}. \tag{20.32}$$

The fluid velocity is a normalized four-velocity with

$$g_{\mu\nu}u^{\mu}u^{\nu} = -1. \tag{20.33}$$

We are using a formulation in general coordinates as appropriate for integrals on a hypersurface. Fluid velocity, temperature and chemical potential then depend on the coordinate x. However, in global thermal equilibrium, they must fulfill certain conditions, namely

$$\nabla_{\mu}\beta_{\nu} + \nabla_{\nu}\beta_{\mu} = 0, \qquad \partial_{\mu}\alpha = 0. \tag{20.34}$$

It is a useful exercise to show that (20.28) reduces to (20.22) for a hypersurface of constant time and a fluid at rest such that $u^{\mu} = (1, 0, 0, 0)$.

20.6 Functional integral representation

Let us now formulate a functional integral representation for the density matrix in equation (20.28). This can be done analogously to the $t = $ const surface. One obtains

$$\rho_{\Sigma}[\phi_{+}, \phi_{-}] = \frac{1}{Z} \int_{\phi_{+}, \phi_{-}} \mathcal{D}\phi \ \exp\left[-S[\phi]\right] \tag{20.35}$$

where

$$S[\phi] = \int d^4x \ \sqrt{g} \ \mathcal{L}_E \tag{20.36}$$

is the Euclidean action on a "torus" formed by the hypersurface Σ and an imaginary time coordinate. Fields are periodic in the sense that

$$\phi(x - i\beta) = \phi(x) \tag{20.37}$$

for bosons while fermions get an additonal minus sign.

The chemical potential is represented by an external gauge field coupled to the conserved current, $A_{\nu}(x) = \mu(x)u_{\nu}(x)$.

Note that the state we describe here is specified on the surface Σ and essentially assumes that the metric $g_{\mu\nu}$ and all parameters such as $\alpha(x)$ were constant in the past. Only then such a global equilibrium state can form. In a time-dependent situation such as an evolving cosmology, this is not the case. The evolving quantum state is in general not the instantaneous thermal equilibrium state.

The standard Minkowski vacuum state follows from the above prescription for $T \to 0$ or $\beta \to \infty$. The infinite interval over the Euclidean time can be split into two parts and they can be rotated towards the real time domain. This leads back to the standard Minkowski space formalism.

20.7 Density matrix for single mode

To gain more intuition, let us investigate a free real scalar field in $1+0$ dimensions with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}m^2\phi^2. \tag{20.38}$$

This can describe a single mode of the electromagnetic field, for example, and is equivalent to the quantum mechanical harmonic oscillator.

The Schrödinger functional for the grand state is now simply

$$\psi_0[\phi] = ce^{-m\phi^2/2} \tag{20.39}$$

with a complex constant c. Accordingly, the density functional in that state

$$\rho_0[\phi_+, \phi_-] = \frac{1}{Z} e^{-m(\phi_+^2 + \phi_-^2)/2}. \tag{20.40}$$

One can directly see that this is a pure state because it factorizes into a ket and a bra contribution. Excited states with n particles or quanta are of the form

$$\rho_n[\phi_+, \phi_-] = \frac{1}{Z_n} H_n(\sqrt{m}\phi_+) H_n(\sqrt{m}\phi_-) e^{-m(\phi_+^2 + \phi_-^2)/2} \tag{20.41}$$

where $H_n(x)$ are the Hermite polynomials

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2, \cdots \tag{20.42}$$

These are still pure states. The corresponding Schrödinger functional would be

$$\psi_n[\phi] = \frac{1}{\sqrt{2^n n!}} H_n(\sqrt{m}\phi) c e^{-m\phi^2/2}. \tag{20.43}$$

Under time evolution, the Schrödinger functionals above would pick up a factor $e^{-im(n+1/2)t}$ which cancels, however, in the density functional.

Another interesting class of states are coherent states. In quantum mechanics they are described by

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{20.44}$$

with complex parameter α.

Here they lead to the density matrix functional

$$\rho_\alpha[\phi_+, \phi_-] = \frac{1}{Z} \exp\left\{ -\frac{1}{2} m \left[\left(\phi_+ - \sqrt{\frac{2}{m}} \mathrm{Re}(\alpha) \right)^2 + \left(\phi_- - \sqrt{\frac{2}{m}} \mathrm{Re}(\alpha) \right)^2 \right] \right\}. \tag{20.45}$$

Again these are pure states. Under time eolution, one must replace $\alpha \to \alpha(t_0) e^{-im(t-t_0)}$ and one finds that $\mathrm{Re}(\alpha(t))$ describes the oscillatory behaviour of classical solutions to the equations of motion. The density matrix $\rho_{\alpha(t)}$ describes Gaussian fluctuations around this mean value.

Finally, let us consider a thermal state. In the quantum mechanical formalism, it is described as

$$\rho = (1-b) \sum_{n=0}^{\infty} b^n |n\rangle\langle n| \tag{20.46}$$

where $b = e^{-m/T}$ is the Boltzmann weight. Here this leads to the density matrix functional

$$\rho_T[\phi_+, \phi_-] = \frac{1}{Z}(1-b) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{b}{2} \right)^n H_n(\sqrt{m}\phi_+) H_n(\sqrt{m}\phi_-) e^{-m(\phi_+^2 + \phi_-^2)/2}. \tag{20.47}$$

Here, one can use a property of the Hermite polynomials (Mehler's formula)

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) H_n(y) \left(\frac{b}{2} \right)^n = \frac{1}{\sqrt{1-b^2}} \exp\left[\frac{2b}{1+b} xy - \frac{b^2}{1-b^2}(x-y)^2 \right]. \tag{20.48}$$

We thus find

$$\begin{aligned}
\rho_T[\phi_+, \phi_-] &= \frac{1}{Z} \exp\left[-\frac{1}{2} m(\phi_+^2 + \phi_-^2) - \frac{b^2}{1-b^2} m(\phi_+ - \phi_-)^2 + \frac{2b}{1+b} m\phi_+\phi_- \right] \\
&= \frac{1}{Z} \exp\left[-\frac{1}{2} m \left(1 + \frac{2b^2}{1-b^2} \right) (\phi_+^2 + \phi_-^2) + \frac{2b}{1-b^2} m\phi_+\phi_- \right]
\end{aligned} \tag{20.49}$$

This does not factor into a ket and a bra part for $b > 0$. It is therefore not a pure state as expected.

Let us summarize this discussion by remarking that the vacuum or ground state, the coherent states, as well as the thermal states all have density matrices $\rho[\phi_+, \phi_-]$ of Gaussian form. This is not the case for single or multiple particle excited states, though.

For free quantum field theories, one can also expect Gaussian states in many circumstances. However, already with non-vanishing interaction this ceases to be the case.

20.8 Higher dimensional Gaussian states

Let us now generalize the situation somewhat and consider a set of fields ϕ_n. The index n can be discrete and can run over a finite set. In this case, we consider a set of modes. Or it could be running over an infinite set. One may even consider n to be an abstract index that combines several indices such as position, flavor and spin.

We assume the Schrödinger functional to be of the form

$$\psi[\phi] = c \exp\left[-\frac{1}{2}(\phi - \bar{\phi})_m h_{mn} (\phi - \bar{\phi})_n + ij_n\phi_n\right] \tag{20.50}$$

with a symmetric and real matrix $h_{mn} = h_{nm}$. The density functional is accordingly

$$\rho[\phi_+, \phi_-] = \frac{1}{Z}\exp\left\{ -\frac{1}{2}(\phi_+ - \bar{\phi})_m h_{mn}(\phi_+ - \bar{\phi})_n \right.$$
$$\left. -\frac{1}{2}(\phi_- - \bar{\phi})_m h_{mn}(\phi_- - \bar{\phi})_n + ij_n(\phi_+ - \phi_-)_n\right\} \tag{20.51}$$

Let us characterize this state by its expectation values and correlation functions. Besides the field ϕ_n, another observable is its conjugate momentum field π_n. In the position space representation, we are working in here, it is represented by a derivative

$$\pi_n = -i\frac{\delta}{\delta\phi_n} \ . \tag{20.52}$$

This operator acts on the Schrödinger functional or density operator. The canonical commutation relations

$$[\phi_m, \pi_n] = i\delta_{mn}, \qquad [\phi_m, \phi_n] = [\pi_m, \pi_n] = 0, \tag{20.53}$$

are automatically fulfilled.

The field expectation value is given by

$$\langle\phi_m\rangle = \frac{1}{Z}\int \mathcal{D}\phi \ \phi_m \rho[\phi, \phi] = \bar{\phi}_m. \tag{20.54}$$

In a similar way, the expectation value for the conjugate momentum can be obtained,

$$\langle\pi_m\rangle = \frac{1}{Z}\int \mathcal{D}\phi \left(-i\frac{\delta}{\delta\phi_{+m}}\rho[\phi_+, \phi_-]\right)_{\phi_+=\phi_-=\phi} = j_m. \tag{20.55}$$

An exercise in Gaussian integration yields the connected correlation functions

$$\langle\phi_m\phi_n\rangle_c = \langle\phi_m\phi_n\rangle - \langle\phi_m\rangle\langle\phi_n\rangle = \frac{1}{2}(h^{-1})_{mn},$$
$$\langle\pi_m, \pi_n\rangle_c = \frac{1}{2}h_{mn},$$
$$\langle\phi_m\pi_n + \pi_n\phi_m\rangle_c = 0, \tag{20.56}$$
$$\langle\phi_m\pi_n - \pi_n\phi_m\rangle_c = [\phi_m, \pi_n] = i\delta_{mn}.$$

If the matrix h_{mn} is diagonal $h_{mn} = \tilde{h}_m \delta_{mn}$ (no sum convention), the different field modes are independent, otherwise they are correlated.

Imagine now that h_{mn} is diagonal. One then has

$$\langle \phi_m^2 \rangle \langle \pi_m^2 \rangle = \frac{1}{4}. \tag{20.57}$$

This is in fact the statement that Heisenberg's uncertainty bound is satisfied.

Note that for a single mode in the ground state, we have

$$\langle \phi^2 \rangle = \frac{1}{2m}, \qquad \langle \pi^2 \rangle = \frac{m}{2}. \tag{20.58}$$

The energy $E = m$ here sets the quantum uncertainty. In quantum optics, it is possible, however, to prepare so-called squeezed states with

$$\langle \phi^2 \rangle = \frac{1}{2h}, \qquad \langle \pi^2 \rangle = \frac{h}{2} \tag{20.59}$$

where $h > m$ or $h < m$. These are still pure states and they are still Gaussian states. They also still satisfy the Heisenberg bound but, for $n > m$, have a reduced uncertainty of the field at the cost of an increased uncertainty of the conjugate momentum. For $n < m$, the uncertainty of π is reduced while the one of ϕ is increased.

For diagonal h_{mn}, the different modes ϕ_m are fully independent and the density matrix $\rho[\phi_+, \phi_-]$ decomposes into a product of independent factors. This indicates that these degrees of freedom are not entangled. The situation is different in the presence of off-diagonal terms in h_{mn}. In that case, there are non-vanishing correlations between fields and between conjugate momenta - but there is also *quantum entanglement*.

20.9 Two-mode squeezed state

As the simplest example for an entangled Gaussian state consider the two-mode squeezed state with Schrödinger functional

$$\psi_r[\phi_1, \phi_2] = c \exp\left[-\frac{e^{2r}}{4} m(\phi_1 - \phi_2)^2 - \frac{e^{-2r}}{4} m(\phi_1 + \phi_2)^2 \right]. \tag{20.60}$$

For $r = 0$, this simply becomes the product state

$$\psi_0[\phi_1, \phi_2] = c \exp\left[-\frac{1}{2} m(\phi_1^2 + \phi_2^2) \right] = c \exp\left[-\frac{1}{2} m\phi_1^2 \right] \exp\left[-\frac{1}{2} m\phi_2^2 \right]. \tag{20.61}$$

For $r > 0$, such a product decomposition is not possible, however. Generalizations of such two-mode squeezed states describe entangled states from inflation in the early universe or the entanglement of Hawking radiation emerging from a black hole with radiation falling into the horizon (for free bosonic theories).

The density matrix for the two-mode system in the squeezed state is

$$\rho_{12}[\phi_{1+}, \phi_{2+}; \phi_{1-}, \phi_{2-}] = \frac{1}{Z} \exp\left\{ -\frac{e^{2r}}{4} m(\phi_{1+} - \phi_{2+})^2 - \frac{e^{-2r}}{4} m(\phi_{1+} + \phi_{2+})^2 \right.$$
$$\left. -\frac{e^{2r}}{4} m(\phi_{1-} - \phi_{2-})^2 - \frac{e^{-2r}}{4} m(\phi_{1-} + \phi_{2-})^2 \right\}. \tag{20.62}$$

20.10 Reduced density matrix

It is instructive to calculate the reduced density matrix for the mode ϕ_1 by taking the partial trace of the density matrix. Quite generally, the reduced density matrix for a subsystem A of a larger system consisting of the parts A and B is given as the partial trace

$$\rho_A = \text{Tr}_B \{\rho_{AB}\}. \tag{20.63}$$

If A and B are entangled and ρ_{AB} describes a pure state, the reduced density matrix is of a mixed state form.

In contrast, for a pure product state $\rho_{AB} = \rho_A \otimes \rho_B$, the reduced density matrix ρ_A is also pure. In the present case, taking the partial trace for the second mode corresponds to

$$
\begin{aligned}
\rho_1[\phi_{1+}, \phi_{1-}] &= \frac{1}{Z} \int \mathcal{D}\phi \; \rho_{12}[\phi_{1+}, \phi; \phi_{1-}, \phi] \\
&= \frac{1}{Z} \int \mathcal{D}\phi \; \exp\Bigg[-\frac{e^{2r} + e^{-2r}}{4} m(\phi_{1+}^2 + \phi_{1-}^2) \\
&\qquad\qquad\qquad + 2m\phi \left(\frac{e^{2r} - e^{-2r}}{4} m(\phi_{1+} + \phi_{1-}) \right) \\
&\qquad\qquad\qquad - m\phi^2 \frac{e^{2r} + e^{-2r}}{2} \Bigg] \\
&= \frac{1}{Z} \exp\Bigg[-\frac{1}{2} m \cosh(2r)(\phi_{1+}^2 + \phi_{1-}^2) \\
&\qquad\qquad + \frac{1}{4} m \cosh(2r) \tanh^2(2r)(\phi_{1+} + \phi_{1-})^2 \Bigg] \\
&\qquad \cdot \int \mathcal{D}\phi \exp\Bigg[-m\cosh(2r) \left(\phi - \frac{1}{2}\tanh(2r)(\phi_{1+} + \phi_{1-}) \right)^2 \Bigg] \\
&= \frac{1}{Z} \exp\Bigg[-\frac{1}{2} m \cosh(2r) \left(1 - \frac{1}{2}\tanh^2(2r) \right) (\phi_{1+}^2 + \phi_{1-}^2) \\
&\qquad\qquad + \frac{1}{2} m \cosh(2r) \tanh^2(2r) \phi_{1+}\phi_{1-} \Bigg].
\end{aligned}
\tag{20.64}
$$

In the last step, we have performed the Gaussian integral over ϕ and dropped an irrelevant factor. As expected, for $r > 0$, the density matrix ρ_1 now is not of pure state form anymore. It does not factor into a ket and a bra because of the term $\propto \phi_{1+}\phi_{1-}$ in the exponent.

Note the resemblance of (20.64) and (20.49). This is an indication that entanglement can sometimes lead to a locally thermal looking state, albeit it is globally pure.

21 Entropies and entanglement entropies

From an information theoretic point of view, entropy is a quantity that can be used to measure the information content of a probability distribution. It can be extended to quantum distributions formulated in terms of density matrices. We now first discuss the information theoretic significance of entropies logically and afterwards the quantum mechanical and quantum field theoretic implementation.

21.1 Shannon's information entropy

To a set of measurement outcomes, or more general realizations of a random variable, one can associate symbols $\{x_1, \ldots, x_N\}$ and probabilities

$$p(x_1), \ldots, p(x_N). \tag{21.1}$$

Of course one has

$$1 = p(x_1) + \ldots + p(x_N) = \sum_x p(x). \tag{21.2}$$

The last equation introduces a short hand notation we will use occasionally.

For two events X and Y with possible outcomes $\{x_m\}$ and $\{y_n\}$ one has a complete description in terms of *joint probabilities*

$$p(x_m, y_n), \tag{21.3}$$

such that $1 = \sum_{x,y} p(x, y)$. One should read the comma as "and". If the two events are *statistically independent* one has

$$p(x, y) = p(x)p(y), \tag{21.4}$$

but that is of course not always the case. More general, the reduced probabilities for one event are

$$p(x) = \sum_y p(x, y), \qquad p(y) = \sum_x p(x, y). \tag{21.5}$$

Assume now that one has already learned the outcome x_0, then the new probability distribution for y is

$$p(y_n | x_0) = \frac{p(x_0, y_n)}{\sum_k p(x_0, y_k)} = \frac{p(x_0, y_n)}{p(x_0)}, \tag{21.6}$$

which is the *conditional probability*. (Read: probability for y_n under the condition that x_0 has been obtained.) One can write

$$p(x_m, y_n) = p(y_n | x_m) \, p(x_m) = p(x_m | y_n) \, p(y_n), \tag{21.7}$$

which directly implies Bayes' theorem,

$$p(x_m | y_n) = \frac{p(y_n | x_m) \, p(x_m)}{p(y_n)}. \tag{21.8}$$

How much information can one learn from an outcome or event realization x? Or, in other words, how large is the information content $i(x)$ associated with the outcome x? Intuitively, the less likely the outcome, the higher the information content. Moreover, for independent events it is natural to take the information content additive,

$$i(p(x, y)) = i(p(x)p(y)) = i(p(x)) + i(p(y)). \tag{21.9}$$

This directly leads to the logarithm, and the definition of the *information content*

$$i(x) = i(p(x)) = -\ln p(x). \tag{21.10}$$

In principle one might add a (positive) prefactor here or, equivalently, take another base for the logarithm. Oftentimes $\log_2$ is used, but we work here with the natural logarithm. For example, throwing an ideal coin corresponds to $p(x_1) = p(x_2) = 1/2$ and by learning the event outcome one learns an amount of information

$$i = -\ln(1/2) = \ln 2, \tag{21.11}$$

corresponding to one bit of information. Note that a very unlikely event outcome with $p \to 0$ has formally infinite information content, $i \to \infty$. On the other side, a certain event outcome with unit probability, $p = 1$, has no information content, $i = 0$.

Shannon's *information entropy* associated to a discrete random variable or event X is the expected amount of information content,

$$H(X) = \langle i(x) \rangle = -\sum_x p(x) \ln p(x). \tag{21.12}$$

Note that the information entropy is a functional of the probability distribution only, $H(X) = H[p(x)]$.

Some properties of information entropy are

i) Non-negativity. Information entropy for discrete random variables is positive semi-definite, $H(X) \geq 0$.

ii) Concavity. Define a random variable X with probability distribution $p(x)$ out of distributions $p_A(x)$ and $p_B(x)$ for the variables X_A and X_B such that

$$p(x) = q\, p_A(x) + (1 - q)\, p_B(x). \tag{21.13}$$

One has then

$$H(X) \geq qH(X_A) + (1 - q)H(X_B). \tag{21.14}$$

Exercise: Show this property.

iii) Permutation invariance. If one relabels the event outcomes by some permutation of indices $x_m \to x_{\pi(m)}$, the information entropy is unchanged. This is directly clear from the definition.

iv) Minimum value. One has $H(X) = 0$ if and only if X is deterministic such that $p(x) = 1$ for one outcome x.

v) Maximum value. For a set of event outcomes of size or cardinality $|X|$ one has

$$H(X) \leq \ln |X|. \tag{21.15}$$

Proof: maximize the probability distribution $p(x)$ with the condition that it remains normalized. This corresponds to finding the extremum of

$$\mathcal{L} = H(X) + \lambda \left(\sum_x p(x) - 1 \right). \tag{21.16}$$

One has

$$\frac{\partial \mathcal{L}}{\partial p(x)} = -\ln p(x) - 1 + \lambda = 0 \qquad \Rightarrow \qquad p(x) = e^{\lambda - 1}. \tag{21.17}$$

This depends only on λ but must be normalized. Normalization leads to $p(x) = 1/|X|$ and $H(X) = \ln |X|$.

21.2 Relative entropy (or Kullback-Leibler divergence)

The relative entropy is a useful quantity to compare two probability distributions $p(x)$ and $q(x)$. It is defined by

$$D(p||q) = \sum_x p(x) \left[\ln p(x) - \ln q(x) \right].$$

(21.18)

This definition works if the support of the function $p(x)$ (the set of values x where $p(x) > 0$) is within the support of the function $q(x)$. In that case there are no points where $p(x) > 0$ but $q(x) = 0$. For cases where this condition is not fulfilled, one can extend the definition of relative entropy in a natural way to $D(p||q) = \infty$. Note that one can write the relative entropy as an expectation value with respect to the probability distribution $p(x)$,

$$D(p||q) = \left\langle \ln \left(\frac{p(x)}{q(x)} \right) \right\rangle_p.$$

(21.19)

The relative entropy tells in some sense how far the distribution function $p(x)$ is from the distribution function $q(x)$. However, it is not defined symmetrically (nor does it satisfy a triangle inequality) and is therefore not a distance measure or metric in the mathematical sense.

Typically, relative entropy is used to compare two probability distributions where $p(x)$ is often the true distribution and $q(x)$ some kind of approximation to it. It then has the meaning of a gain in information when one replaces the (approximate) model distribution $q(x)$ by the true (or more accurate) distribution $p(x)$. It can also denote a loss of information, or added uncertainty, if one works with $q(x)$ instead of $p(x)$.

To illustrate the asymmetry in the definition, consider the following two examples.

i) Take $p(x_1) = 1$ and $p(x_2) = p(x_3) = \ldots = 0$ and compare this to $q(x_1) = 1 - \epsilon$, $q(x_2) = \epsilon$ and $q(x_3) = \ldots = q(x_N) = 0$. One has

$$D(p||q) = -\ln(1 - \epsilon) \approx \epsilon.$$

(21.20)

The gain in knowledge from $q(x)$ to $p(x)$ is moderate and vanishes for $\epsilon \to 0$ because $q(x)$ is already very close to $p(x)$ on all outcomes x allowed by the true probability distribution $p(x)$ (i. e. on x_1). An experiment that has been designed in an optimal way based on the (wrong) prior distribution $q(x)$ will nevertheless find the correct distribution $p(x)$ rather efficiently.

ii) Take instead $p(x_1) = 1 - \epsilon$, $p(x_2) = \epsilon$ and $p(x_3) = \ldots = p(x_N) = 0$ and compare this to $q(x_1) = 1$ and $q(x_2) = \ldots = q(x_N) = 0$. One has now formally

$$D(p||q) = (1 - \epsilon) \ln \left(\frac{1 - \epsilon}{1} \right) + \epsilon \ln \left(\frac{\epsilon}{0} \right) = \infty.$$

(21.21)

Here the gain in information when replacing the model distribution $q(x)$ by the true distribution $p(x)$ is much larger. The model distribution $q(x)$ vanishes on the state x_2 which has actually non-vanishing probability according to $p(x)$. It is very difficult to find out about this and one needs formally infinitely good statistics in an experiment that is optimized based on the prior distribution $q(x)$ (assuming misguidedly that x_2 is never realized).

An important property of relative entropy is its non-negativity,

$$D(p||q) \geq 0.$$

(21.22)

This follows from the inequality $\ln(x) \leq x - 1$. One sees directly

$$D(p||q) = \sum_x p(x) \ln \left(\frac{p(x)}{q(x)} \right) = -\sum_x p(x) \ln \left(\frac{q(x)}{p(x)} \right) \geq \sum_x p(x) \left(1 - \frac{q(x)}{p(x)} \right) = 0.$$

(21.23)

Relative entropy has the advantage that it generalizes favorably to continuous probability distributions. Consider the continuum limit

$$p(x_m) \to \mathcal{P}(x)dx, \qquad q(x_m) \to \mathcal{Q}(x)dx, \tag{21.24}$$

with *probability densities* $\mathcal{P}(x)$ and $\mathcal{Q}(x)$. The relative entropy becomes

$$D(p||q) = \int dx\, \mathcal{P}(x) \left[\ln(\mathcal{P}(x)dx) - \ln(\mathcal{Q}(x)dx)\right] = \int dx\, \mathcal{P}(x) \left[\ln \mathcal{P}(x) - \ln \mathcal{Q}(x)\right]. \tag{21.25}$$

In contrast, Shannon's entropy has the formal continuum limit

$$H(X) \to \int dx\, \mathcal{P}(x) \ln(\mathcal{P}(x)dx), \tag{21.26}$$

which is not very well defined.

The relative entropy is positive semi-definite and one has $D(p||q) = 0$ if and only if $p(x) = q(x)$. However, it is not a distance measure in the mathematical sense because it is not symmetric, $D(p||q) \neq D(q||p)$. In the limit where p and q are very close, it satisfies the properties of a metric, however. To make this more precise, consider a set of probability distributions $p(\theta)(x)$ where θ is a (multi-dimensional) parameter. Close to some point θ_0 one may expand

$$D(p(\theta)||p(\theta_0)) = \frac{1}{2}(\theta^j - \theta_0^j)(\theta^k - \theta_0^k)g_{jk}(\theta_0) + \dots. \tag{21.27}$$

The constant and linear terms vanish because $D(p||q)$ is positive semi-definite and vanishes at $p = q$. The object $g_{jk}(\theta_0)$ is known as the *Fisher information metric*. It is by construction symmetric and serves as a Euclidean, positive semi-definite metric on the space of parameters θ of probability distributions $p(\theta)$. One may also see this as a two-form constructed via the pull-back of the map $\theta \to p(\theta)$ from a metric directly defined on the space of probability functions. The latter is obtained from the expansion

$$D(q + \delta q||q) = \sum_x \frac{1}{2q(x)}\delta q(x)^2 + \dots. \tag{21.28}$$

We have set here $p(x) = q(x) + \delta q(x)$ and used that both p and q are normalized such that $\sum_x \delta q(x) = 0$.

21.3 von Neumann's quantum entropy

The *quantum entropy* or *von Neumann entropy* is defined for a given quantum density matrix (or density operator) ρ as

$$S(\rho) = -\text{tr}\left\{\rho \ln \rho\right\}. \tag{21.29}$$

The logarithm of an operator is here defined via its eigenvalues. Recall that $\rho = \rho^\dagger$ is hermitean and can always be diagonalized such that it has the form

$$\rho = p_j |j\rangle\langle j|, \tag{21.30}$$

where the states $|j\rangle$ are orthogonal and normalized, $\langle j|k\rangle = \delta_{jk}$. In this basis one has

$$S(\rho) = -\sum_j p_j \ln p_j = H(p) \tag{21.31}$$

with Shannon entropy H. The quantum entropy has the properties

i) Non-negativity.
$$S(\rho) \geq 0. \tag{21.32}$$
This follows from the fact that the density matrix has eigenvalues in the range $0 \leq p_j \leq 1$. Of course, $S(\rho) = 0$ for a pure state $\rho = |\psi\rangle\langle\psi|$ while mixed states have $S > 0$.

ii) Maximum value. Occurs for maximally mixed states. In an N-dimensional Hilbert space this corresponds to $\rho = \text{diag}(1/N)$ and $S(\rho) = \ln N$.

iii) Invariance under unitary transformations. The density operator transforms as
$$\rho \to U\rho U^\dagger, \tag{21.33}$$
and the von Neumann entropy is invariant,
$$S(U\rho U^\dagger) = S(\rho). \tag{21.34}$$
This is immediately clear because unitary transformations due not change the eigenvalues.

21.4 Quantum relative entropy

The quantum version of relative entropy is defined for two normalized density matrices ρ and σ by
$$S(\rho||\sigma) = \text{tr}\left\{\rho\left[\ln\rho - \ln\sigma\right]\right\}. \tag{21.35}$$
As in the classical case, this holds for $\sup(\rho) \subseteq \sup(\sigma)$ and is extended naturally by setting $S(\rho||\sigma)$ otherwise. Quantum relative entropy has rather useful properties and should presumably be used more often in quantum field theory.

21.5 Rényi entropy

The *quantum Rényi entropy* is defined for a parameter $N > 0$ (not necessarily integer) by
$$S_N(\rho) = \frac{1}{1-N} \ln\text{tr}\left\{\rho^N\right\}. \tag{21.36}$$
Here we have assumed that ρ is normalized, $\text{tr}\rho = 1$. Otherwise, one must include appropriate normalization factors in (21.36).

As a special case, for $N \to 1$ the Rényi entropy becomes von Neumanns entropy,[2]
$$\begin{aligned} S_N(\rho) &= \frac{1}{1-N}\ln\text{tr}\left\{e^{N\ln\rho}\right\} = \frac{1}{1-N}\ln\text{tr}\left\{\rho\left(1 + (N-1)\ln\rho + \ldots\right)\right\} \\ &\overset{N\to 1}{\to} -\text{tr}\left\{\rho\ln\rho\right\} = S(\rho). \end{aligned} \tag{21.37}$$
One can sometimes calculate the Rényi entropy for arbitrary integer values $N \geq 2$ and determine the von Neumann entropy via analytic continuation $N \to 1$.

Note that for $N = 2$ one has simply
$$S_2(\rho) = -\ln\text{tr}\left\{\rho^2\right\}. \tag{21.38}$$
This is often a relatively simple quantity to compute (or measure experimentally) and allows to distinguish pure states with $S_2 = 0$ from mixed states with $S_2 > 0$.

One can also define a generalization of relative entropy in a similar way, the *quantum Rényi relative entropy*
$$S_N(\rho||\sigma) = \frac{1}{N-1}\ln\frac{\text{tr}\left\{\rho\sigma^{N-1}\right\}}{\text{tr}\left\{\rho^N\right\}} = \frac{1}{N-1}\ln\text{tr}\left\{\rho\sigma^{N-1}\right\} - S_N(\rho). \tag{21.39}$$
In the limit $N \to 1$ the quantum Rényi relative entropy approaches the quantum relative entropy, $S_N(\rho||\sigma) \overset{N\to 1}{\to} S(\rho||\sigma)$. *Exercise*: verify this.

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[2] See for instance "Geometric and Renormalized Entropy in Conformal Field Theory" by Holzhey et al. (1994) or "Entanglement Entropy and Quantum Field Theory" by Calabrese and Cardy (2004).

21.6 Joint quantum entropy

The density operator of a composite system $A + B$ is ρ_{AB} and has the joint entropy

$$S(\rho_{AB}) = -\text{tr}\left\{\rho_{AB}\ln\rho_{AB}\right\}, \tag{21.40}$$

where the trace goes over both systems, $\text{tr} = \text{tr}_A\text{tr}_B$. If there is no doubt, we will denote the full density matrix simply as $\rho_{AB} = \rho$ and the joint entropy as $S(\rho)$.

21.7 Entanglement entropy

The reduced density matrix for subsystem A of the full system $A + B$ is defined by

$$\rho_A = \text{tr}_B\{\rho\}. \tag{21.41}$$

The associated von Neumann entropy is also known as the *entanglement entropy*

$$S_A = -\text{tr}_A\left\{\rho_A\ln\rho_A\right\}. \tag{21.42}$$

In a similar way, the reduced density matrix for system B is $\rho_B = \text{tr}_A\{\rho\}$ and the associated entropy is $S_B = -\text{tr}_B\{\rho_B\ln\rho_B\}$.

For a pure state $\rho = |\phi\rangle\langle\phi|$ one can write in the *Schmidt basis*

$$|\phi\rangle = \sum_j \sqrt{\lambda_j}\,|j_A\rangle|j_B\rangle, \tag{21.43}$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$. The reduced density matrices are

$$\begin{aligned}
\rho_A &= \sum_j \lambda_j |j_A\rangle\langle j_A|, \\
\rho_B &= \sum_j \lambda_j |j_B\rangle\langle j_B|.
\end{aligned} \tag{21.44}$$

One has then for the entanglement entropy

$$S_A = S_B = -\sum_j \lambda_j \ln\lambda_j, \tag{21.45}$$

and in particular both entropies are equal. (This is not necessarily the case for mixed states $\rho \neq |\phi\rangle\langle\phi|$.) For a product state, only one coefficient $\lambda_0 = 1$ is non-vanishing and $S_A = S_B = 0$. Entangled states have several non-vanishing Schmidt basis coefficients and in this sense $S_A = S_B > 0$ is here a measure for the amount of entanglement between subsystems A and B.

21.8 Entropies in quantum field theory

The definitions for the Rényi entropy in eq. (21.36) with the von Neumann entropy as a limit and the Rényi relative entropy in eq. (21.39) with the quantum relative entropy in eq. (21.35) as a limit, can directly be used in quantum field theory. For example, the Rényi entropy can be calculated in terms of the density matrix function $S[\phi_+, \phi_-]$ as

$$S_N(\rho) = \frac{1}{1-N}\ln\text{tr}\left\{\rho^N\right\} = \frac{1}{1-N}\ln\left[\int \mathcal{D}\phi_1\cdots\mathcal{D}\phi_N\ \rho[\phi_1, \phi_2]\rho[\phi_2, \phi_3]\cdots\rho[\phi_N, \phi_1]\right]. \tag{21.46}$$

We have again assumed standard normalization, $\int \mathcal{D}\phi\ \rho[\phi, \phi] = 1$.

One can also extend this to entanglement entropies or relative entanglement entropies when the density functionals result from taking partial traces, for example over some regions in space.

For example, one may define a density matrix functional for some region A on some Cauchy hypersurface Σ by taking the trace over the complement region $B = \Sigma - A$,

$$\rho_A = \text{tr}_B \left\{ \rho \right\}. \tag{21.47}$$

The corresponding von Neumann entropy

$$S_A = -\text{tr}_A \left\{ \rho_A \ln \rho_A \right\} \tag{21.48}$$

is then the entanglement entropy quantifying entanglement between regions A and B. One should note that such entanglement entropies are typically divergent in the ultraviolett because of very many entangled modes.

Relative entanglement entropies, however, have the potential to be well defined and finite. It is an interesting topic of current research to formulate quantum field dynamics in terms of relative entanglement entropies.

21.9 Gaussian density matrices

Let us discuss the determination of entanglement entropies more concretely for a Gaussian density matrix. Without essential loss of generality, we will assume $\bar{\phi}_m = \langle \phi_m \rangle = 0$ and also $\langle \pi_m \rangle = 0$. Again we use an abstract index notation where m includes discrete and continuous indices, for example positions on a Cauchy hypersurface or some subregion of it.

We take the density matrix to be of the form

$$\rho[\phi_+, \phi_-] = \frac{1}{Z} \exp \left[-\frac{1}{2} (\phi_+, \phi_-)_m \begin{pmatrix} h_{mn} & -\Sigma_{mn} \\ -\Sigma_{mn} & h_{mn} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}_n \right]. \tag{21.49}$$

For $\Sigma_{mn} = 0$, this would factorize into a ket and a bra contribution and describe a pure state. For $\Sigma_{mn} \neq 0$ this is not the case, however, and $\rho[\phi_+, \phi_-]$ describes a mixed state. This could be the result of incomplete knowledge, for example at non-zero temperature, or a result of entanglement if ρ describes a subsystem.

To calculate the Rényi entropy, we need

$$\frac{\text{tr} \left\{ \rho^N \right\}}{\text{tr} \left\{ \rho \right\}^N} = \frac{\int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_N \ \rho[\phi_1, \phi_2] \cdots \rho[\phi_N, \phi_1]}{\left(\int \mathcal{D}\phi \ \rho[\phi, \phi] \right)^N}. \tag{21.50}$$

This calculation can be done here rather directly because all functional integrals are Gaussian. The integral in the numerator can be written as

$$\int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_N \ \exp \left[-\frac{1}{2} (\phi_1, \cdots, \phi_N)_m \begin{pmatrix} 2h & -\Sigma & & & -\Sigma \\ -\Sigma & 2h & -\Sigma & & \\ & -\Sigma & 2h & -\Sigma & \\ & & & \ddots & \\ -\Sigma & & & & 2h \end{pmatrix}_{mn} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_N \end{pmatrix}_n \right] \tag{21.51}$$

Similarly, the denominator can be written as

$$\int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_N \ \exp \left[-\frac{1}{2} (\phi_1, \cdots, \phi_N)_m \begin{pmatrix} (2h - 2\Sigma) & & \\ & \ddots & \\ & & (2h - 2\Sigma) \end{pmatrix}_{mn} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}_n \right] \tag{21.52}$$

Note that the abstract field indices m and n are not to be confused with the "replica" indices $1, \cdots, N$. The result of the functional integral will be the determinant of the matrix (21.51) divided by the determinant of the matrix in (21.52) to the power $-1/2$. It can be stated as

$$\frac{\text{tr} \left\{ \rho^N \right\}}{\text{tr} \left\{ \rho \right\}^N} = \exp \left[-\frac{1}{2} \text{tr} \ln \det(\mathbf{M}_N) \right] \tag{21.53}$$

which contains the N-dimensional cyclic matrix (with entries being matrices in abstract index space)

$$\mathbf{M}_N = (1 + 2a)\mathbf{1}_N - a\mathbf{Z}_N - a\mathbf{Z}_N^T. \tag{21.54}$$

Here we defined $\mathbf{1}_N$ to be the N-dimensional unit matrix and $\mathbf{Z}_N$ to be the N-dimensional cyclic matrix $(\mathbf{Z}_N)_j = \delta_{(j+1)k}$. Here j, k are in the range $1, \cdots, N$ and the index $j = N + 1$ is to be identified with the index $j = 1$. We also use

$$a_{mn} := \left[(2h - 2\Sigma)^{-1} \, \Sigma \right]_{mn} . \tag{21.55}$$

One can furthermore write

$$\mathbf{M}_N = \mathbf{A}_N(a)\mathbf{A}_N^T(a^T) \tag{21.56}$$

with

$$\mathbf{A}_N(a) = \left(\sqrt{\frac{1}{4} + a} + \frac{1}{2} \right) \mathbf{1}_N - \left(\sqrt{\frac{1}{4} + a} - \frac{1}{2} \right) \mathbf{Z}_N. \tag{21.57}$$

The determinant of $\mathbf{A}_N(a)$ is found to be

$$\det \left(\mathbf{A}_N(a) \right) = \left(\sqrt{\frac{1}{4} + a} + \frac{1}{2} \right)^N - \left(\sqrt{\frac{1}{4} + a} - \frac{1}{2} \right)^N \tag{21.58}$$

and one can use

$$\det(\mathbf{M}_N) = \det(\mathbf{A}_N)^2. \tag{21.59}$$

Collecting terms, one finds for the Rényi entropy

$$S_N(\rho) = \frac{1}{N-1} \left\{ \text{tr} \ln \left(\left(\sqrt{\frac{1}{4} + a} + \frac{1}{2} \right)^N - \left(\sqrt{\frac{1}{4} + a} - \frac{1}{2} \right)^N \right) \right\} . \tag{21.60}$$

The remaining trace goes over the (abstract) field indices m, n.

In the present case, one can actually take the limit $N \to 1$ and one finds for the von Neumann entropy $S(\rho) := \lim_{N \to 1} S_N(\rho)$,

$$S(\rho) = \text{tr} \left\{ \left(\sqrt{\frac{1}{4} + a} + \frac{1}{2} \right) \ln \left(\sqrt{\frac{1}{4} + a} + \frac{1}{2} \right) - \left(\sqrt{\frac{1}{4} + a} - \frac{1}{2} \right) \ln \left(\sqrt{\frac{1}{4} + a} - \frac{1}{2} \right) \right\} . \tag{21.61}$$

In a similar way, one can treat the relative entropy between two Gaussian density matrices.

Let us emphasize again that what has been sketched here is only a first step of a more detailed information theoretic approach to quantum field theory. Many questions remain open so far but will likely be addressed by current research in the next few years.