Functional renormalization group for the effective average action

physics at different length scales

- microscopic theories : where the laws are formulated
- effective theories : where observations are made
- effective theory may involve different degrees of freedom as compared to microscopic theory
- example: the motion of the earth around the sun does not need an understanding of nuclear burning in the sun

QCD :
Short and long distance
degrees of freedom are different !

Short distances : quarks and gluons
Long distances : baryons and mesons

How to make the transition?

confinement/chiral symmetry breaking

collective
degrees of freedom

Hubbard model

- Electrons on a cubic lattice

  here : on planes (d = 2)

- Repulsive local interaction if two electrons are on the same site
- Hopping interaction between two neighboring sites

Hubbard model

Functional integral formulation

$$Z[\eta] = \int_{\hat\psi(\beta)=-\hat\psi(0),\hat\psi^*(\beta)=-\hat\psi^*(0)} \mathcal{D}(\hat\psi^*(\tau),\hat\psi(\tau))$$
$$\exp\Big(-\int_0^\beta d\tau\Big(\sum_{\mathbf{x}} \hat\psi^\dagger_{\mathbf{x}}(\tau)\big(\frac{\partial}{\partial\tau}-\mu\big)\hat\psi_{\mathbf{x}}(\tau)$$
$$+\sum_{\mathbf{xy}} \hat\psi^\dagger_x(\tau)\mathcal{T}_{\mathbf{xy}}\hat\psi_{\mathbf{y}}(\tau)$$
$$+\frac{1}{2}U\sum_{\mathbf{x}}(\hat\psi^\dagger_{\mathbf{x}}(\tau)\hat\psi_{\mathbf{x}}(\tau))^2$$
$$-\sum_{\mathbf{x}}(\eta^\dagger_{\mathbf{x}}(\tau)\hat\psi_{\mathbf{x}}(\tau)+\eta^T_{\mathbf{x}}(\tau)\hat\psi^*_{\mathbf{x}}(\tau))\Big)\Big)$$

next neighbor interaction

$$\mathcal{T}_{xy} = \begin{cases} -t & \text{, if } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are nearest neighbors} \\ 0 & \text{, else} \end{cases}$$

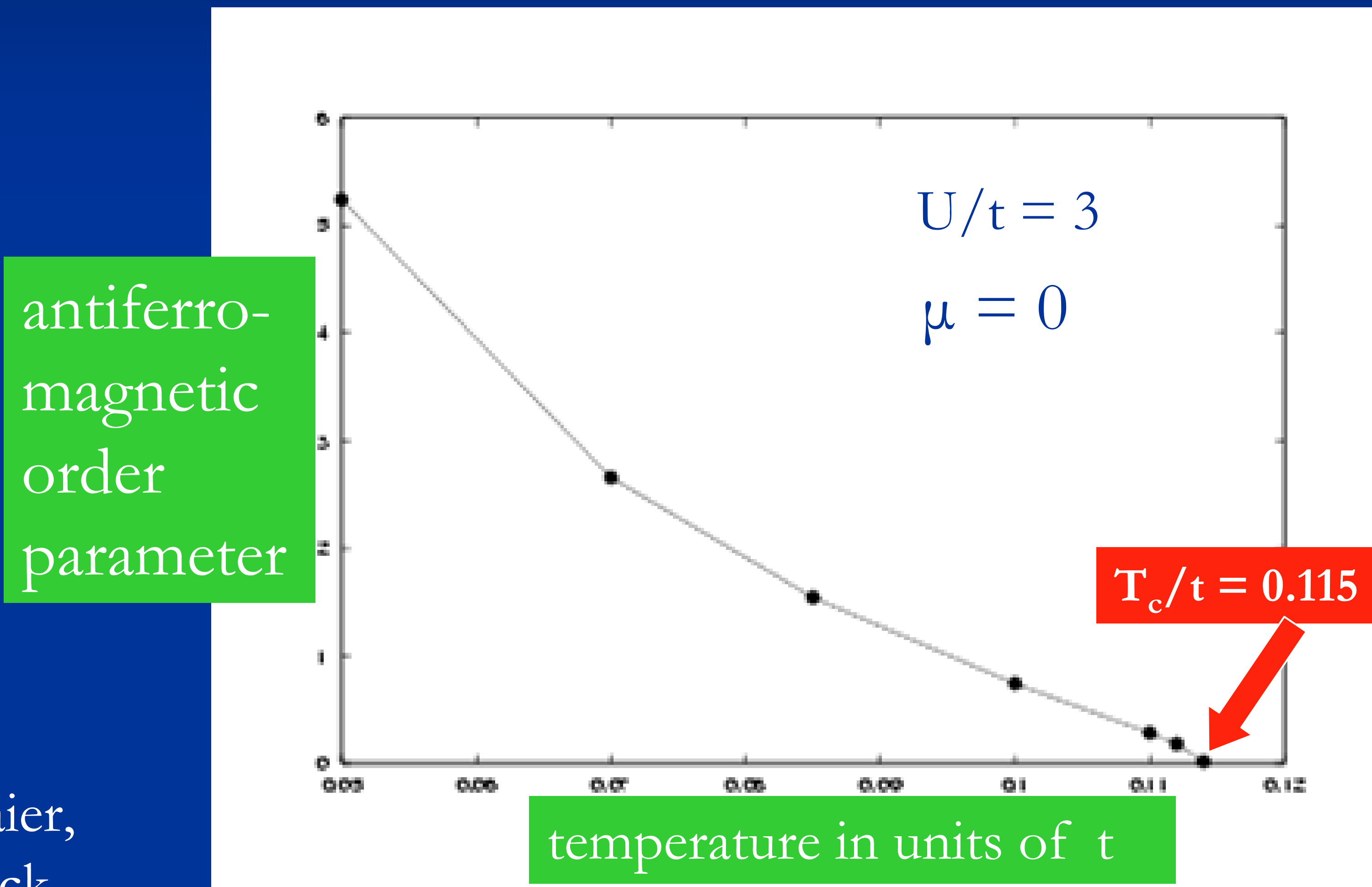
U > 0 :
repulsive local interaction

External parameters
T : temperature
μ : chemical potential
(doping)

In solid state physics :
“ model for everything “

- Antiferromagnetism
- High T_c superconductivity
- Metal-insulator transition
- Ferromagnetism

Antiferromagnetism in d=2 Hubbard model



T.Baier, E.Bick,…

antiferromagnetic order is finite size effect

- here size of experimental probe 1 cm
- vanishing order for infinite volume
- consistency with Mermin-Wagner theorem
- dependence on probe size very weak

Collective degrees of freedom are crucial !

for $T < T_c$

- nonvanishing order parameter

$$\tilde{\vec{m}}(X) = \hat{\psi}^{\dagger}(X)\vec{\sigma}\hat{\psi}(X)$$

$$\hat{\vec{m}}(Q) \rightarrow \vec{a}\delta(Q - \Pi)$$

- gap for fermions

- low energy excitations:
  antiferromagnetic spin waves

effective theory / microscopic theory

- sometimes only distinguished by different values of couplings
- sometimes different degrees of freedom
- need for methods that can cope with such situations

Functional Renormalization Group

describes flow of effective action from small to large length scales

perturbative renormalization : case where only couplings change , and couplings are small

How to come from quarks and gluons to baryons and mesons ?

How to come from electrons to spin waves ?

Find effective description where relevant degrees of freedom depend on momentum scale or resolution in space.

Microscope with variable resolution:

- High resolution , small piece of volume: quarks and gluons
- Low resolution, large volume : hadrons

From

Microscopic Laws

(Interactions, classical action)

to

Fluctuations!

Macroscopic Observation

**(Free energy functional,
effective action)**

- block spins

  Kadanoff, Wilson

- exact renormalization group equations

  Wilson, Kogut

  Wegner, Houghton

  Weinberg

  Polchinski

  Hasenfratz^2

- Lattice finite size scaling

  Lüscher,...

- coarse grained free energy/average action

effective average action

Unified picture for scalar field theories with symmetry O(N) in arbitrary dimension **d** and arbitrary **N**

linear or nonlinear sigma-model for
chiral symmetry breaking in QCD
or:
scalar model for antiferromagnetic spin waves
(linear O(3) – model)

fermions will be added later

Effective potential includes all fluctuations

Average potential U_k

$\equiv$ scale dependent effective potential
$\equiv$ coarse grained free energy

$\boxed{\text{Only fluctuations with momenta } q^2 > k^2 \text{ included}}$

k: infrared cutoff for fluctuations, "average scale"

Λ: characteristic scale for microphysics

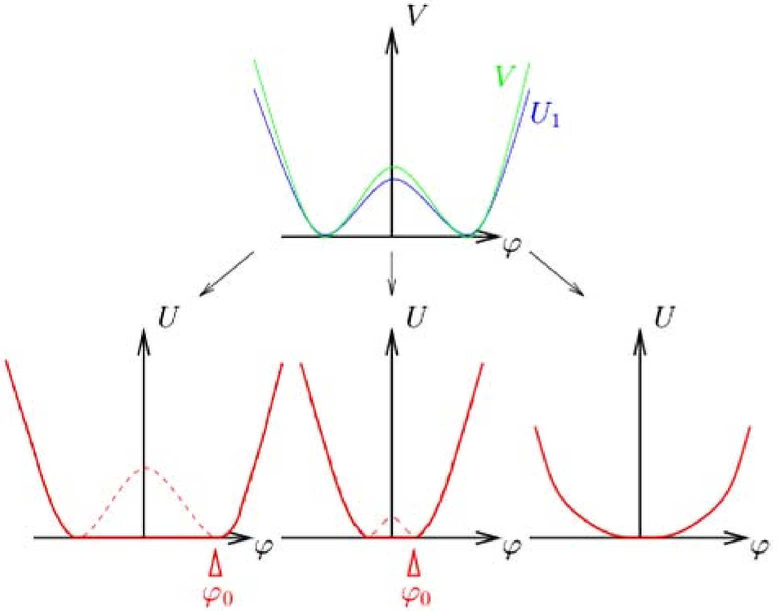
$$U_\Lambda \approx S \to U_0 \equiv U$$

Scalar field theory

$\varphi_a(x)$: magnetization, density, chemical concentration, Higgs field, meson field, inflaton, cosmon

$O(N)$-symmetry:

$$S = \int d^d x \left\{ \frac{1}{2} \partial_\mu \varphi_a \partial_\mu \varphi_a + V(\rho) \right\}; \quad \rho = \frac{1}{2} \varphi_a \varphi_a$$



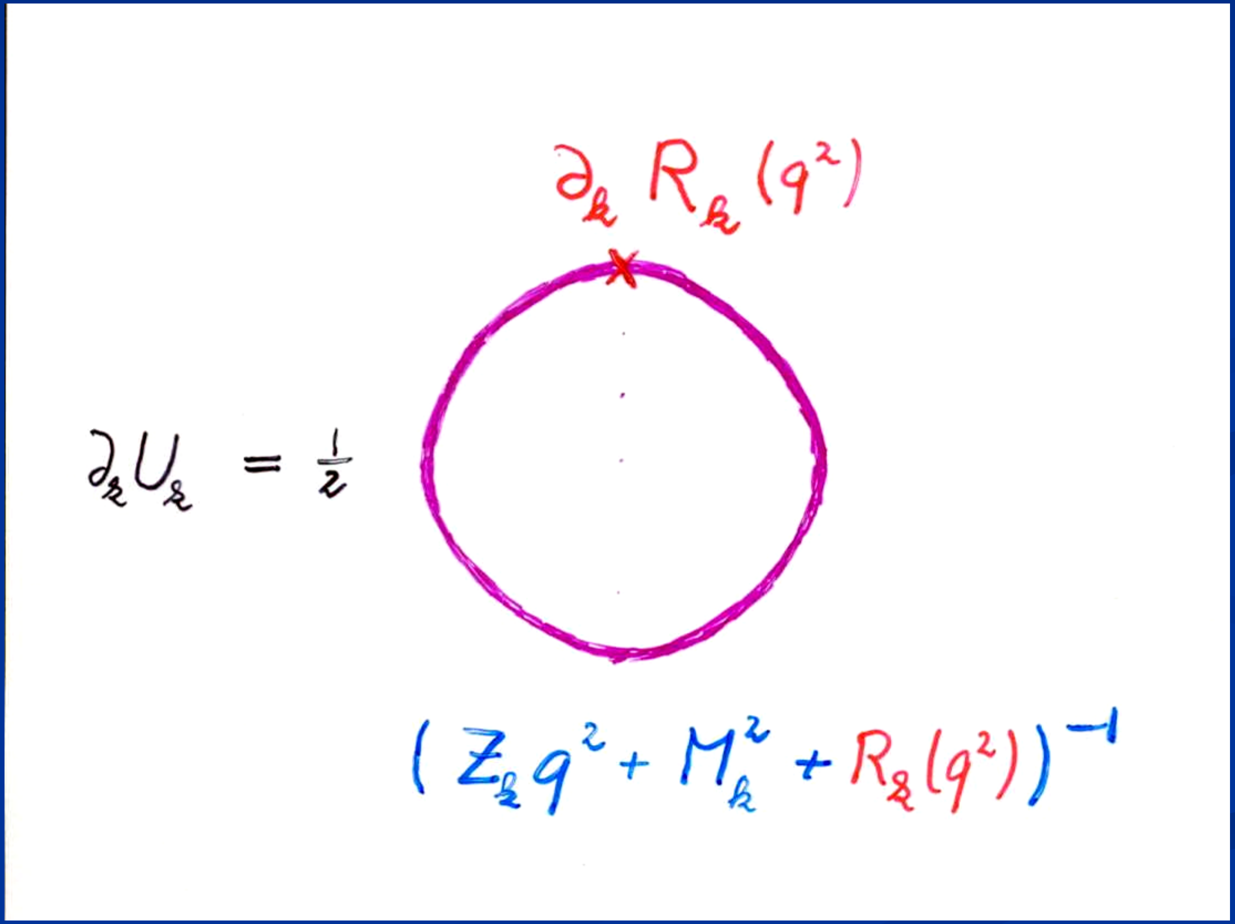
Flow equation for average potential

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

Simple one loop structure – nevertheless (almost) exact



Infrared cutoff

R_k : IR-cutoff

e.g $R_k = \dfrac{Z_k q^2}{e^{q^2/k^2} - 1}$

or $R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$ (Litim)

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Flow equation for U_k

$$\boxed{\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^dq}{(2\pi)^d}\frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}}$$

'91

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

R_k : IR-cutoff

e.g $\quad R_k = \dfrac{Z_k q^2}{e^{q^2/k^2} - 1}$

or $\quad R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$ (Litim)

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Partial differential equation for function U(k,φ) depending on two (or more) variables

$Z_k = c\, k^{-\eta}$

Regularisation

For suitable R_k :

$$R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$
$$R_k = Z_k (k^2 - q^2) \Theta(k^2 - q^2)$$

- Momentum integral is ultraviolet and infrared finite
- Numerical integration possible
- Flow equation defines a regularization scheme (ERGE –regularization)

$$\boxed{\partial_k U_k(\varphi) = \frac{1}{2} \sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}}$$

Integration by momentum shells

$$\boxed{\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}}$$

Momentum integral

is dominated by

$q^2 \sim k^2$.

Flow only sensitive to

physics at scale k

Wave function renormalization and anomalous dimension

Z_k: wave function renormalization

$$k\partial_k Z_k = -\eta_k Z_K$$

η_k: anomalous dimension

$$t = \ln(k/\Lambda)$$

$$\partial_t \ln Z = -\eta$$

for $Z_k\,(\varphi, q^2)$: flow equation is **exact !**

Scaling form of evolution equation

$$u = \frac{U_k}{k^d}$$

$$\tilde{\rho} = Z_k k^{2-d} \rho$$

$$u' = \frac{\partial u}{\partial \tilde{\rho}} \quad \text{etc.}$$

$$\boxed{\begin{aligned} \partial_t u|_{\tilde{\rho}} = & -du + (d-2+\eta)\tilde{\rho}u' \\ & +2v_d\{l_0^d(u'+2\tilde{\rho}u'';\eta) \\ & \quad +(N-1)\, l_0^d(u';\eta)\} \end{aligned}}$$

$$v_d^{-1} = 2^{d+1}\pi^{d/2}\,\Gamma\left(\frac{d}{2}\right)$$

linear cutoff:

$$l_0^d(w;\eta) = \frac{2}{d}\left(1 - \frac{\eta}{d+2}\right)\frac{1}{1+w}$$

On r.h.s. :
neither the scale k
nor the wave function
renormalization Z
appear explicitly.

Scaling solution:
no dependence on t;
corresponds
to second order
phase transition.

Tetradis …

decoupling of heavy modes

$$l_0^d(w;\eta) = \frac{2}{d}\left(1 - \frac{\eta}{d+2}\right)\frac{1}{1+w}$$

threshold functions vanish for large w : large mass as compared to k

$$\boxed{\begin{aligned}\partial_t u|_{\tilde\rho} = & -du + (d-2+\eta)\tilde\rho u' \\ & +2v_d\{l_0^d(u'+2\tilde\rho u'';\eta) \\ & \quad +(N-1)\,l_0^d(u';\eta)\}\end{aligned}}$$

Flow involves effectively only modes with mass smaller or equal k

unnecessary heavy modes are eliminated automatically

⟹ effective theories

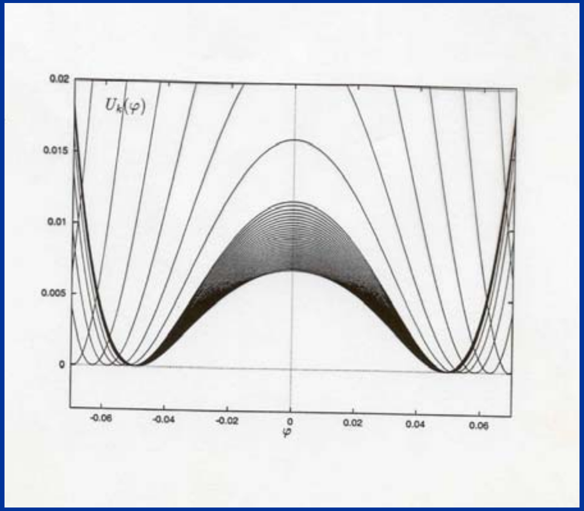
addition of new collective modes still needs to be done

unified approach

- choose N
- choose d
- choose initial form of potential
- run !

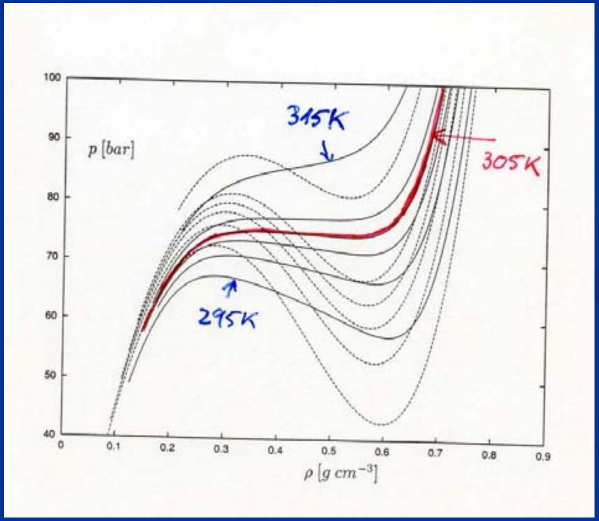
Flow of effective potential

Ising model



S.Seide …

CO_2



Experiment :

T_* =304.15 K
p_* =73.8.bar
ρ_* = 0.442 g cm-2

Critical exponents

$d = 3$

Critical exponents ν and η

N	ν		η	
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
3	0.704	0.7102	0.049	0.0380
4	0.739	0.7474	0.047	0.0363
10	0.881	0.886	0.028	0.025
100	0.990	0.980	0.0030	0.003
		↑		↑

"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

Critical exponents , d=3

N	ν ERGE	ν world	η ERGE	η world
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
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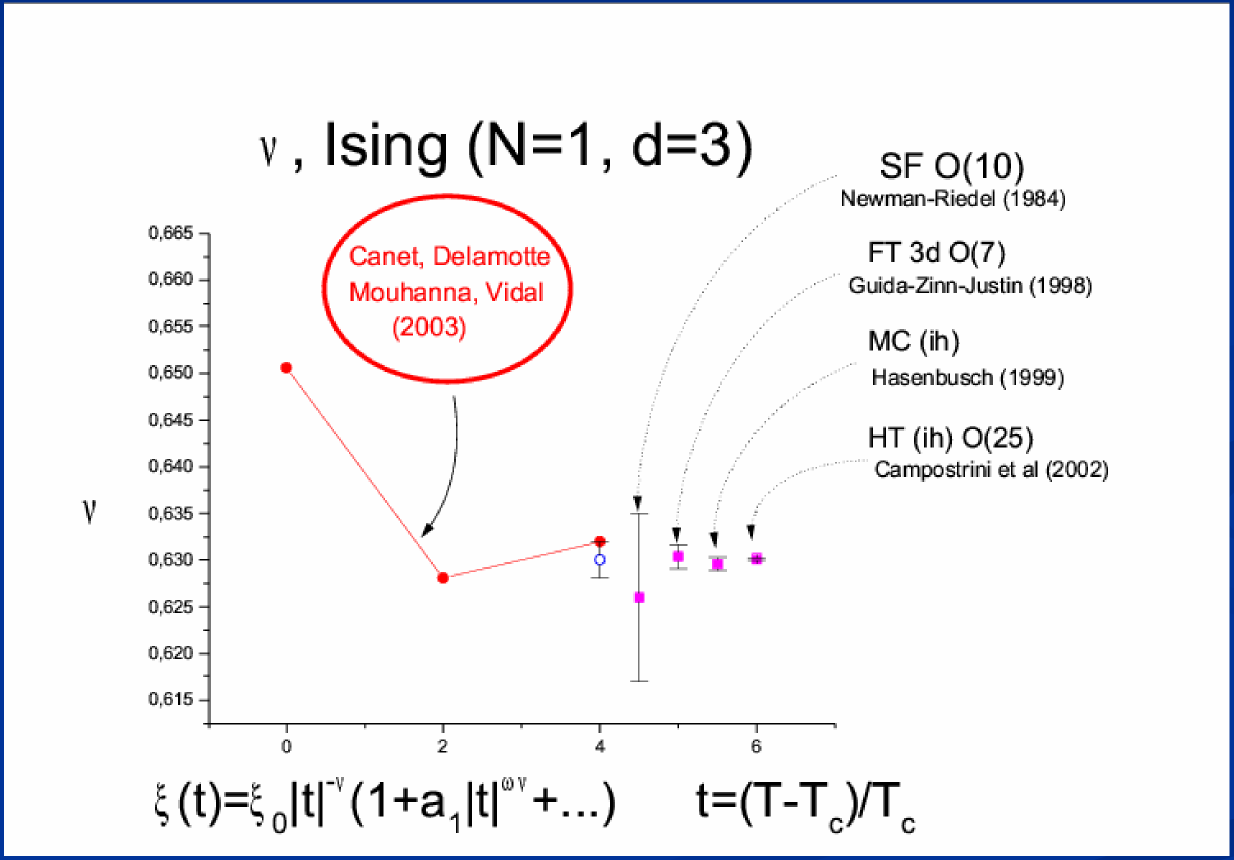
"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

derivative expansion

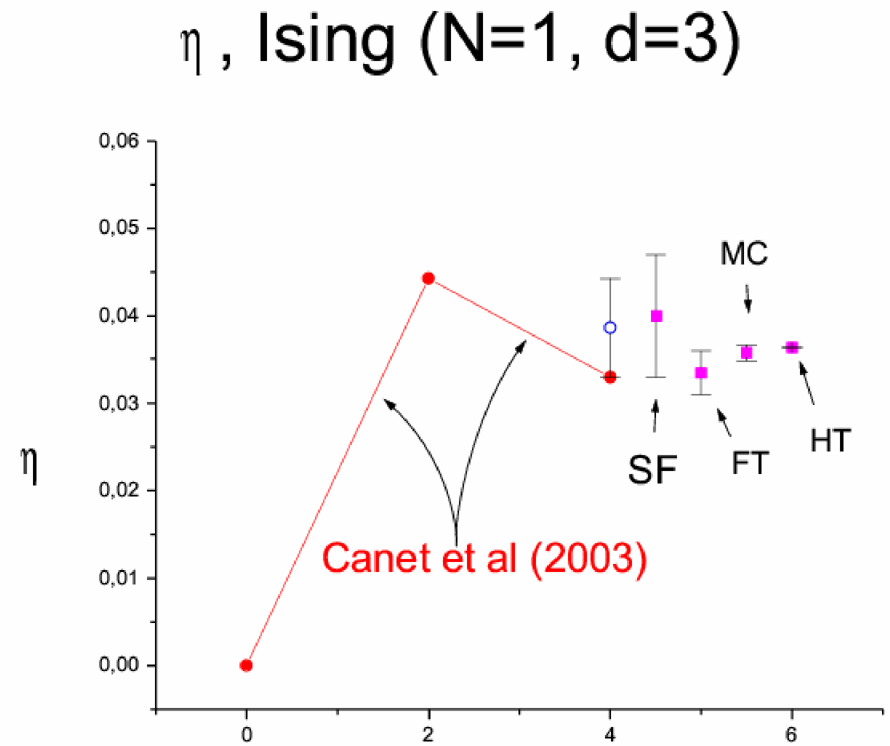
- good results already in lowest order in derivative expansion : one function u to be determined
- second order derivative expansion - include field dependence of wave function renormalization : three functions to be determined

apparent convergence of derivative expansion



from talk by Bervilliers

anomalous dimension



$$\Gamma^{(2)}[q] \underset{q\to\infty}{\propto} q^{2-\eta} \quad \text{for } T = T_c$$

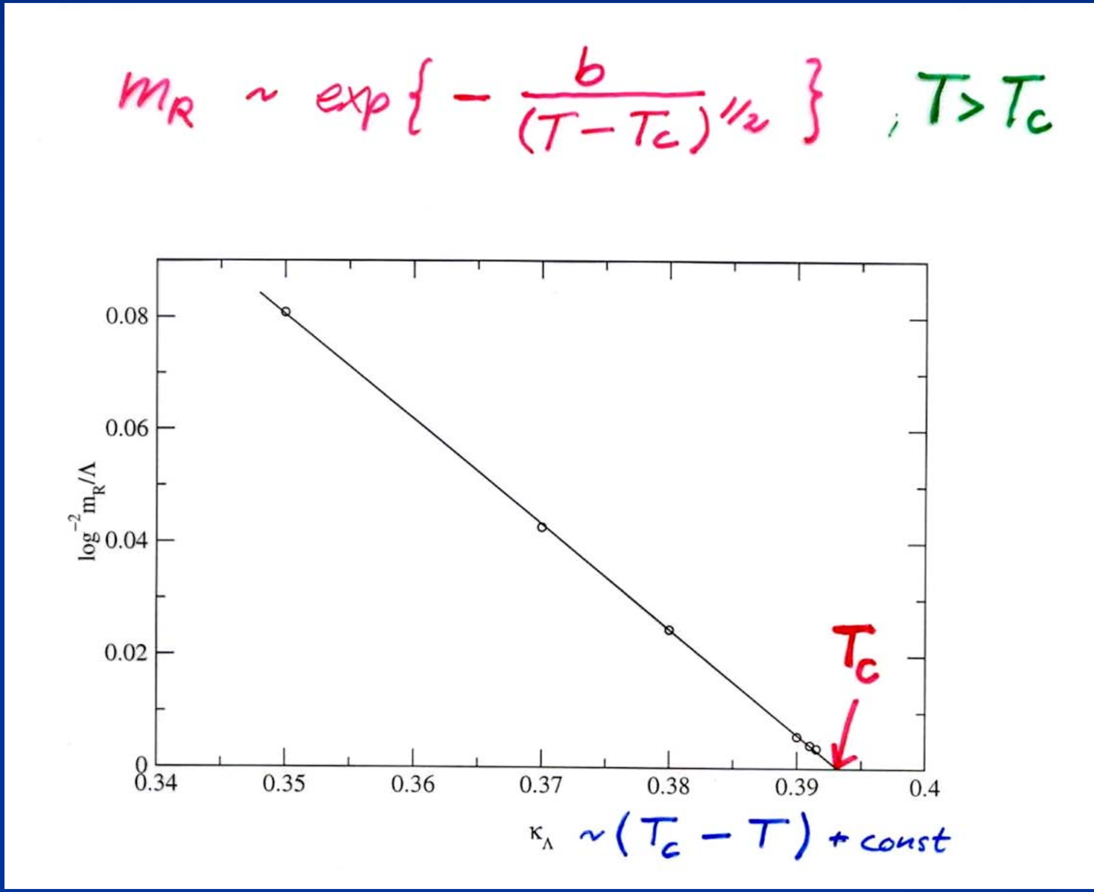
Solution of partial differential equation :

yields highly nontrivial non-perturbative results despite the one loop structure !

Example:

Kosterlitz-Thouless phase transition

Essential scaling : d=2,N=2



- Flow equation contains correctly the non-perturbative information !
- (essential scaling usually described by vortices)

Von Gersdorff …

Kosterlitz-Thouless phase transition (d=2,N=2)

Correct description of phase with

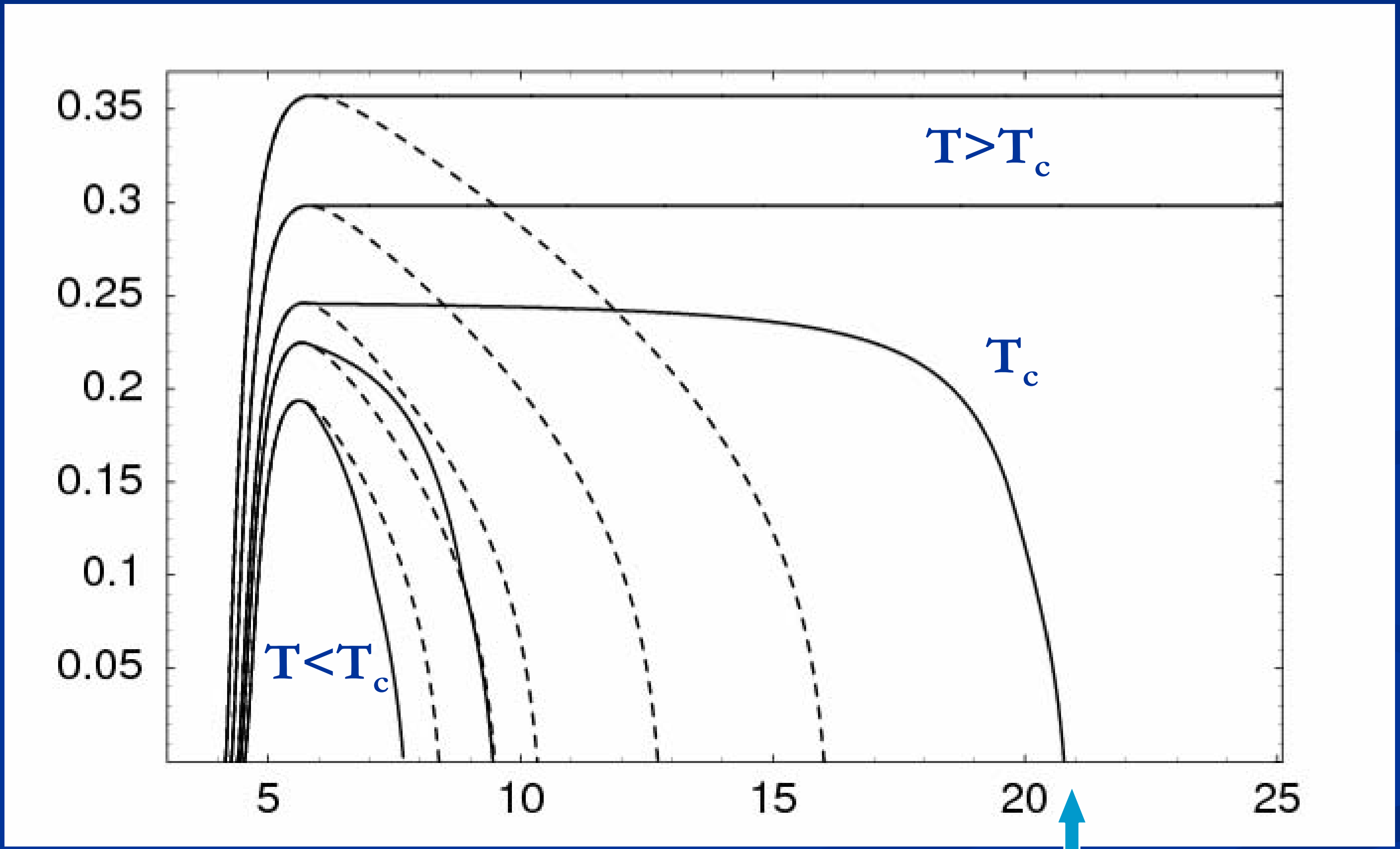
Goldstone boson

(infinite correlation length)

for $T<T_c$

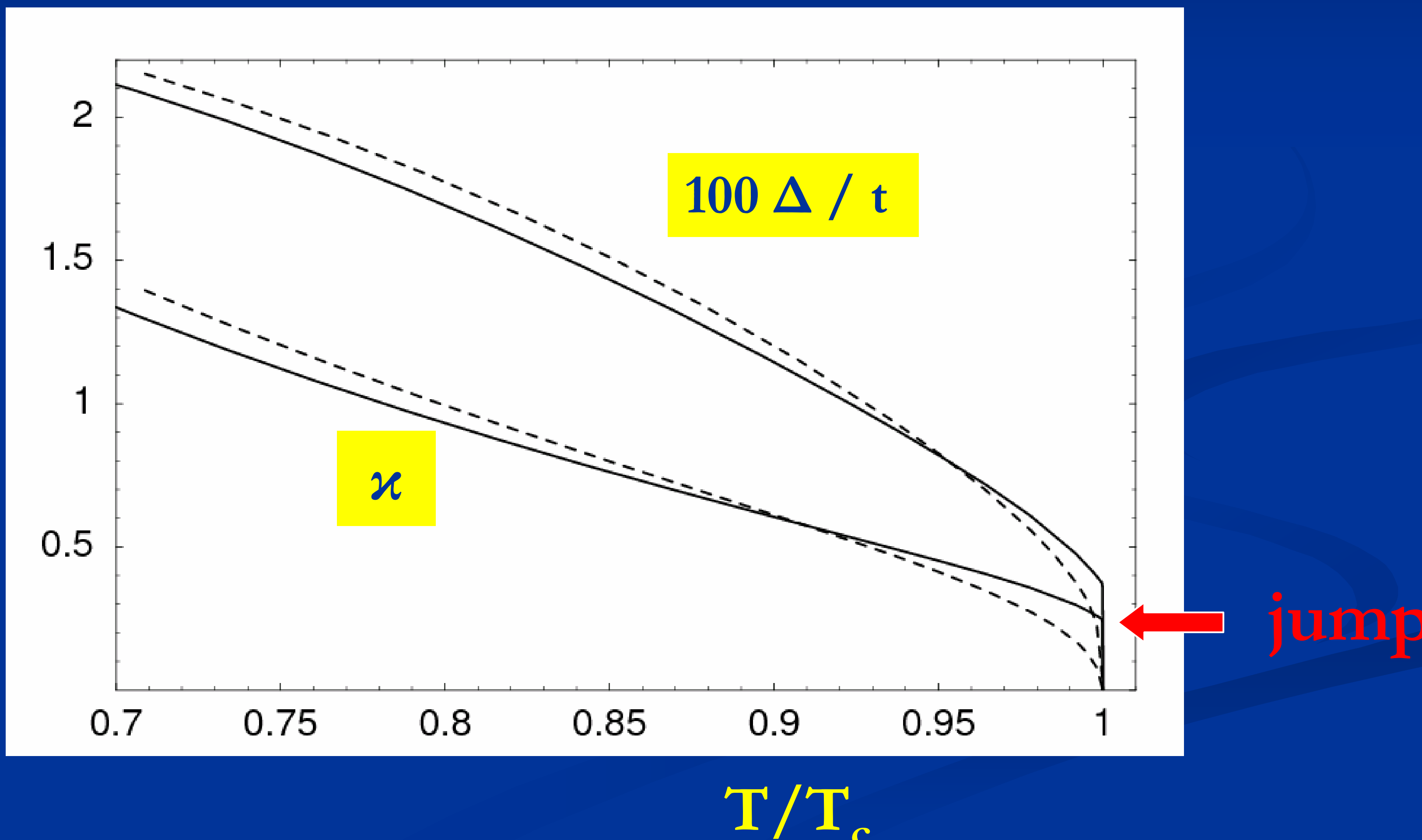
Running renormalized d-wave superconducting order parameter $\varkappa$ in doped Hubbard model

$\varkappa$

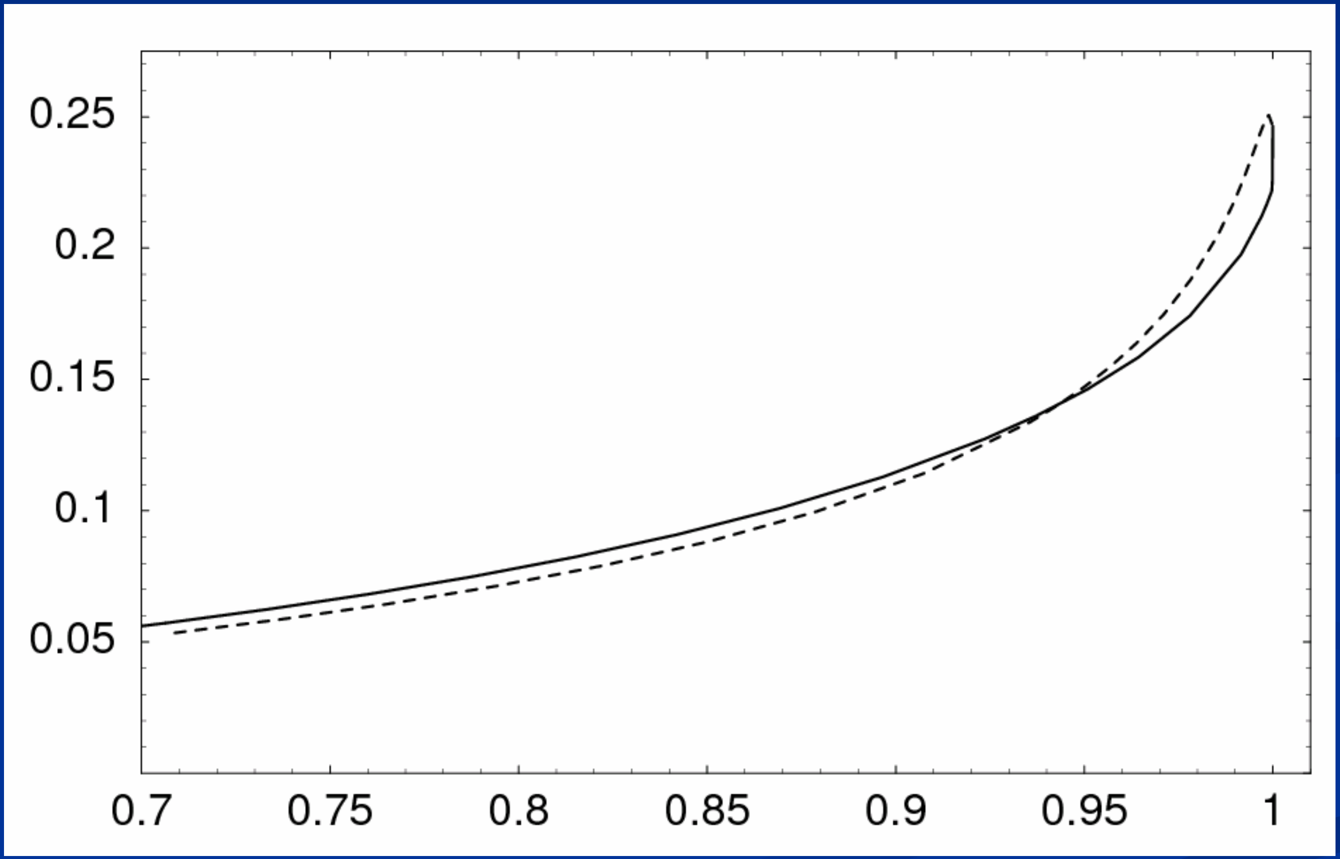


C.Krahl,… **- ln (k/Λ)** macroscopic scale 1 cm

Renormalized order parameter ϰ and gap in electron propagator Δ in doped Hubbard model



Temperature dependent anomalous dimension η



η

T/T_c

convergence and errors

- for precise results: systematic derivative expansion in second order in derivatives
- includes field dependent wave function renormalization $Z(\varrho)$
- fourth order : similar results
- apparent fast convergence : no series resummation
- rough error estimate by different cutoffs and truncations

Effective average action

and

exact renormalization group equation

Generating functional

generating functional for connected Green's functions in presence of quadratic infrared cutoff

$$W_k[j] = \ln \int \mathcal{D}\chi \exp\left(-S[\chi] - \Delta_k S[\chi] + \int d^d x\, j_a \chi_a\right)$$

$$\Delta_k S = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d} R_k(q^2)\chi_a(-q)\chi_a(q)$$

$$\text{e.g.} \quad R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$

$$\lim_{k\to 0} R_k = 0$$

$$R_{k\to\infty} \to \infty$$

Effective average action

$$\Gamma_k[\varphi] = -W_k[j] + \int d^d x \, j_a \varphi_a - \Delta_k S[\varphi]$$

$\Gamma_0[\varphi]$: quantum effective action

generates 1PI vertices

free energy: $F = \Gamma T + \mu n V$

Γ_k includes all fluctuations (quantum, thermal) with $q^2 > k^2$

Γ_Λ specifies microphysics

$$\varphi_a = \langle \chi_a \rangle = \frac{\delta W_k}{\delta j_a}$$

Loop expansion : perturbation theory with infrared cutoff in propagator

Quantum effective action

for $k \to 0$
all fluctuations (quantum + thermal)
are included

knowledge of $\Gamma_{k \to 0} \hat{=}$ solution of model

Exact renormalization group equation

Exact flow equation

for scale dependence of average action

$$\boxed{\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1} \partial_k R_k\right\}}$$

'92

$$\left(\Gamma_k^{(2)}\right)_{ab}(q, q') = \frac{\delta^2 \Gamma_k}{\delta\varphi_a(-q)\delta\varphi_b(q')}$$

$\mathrm{Tr} : \sum_a \int \frac{d^d q}{(2\pi)^d}$

(fermions : STr)

Proof of exact flow equation

$$
\begin{aligned}
\partial_k \Gamma\big|_\phi &= -\partial_k W\big|_j - \partial_k \Delta_k S[\varphi] \\
&= \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k(\langle\phi\phi\rangle - \langle\phi\rangle\langle\phi\rangle)\right\} \\
&= \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k W_k^{(2)}\right\}
\end{aligned}
$$

$$W_k^{(2)}(\Gamma_k^{(2)} + R_k) = \mathbb{1}$$

$$(\Delta_k S^{(2)} \equiv R_k)$$

$$\Longrightarrow$$

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k(\Gamma_k^{(2)} + R_k)^{-1}\right\}$$

Truncations

Functional differential equation –

cannot be solved exactly

Approximative solution by truncation of

most general form of effective action

non-perturbative systematic expansions

derivative expansion

Tetradis,...; Morris

$O(N)$-model:

$$\Gamma_k = \int d^d x \{ U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \varphi_a \partial_\mu \varphi_a + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial_\mu \rho + \cdots \}$$

$$(N = 1 : \quad Y_k \equiv 0)$$

field expansion
(flow eq. for 1PI vertices)

Weinberg; Ellwanger,...

$$\Gamma_k = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{j=0}^{n} d^d x_j \, \Gamma_k^{(n)}(x_1, x_2, \ldots, x_n) \prod_{j=0}^{n} (\phi(x_j) - \phi_0)$$

Expansion in canonical dimension of couplings

Lowest order:

$$d = 4 : \quad \rho_0, \bar{\lambda}, Z$$

$$d = 3 : \quad \rho_0, \bar{\lambda}, \bar{\gamma}, Z$$

$$U = \frac{1}{2}\bar{\lambda}(\rho - \rho_0)^2 + \frac{1}{6}\bar{\gamma}(\rho - \rho_0)^3$$

works well for $O(N)$ models

Tetradis,...; Tsypin

polynomial expansion of potential converges

if expanded around ρ_0

Tetradis,...; Aoki et al.

Exact flow equation for effective potential

- Evaluate exact flow equation for homogeneous field φ .
- R.h.s. involves exact propagator in homogeneous background field φ.

many models have been studied along these lines …

- several fields
- complicated phase structure (e.g. ^{3}He)
- replica trick N=0
- shift in critical temperature for Bose-Einstein condensate with interaction (needs resolution for momentum dependence of propagator)
- gauge theories

disordered systems

→ Canet , Delamotte , Tissier , …

including fermions :

no particular problem !

Universality in ultra-cold fermionic atom gases

with

S. Diehl , H.Gies , J.Pawlowski

BEC – BCS crossover

Bound molecules of two atoms
on microscopic scale:

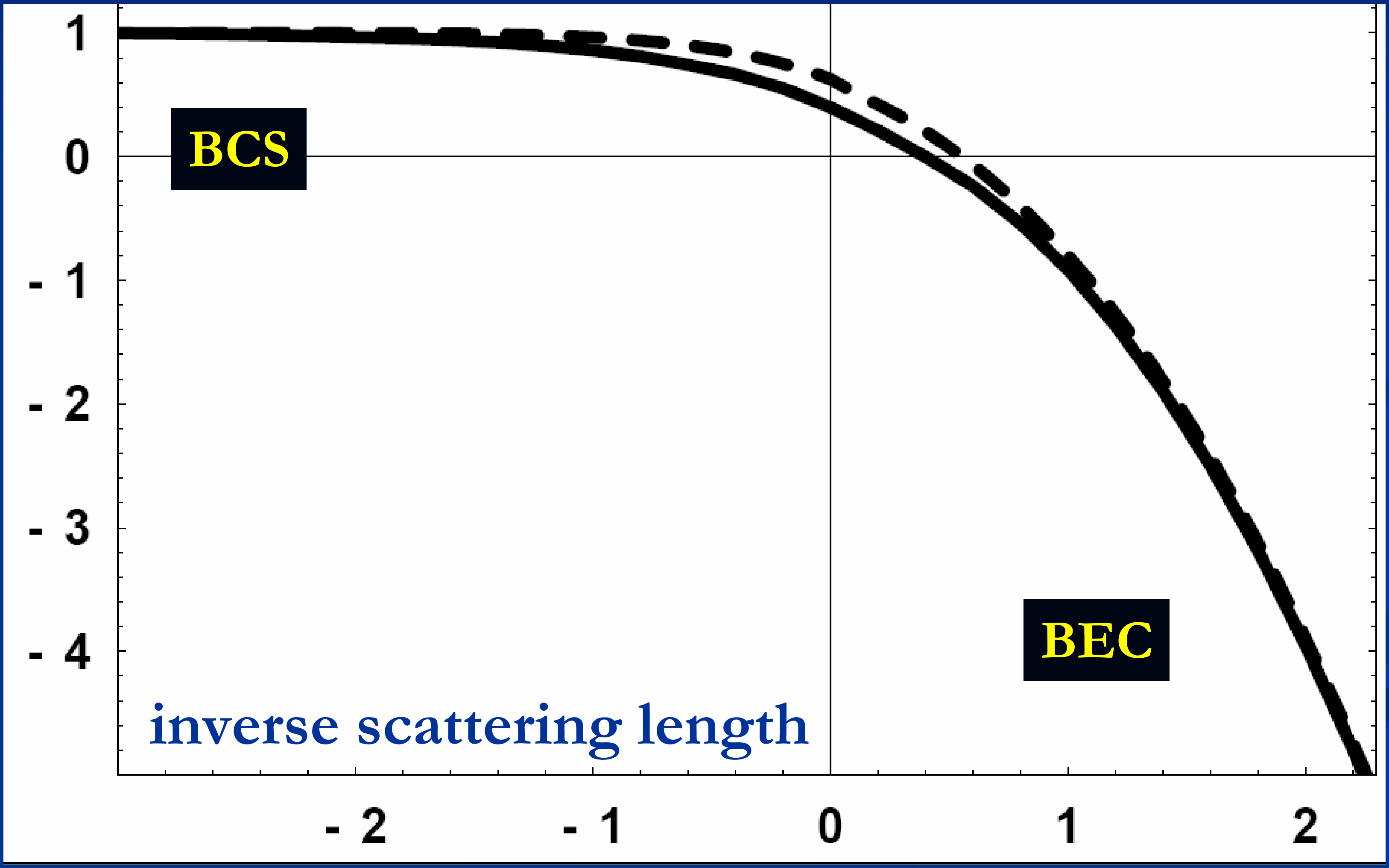
Bose-Einstein condensate (BEC) for low T

Fermions with attractive interactions
(molecules play no role) :

BCS – superfluidity at low T
by condensation of Cooper pairs

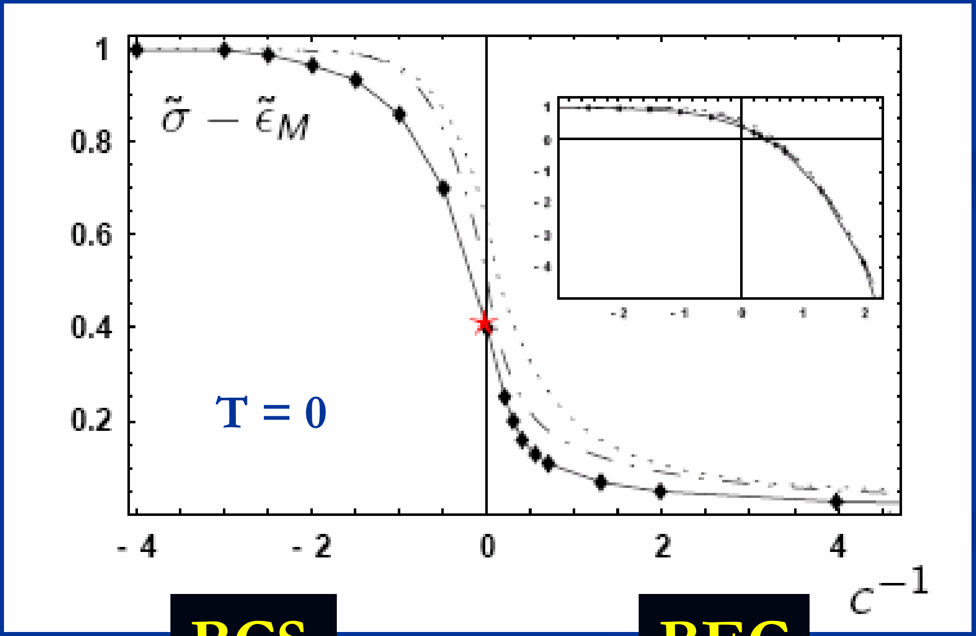
Crossover by Feshbach resonance
as a transition in terms of external magnetic field

chemical potential



BEC – BCS crossover

- qualitative and partially quantitative theoretical understanding
- mean field theory (MFT) and first attempts beyond



BCS

BEC

concentration : **c = a k$_F$**

reduced chemical potential : **σ˜ = μ/ε_F**

Fermi momemtum : **k_F**

Fermi energy : **ε_F**

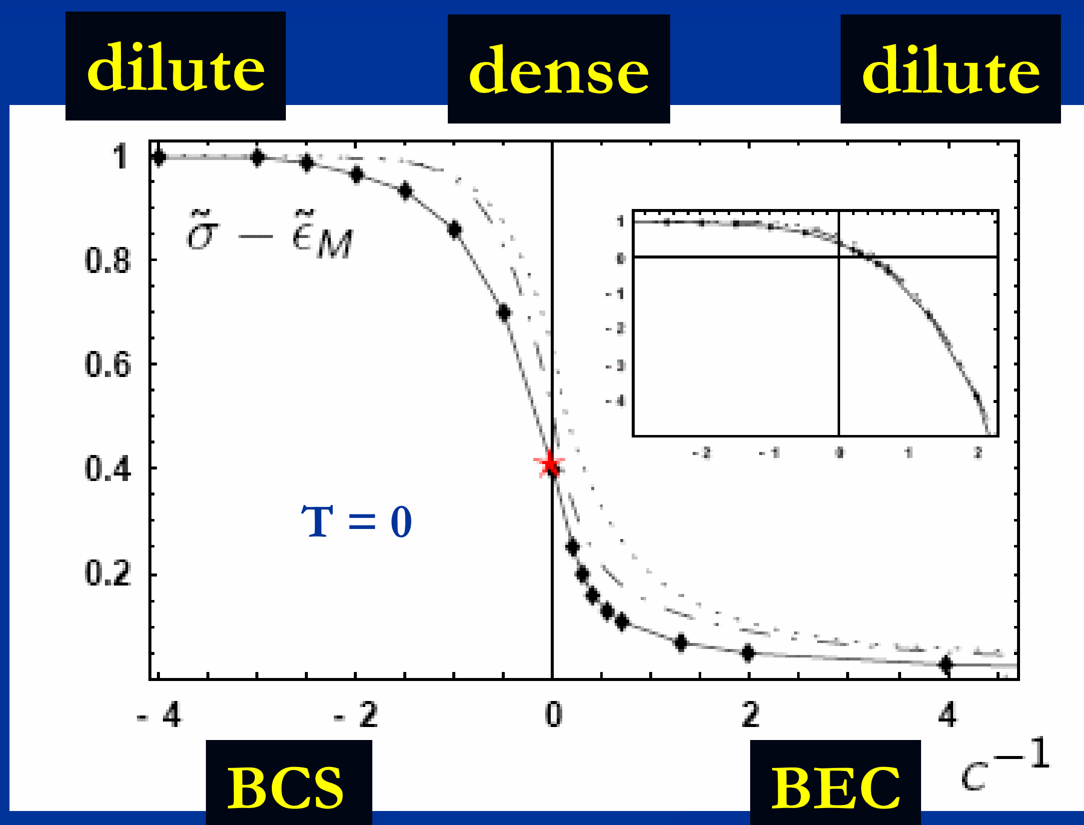
binding energy :

$$\tilde{\epsilon}_M = -\theta(c^{-1})c^{-2}$$

concentration

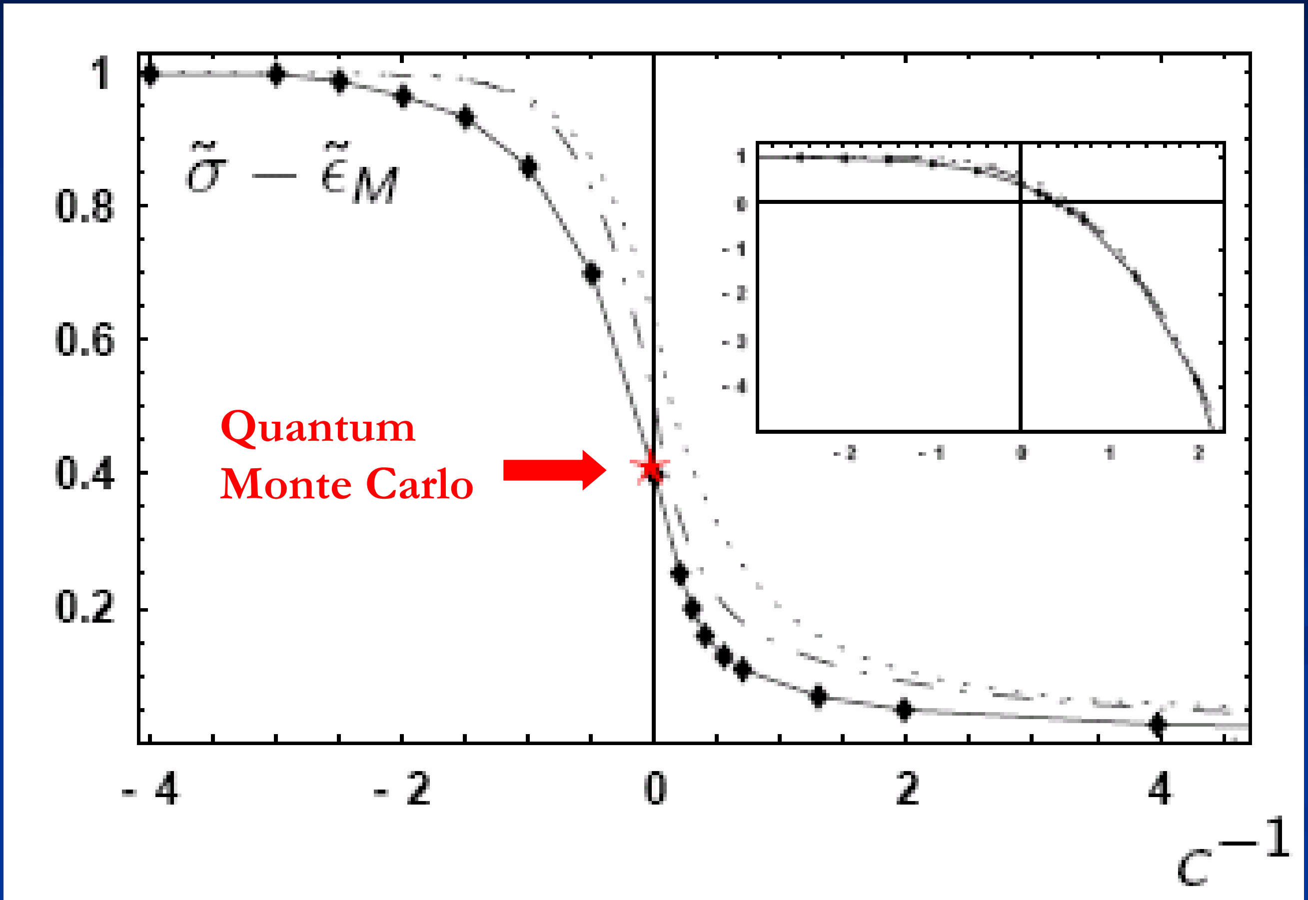
- $c = a\, k_F$, $a(B)$: scattering length
- needs computation of density $n = k_F^3/(3\pi^2)$

non-interacting Fermi gas



non-interacting Bose gas

different methods



- Compare RGE (diamonds), SDE (dashed-dotted) and MFT (dashed) approximation schemes.

- Compare to QMC calculations at $c^{-1} = 0$

	QMC	RGE	SDE	MFT
$\tilde{\sigma}$	$0.44(2)^*, 0.42(2)^\dagger$	0.40	0.50	0.63

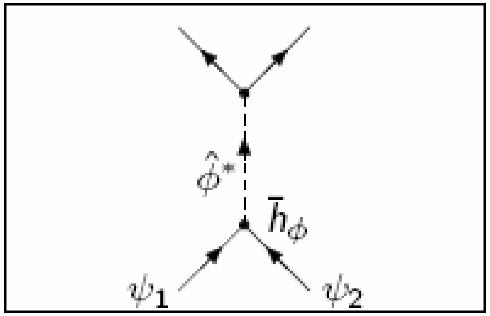
(* Carlson *et al.*, PRL 91, 050401 (2003),
† Giorgini et al., PRL 93, 200404 (2004)).

QFT for non-relativistic fermions

- functional integral, action

$$S = \int_x \{\psi^\dagger(\partial_\tau - \frac{\Delta}{2M} - \sigma)\psi$$
$$+\varphi^*(\partial_\tau - \frac{\Delta}{4M} + \bar{\nu}_\Lambda - 2\sigma)\varphi$$
$$-\bar{h}_\varphi(\varphi^*\psi_1\psi_2 - \varphi\psi_1^*\psi_2^*)\}$$

Molecule exchange



perturbation theory:
Feynman rules

τ : euclidean time on torus with circumference 1/T
σ : effective chemical potential

parameters

- detuning ν(B)

$$\bar{\nu}_\Lambda = \bar{\nu}_{\Lambda,0} + \bar{\mu}_B(B - B_0)$$

$$\frac{\partial \bar{\nu}_\Lambda}{\partial B} = \bar{\mu}_B$$

$$\begin{aligned} S = \int_x \{ & \psi^\dagger(\partial_\tau - \frac{\Delta}{2M} - \sigma)\psi \\ & + \varphi^*(\partial_\tau - \frac{\Delta}{4M} + \bar{\nu}_\Lambda - 2\sigma)\varphi \\ & - \bar{h}_\varphi(\varphi^*\psi_1\psi_2 - \varphi\psi_1^*\psi_2^*)\} \end{aligned}$$

- Yukawa or Feshbach coupling h_φ

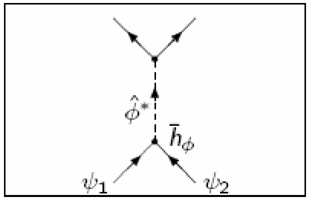
fermionic action

equivalent fermionic action , in general not local

$$S_F = \int_x \psi^\dagger\left(\partial_\tau - \frac{\Delta}{2M} - \sigma\right)\psi + S_{\rm int}$$

$$S_{\rm int} = -\frac{1}{2}\int_{Q_1,Q_2,Q_3} \left(\psi^\dagger(-Q_1)\psi(Q_2)\right)\left(\psi^\dagger(Q_4)\psi(-Q_3)\right) \frac{\bar h_\varphi^2}{\bar\nu_\Lambda - 2\sigma + (\vec q_1 - \vec q_4)^2/4M + 2\pi i T(n_1 - n_4)}$$

Molecule exchange



scattering length a

$$\bar{\lambda} = -\frac{\bar{h}_\varphi^2}{\bar{\nu}_\Lambda}$$

$$a = M\,\lambda/4\pi$$

- broad resonance : pointlike limit
- large Feshbach coupling

$$\bar{h}_\varphi^2 \to \infty,\ \bar{\nu}_\Lambda \to \infty,\ \bar{\lambda}\ \text{fixed}$$

$$S_{\rm int} = -\frac{1}{2}\int_{Q_1,Q_2,Q_3} (\psi^\dagger(-Q_1)\psi(Q_2))(\psi^\dagger(Q_4)\psi(-Q_3))$$
$$\frac{\bar{h}_\varphi^2}{\bar{\nu}_\Lambda - 2\sigma + (\vec{q}_1 - \vec{q}_4)^2/4M + 2\pi i T(n_1 - n_4)}$$

collective di-atom states

collective degrees of freedom

can be introduced by

partial bosonisation

(Hubbard - Stratonovich transformation)

units and dimensions

- $c = 1$; $\hbar = 1$; $k_B = 1$
- momentum ~ length^{-1} ~ mass ~ eV
- energies : 2ME ~ (momentum)^{2}

  (M : atom mass)
- typical momentum unit : Fermi momentum
- typical energy and temperature unit : Fermi energy
- time ~ (momentum) $^{-2}$
- **canonical dimensions different from relativistic QFT !**

rescaled action

$$S = \int_{\hat{x}} \{\hat{\psi}^\dagger(\hat{\partial}_\tau - \hat{\Delta} - \hat{\sigma})\hat{\psi}$$
$$+\hat{\varphi}^*(\hat{\partial}_\tau - \frac{1}{2}\hat{\Delta} + \hat{\nu} - 2\hat{\sigma})\hat{\varphi}$$
$$-\hat{h}_\varphi(\hat{\varphi}^*\hat{\psi}_1\hat{\psi}_2 - \hat{\varphi}\hat{\psi}_1^*\hat{\psi}_2^*)\}$$

$$\hat{\psi} = \hat{k}^{-3/2}\psi \ , \ \hat{\varphi} = \hat{k}^{-3/2}\varphi,$$
$$\hat{x} = \hat{k}x \ , \ \hat{\tau} = \frac{\hat{k}^2}{2M}\tau,$$
$$\hat{\sigma} = \frac{2M\sigma}{\hat{k}^2} \ , \ \hat{h}_\varphi = \frac{2M\bar{h}_\varphi}{\sqrt{\hat{k}}}$$

- M drops out
- all quantities in units of k_F if

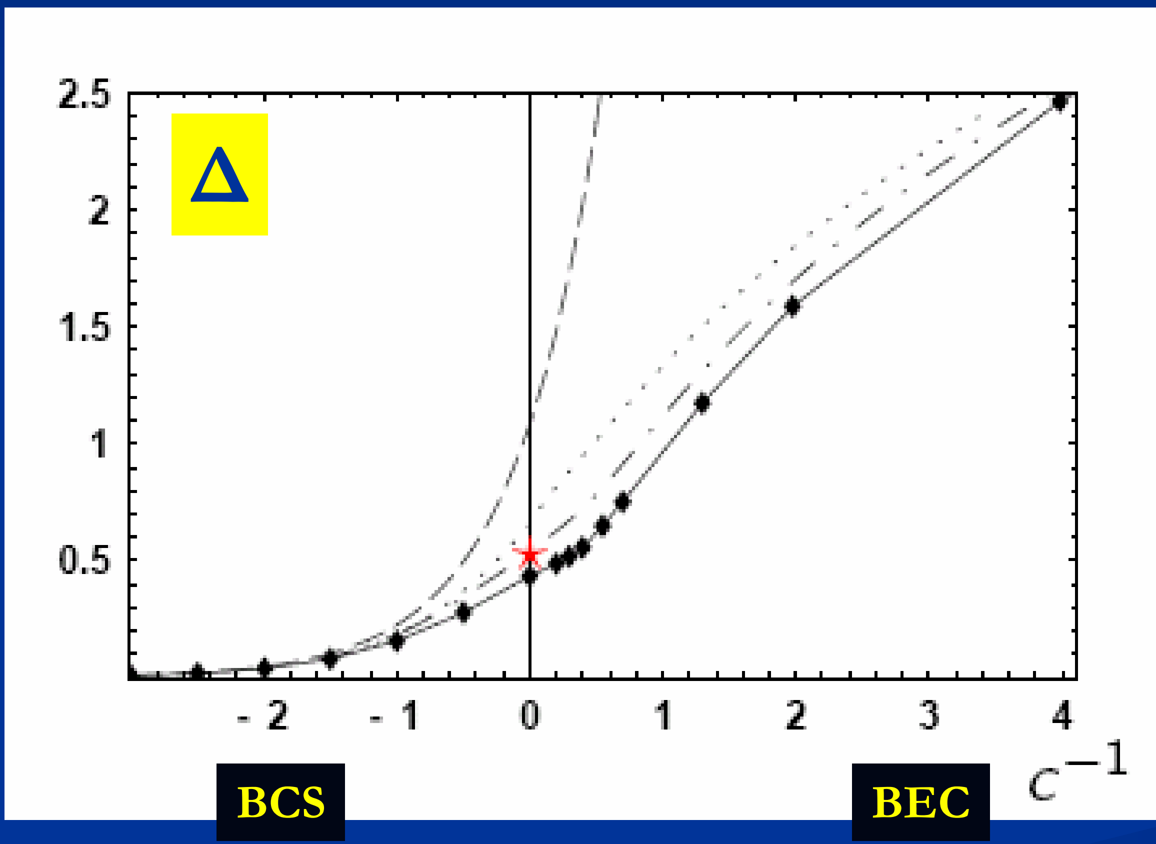
$$\hat{k} = k_F$$

effective action

- integrate out all quantum and thermal fluctuations
- quantum effective action
- generates full propagators and vertices
- richer structure than classical action

$$\begin{aligned}\Gamma = \int_x \{ &\psi^\dagger(\partial_\tau - A_\psi \Delta - \sigma)\psi \\ &+\varphi^*(\partial_\tau - A_\varphi \Delta)\varphi + u(\varphi) \\ &-h_\varphi(\varphi^* \psi_1 \psi_2 - \varphi \psi_1^* \psi_2^*) + \ldots \}\end{aligned}$$

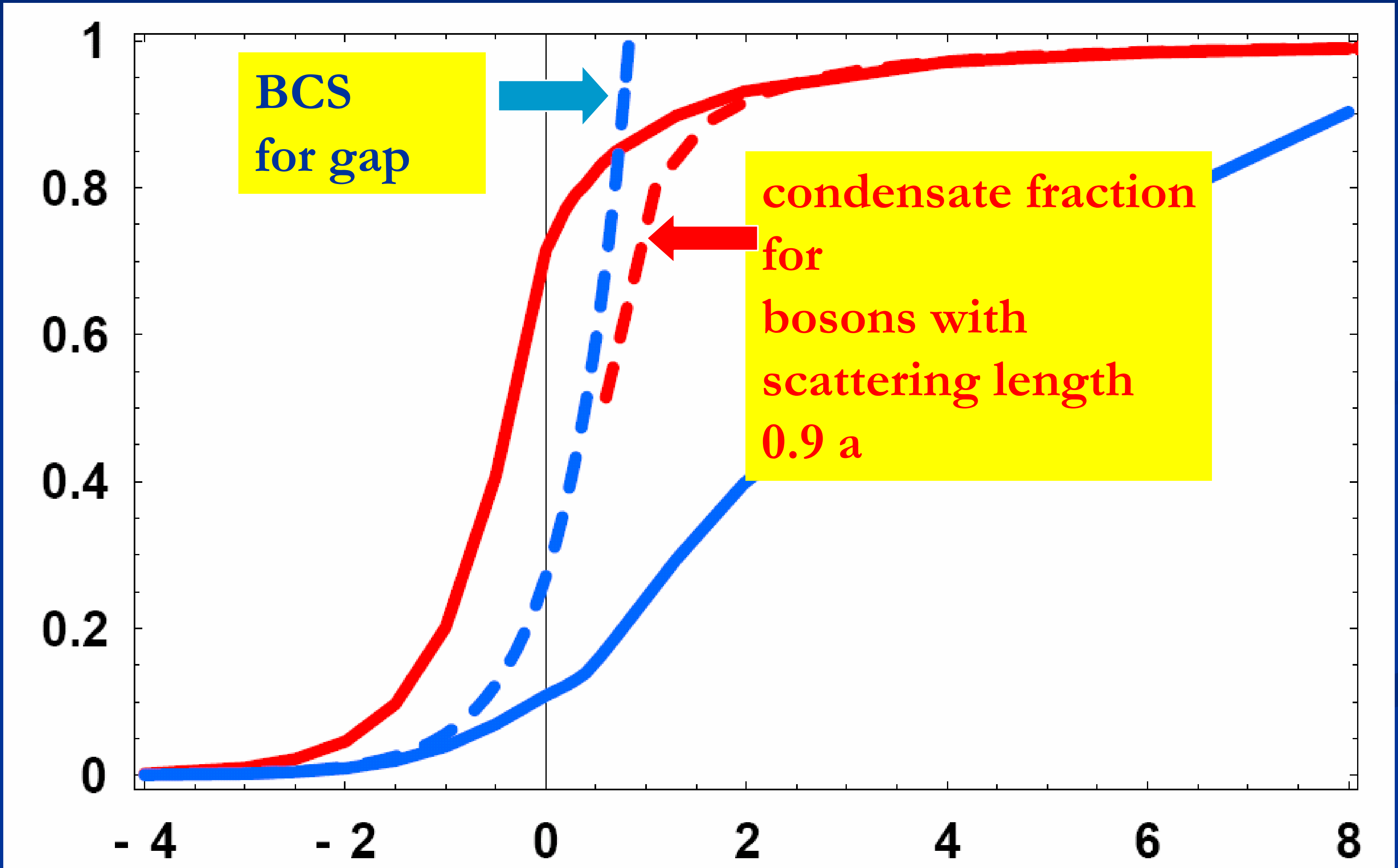
gap parameter



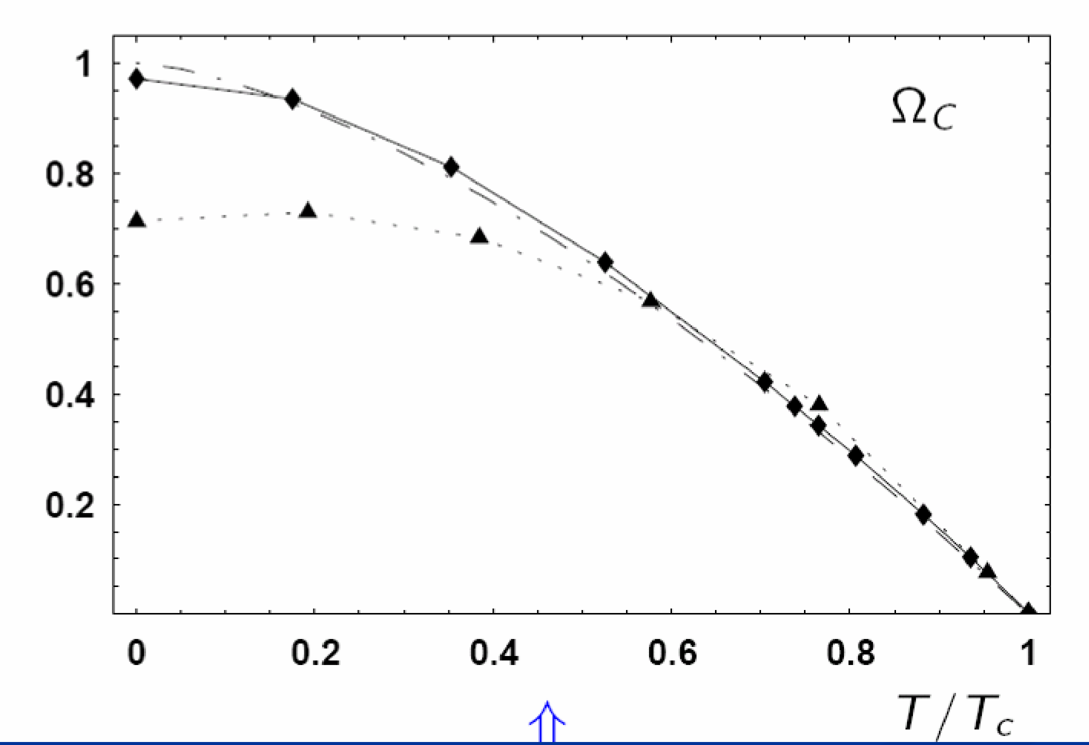
- BCS regime: recover BCS gap result $\Delta/\Delta^{BCS}(c^{-1}) \approx 0.9$ for $c^{-1} < -2$.

- MFT (dashed): No boson interactions.
- SDE (dashed-dotted): Overestimates interactions, $a_M = 2$.

limits

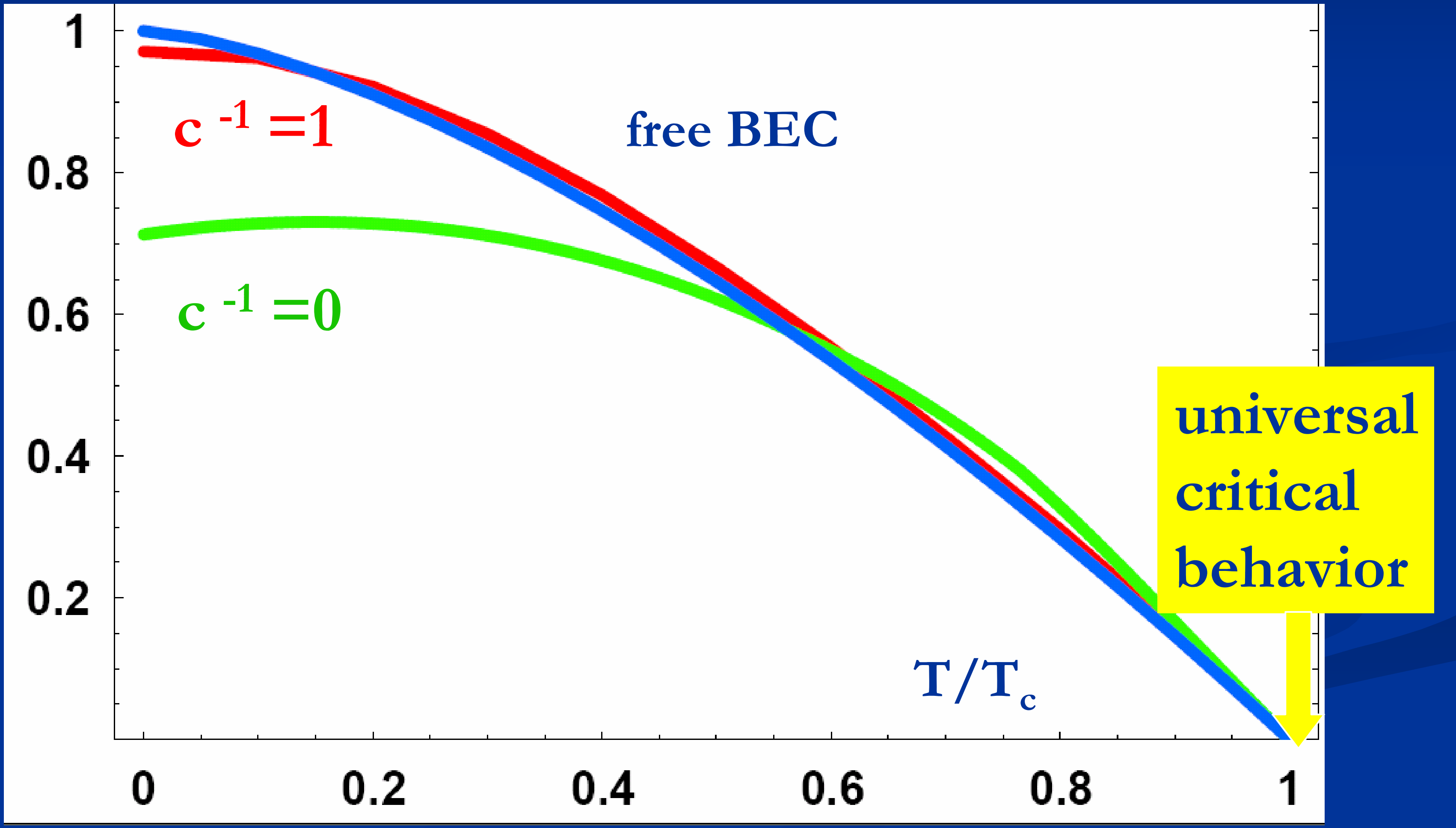

BCS
for gap
condensate fraction
for
bosons with
scattering length
0.9 a
1
0.8
0.6
0.4
0.2
0
- 4
- 2
0
2
4
6
8

temperature dependence of condensate



- Compare free BE condensate fraction to result for $c^{-1} = 0$ (resonance, triangles) and $c^{-1} = 1$ (BEC regime, diamonds).
- Low temperature: Condensate fraction strongly depends on c^{-1}.
- Close to criticality:
  - Second order phase transition.
  - Similar approach to T_c: dominance of boson fluctuations, system attracted to universal critical point.

condensate fraction : second order phase transition



changing degrees of freedom

Antiferromagnetic order in the Hubbard model

A functional renormalization group study

T.Baier, E.Bick, …

Hubbard model

Functional integral formulation

$$
\begin{aligned}
Z[\eta] \;=\; & \int_{\hat\psi(\beta)=-\hat\psi(0),\hat\psi^*(\beta)=-\hat\psi^*(0)} \mathcal{D}(\hat\psi^*(\tau),\hat\psi(\tau)) \\
& \exp\Big(- \int_0^\beta d\tau \big(\sum_{\mathbf{x}} \hat\psi^\dagger_{\mathbf{x}}(\tau) (\frac{\partial}{\partial\tau} - \mu) \hat\psi_{\mathbf{x}}(\tau) \\
& \quad + \sum_{\mathbf{xy}} \hat\psi^\dagger_x(\tau) \mathcal{T}_{\mathbf{xy}} \hat\psi_{\mathbf{y}}(\tau) \\
& \quad + \frac{1}{2} U \sum_{\mathbf{x}} (\hat\psi^\dagger_{\mathbf{x}}(\tau)\hat\psi_{\mathbf{x}}(\tau))^2 \\
& \quad - \sum_{\mathbf{x}} (\eta^\dagger_{\mathbf{x}}(\tau)\hat\psi_{\mathbf{x}}(\tau) + \eta^T_{\mathbf{x}}(\tau)\hat\psi^*_{\mathbf{x}}(\tau)) \big) \Big)
\end{aligned}
$$

next neighbor interaction

$$
\mathcal{T}_{xy} = \begin{cases} -t & \text{, if } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are nearest neighbors} \\ 0 & \text{, else} \end{cases}
$$

External parameters
T : temperature
μ : chemical potential
(doping)

U > 0 :
repulsive local interaction

lattice propagator

$$S_{F,\mathrm{kin}} = \sum_{Q} \hat{\psi}^{\dagger}(Q)(i\omega_F - \mu - 2t(\cos q_1 + \cos q_2))\hat{\psi}(Q),$$

Fermion bilinears

$$\tilde{\rho}(X) = \hat{\psi}^\dagger(X)\hat{\psi}(X),$$
$$\tilde{\vec{m}}(X) = \hat{\psi}^\dagger(X)\vec{\sigma}\hat{\psi}(X)$$

Introduce sources for bilinears

$$S_F = S_{F,\mathrm{kin}} + \frac{1}{2}U(\hat{\psi}^\dagger\hat{\psi})^2 - J_\rho\tilde{\rho} - \vec{J}_m\tilde{\vec{m}}$$

Functional variation with respect to sources J yields expectation values and correlation functions

$$Z = \int \mathcal{D}(\psi^*, \psi)\exp\left(-(S_F + S_\eta)\right)$$
$$S_\eta = -\eta^\dagger\psi - \eta^T\psi^*$$

Partial Bosonisation

- collective bosonic variables for fermion bilinears
- insert identity in functional integral
  (Hubbard-Stratonovich transformation)
- replace four fermion interaction by equivalent bosonic interaction (e.g. mass and Yukawa terms)
- problem : decomposition of fermion interaction into bilinears not unique (Grassmann variables)

$$(\hat{\psi}^\dagger(X)\hat{\psi}(X))^2 = \tilde{\rho}(X)^2 = -\frac{1}{3}\tilde{\vec{m}}(X)^2$$

Partially bosonised functional integral

$$Z[\eta, \eta^*, J_\rho, \vec{J}_m] = \int \mathcal{D}(\psi^*, \psi, \hat{\rho}, \hat{\vec{m}}) \exp\left(-(S + S_\eta + S_J)\right)$$

$$S = S_{F,\mathrm{kin}} + \frac{1}{2}U_\rho \hat{\rho}^2 + \frac{1}{2}U_m \hat{\vec{m}}^2 - U_\rho \hat{\rho}\tilde{\rho} - U_m \hat{\vec{m}}\tilde{\vec{m}},$$
$$S_J = -\ J_\rho \hat{\rho} - \vec{J}_m \hat{\vec{m}}$$

equivalent to
fermionic functional integral

if $U = -U_\rho + 3U_m$

Bosonic integration is Gaussian

or:

solve bosonic field equation as functional of fermion fields and reinsert into action

$$\hat{\rho} = \tilde{\rho} + \frac{J_\rho}{U_\rho}, \qquad \hat{\vec{m}} = \tilde{\vec{m}} + \frac{\vec{J}_m}{U_m}$$

fermion – boson action

$$S = S_{F,\mathrm{kin}} + S_B + S_Y + S_J,$$

fermion kinetic term

$$S_{F,\mathrm{kin}} = \sum_Q \hat{\psi}^\dagger(Q)(i\omega_F - \mu - 2t(\cos q_1 + \cos q_2))\hat{\psi}(Q),$$

boson quadratic term ("classical propagator")

$$S_B = \frac{1}{2}\sum_Q \left(U_\rho \hat{\rho}(Q)\hat{\rho}(-Q) + U_m \hat{\vec{m}}(Q)\hat{\vec{m}}(-Q)\right),$$

Yukawa coupling

$$S_Y = -\sum_{QQ'Q''} \delta(Q - Q' + Q'') \times (U_\rho \hat{\rho}(Q)\hat{\psi}^\dagger(Q')\hat{\psi}(Q'') + U_m \hat{\vec{m}}(Q)\hat{\psi}^\dagger(Q')\vec{\sigma}\hat{\psi}(Q'')),$$

source term

$$S_J = -\sum_Q \Big(J_\rho(-Q)\hat{\rho}(Q) + \vec{J}_m(-Q)\hat{\vec{m}}(Q) \Big)$$

is now linear in the bosonic fields

Mean Field Theory (MFT)

Evaluate Gaussian fermionic integral in background of bosonic field , e.g.

$$
\begin{aligned}
\hat{\rho}(Q) &\to \rho\delta(Q) \\
\hat{\vec{m}}(Q) &\to \vec{a}\delta(Q-\Pi)
\end{aligned}
$$

$$
\begin{aligned}
Z_{\mathrm{MF}} &= \int \mathcal{D}(\hat{\psi}^*, \hat{\psi}) \exp(-S_{\mathrm{MF}}), \\
S_{\mathrm{MF}} &= \sum_Q \hat{\psi}^\dagger(Q)(i\omega_F - \mu - 2t(\cos q_1 + \cos q_2))\hat{\psi}(Q) \\
&\quad - \sum_Q (U_\rho \rho \hat{\psi}^\dagger(Q)\hat{\psi}(Q) + U_m \vec{a}\hat{\psi}^\dagger(Q+\Pi)\vec{\sigma}\hat{\psi}(Q)) \\
&\quad + \frac{V_2}{2T}(U_\rho \rho^2 + U_m \vec{a}^2) - J_\rho(0)\rho - \vec{J}_m(-\Pi)\vec{a}
\end{aligned}
$$

$$
\Gamma_{\mathrm{MF}} = -\ln Z_{\mathrm{MF}} + J_\rho(0)\rho + \vec{J}_m(-\Pi)\vec{a}
$$

Effective potential in mean field theory

$$U(\rho,\vec{a}) = \frac{T\Gamma}{V_2} = \frac{1}{2}(U_\rho \rho^2 + U_m \vec{a}^2) + \Delta U(\rho,\vec{a})$$

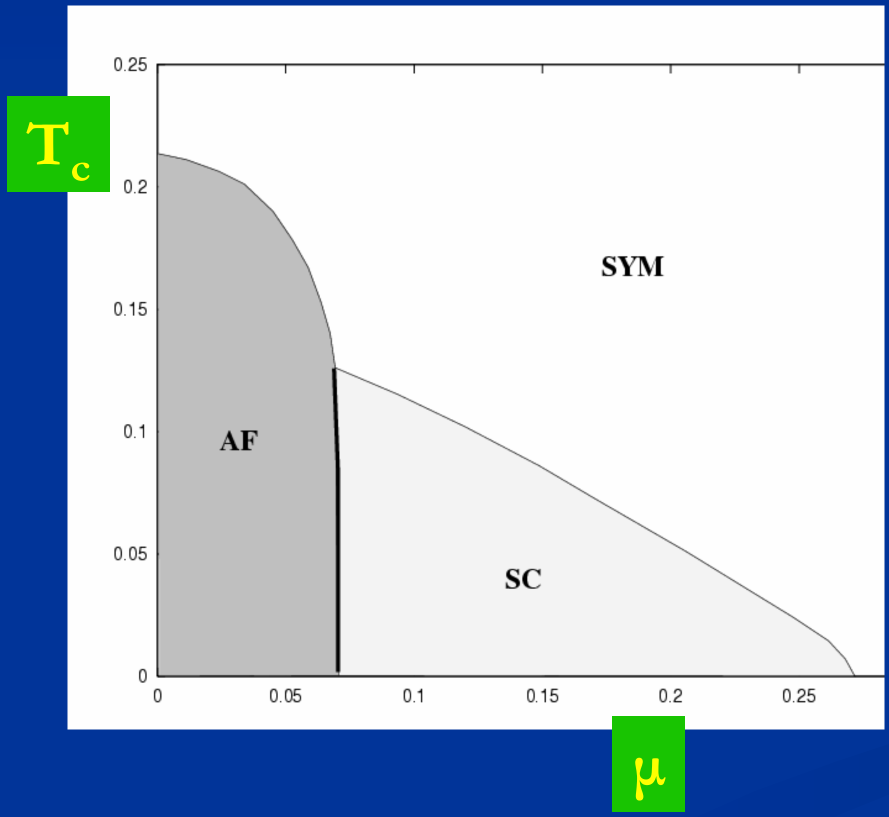
$$\Delta U(\rho,\vec{a}) = -\frac{T}{V_2}\ln\int \mathcal{D}(\hat{\psi}^*,\hat{\psi})\exp(-S_\Delta),$$

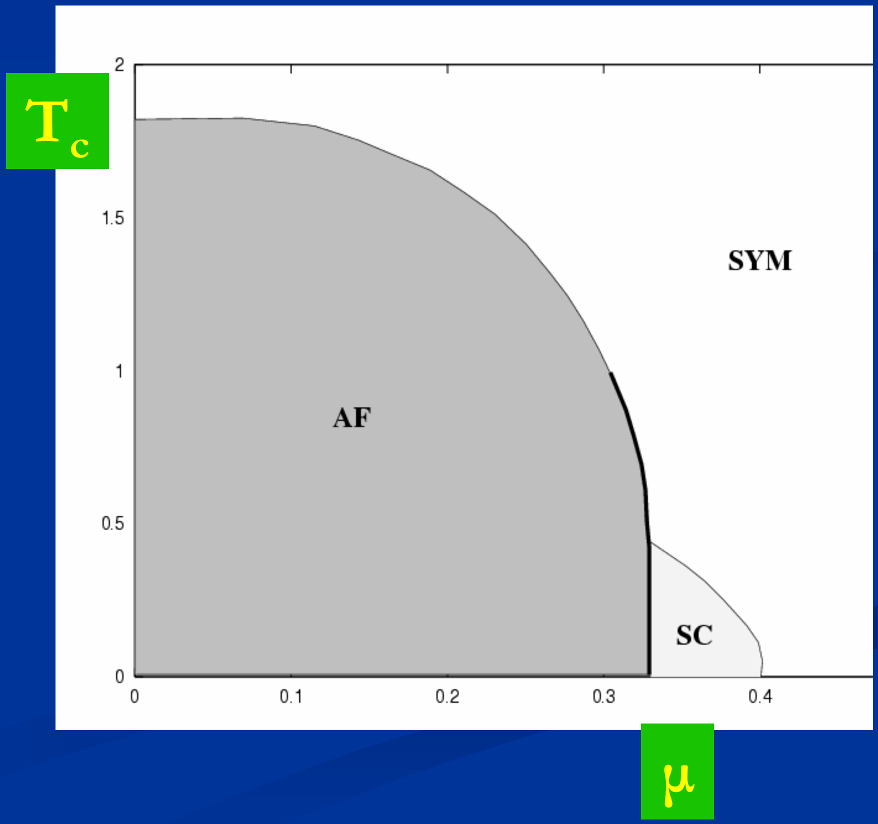
$$S_\Delta = \sum_Q \left(\hat{\psi}^\dagger(Q)P(Q)\hat{\psi}(Q) - U_m\vec{a}\hat{\psi}^\dagger(Q+\Pi)\vec{\sigma}\hat{\psi}(Q)\right)$$

$$\begin{aligned} P(Q) &= i\omega_F - \mu_{\text{eff}} - 2t(\cos q_1 + \cos q_2), \\ \mu_{\text{eff}} &= \mu + U_\rho \rho. \end{aligned}$$

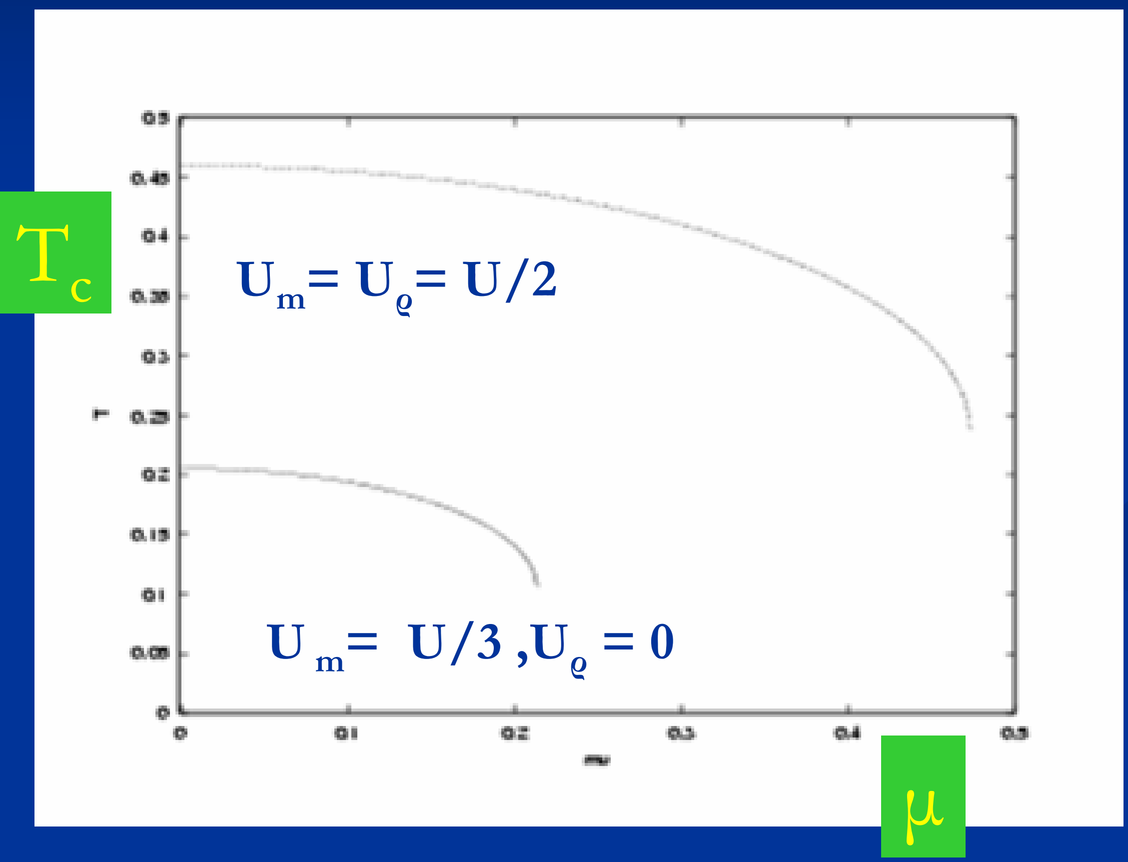
Mean field phase diagram

for two different choices of couplings – same U !





Mean field ambiguity



mean field phase diagram

Artefact of approximation …

cured by inclusion of bosonic fluctuations

J.Jaeckel,…

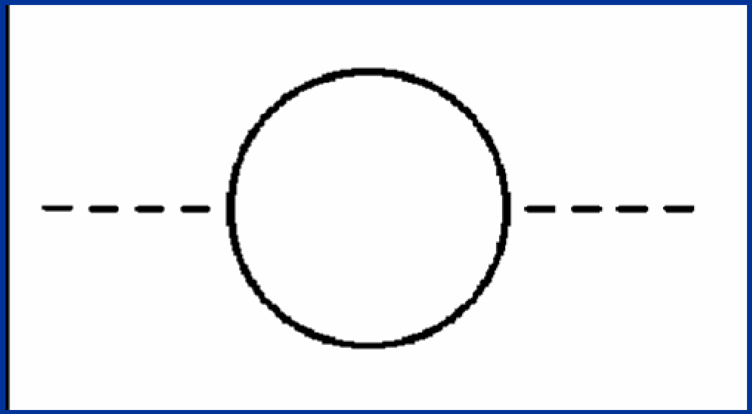
$$U = -U_\rho + 3U_m$$

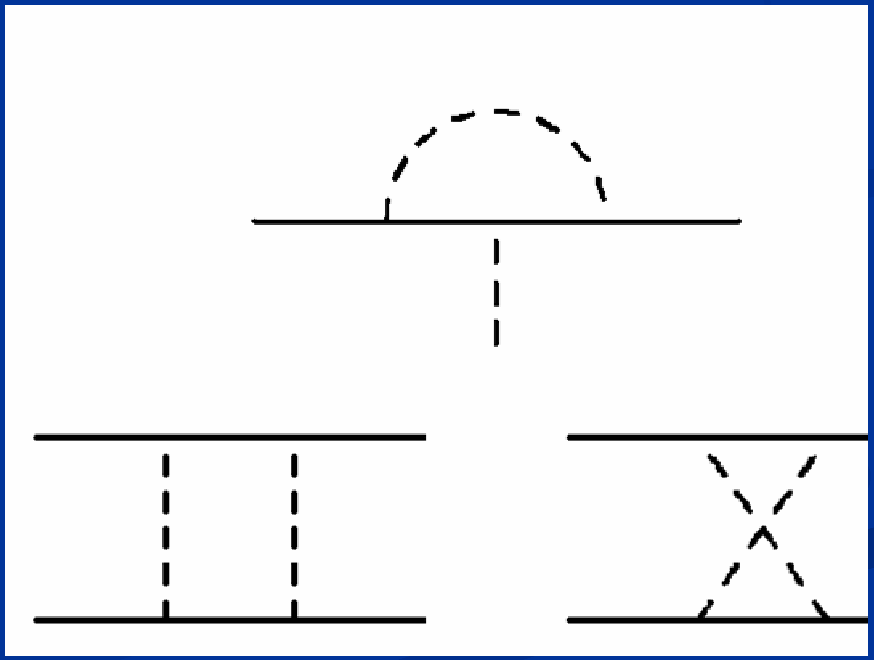
Rebosonization and the mean field ambiguity

Bosonic fluctuations

fermion loops

boson loops

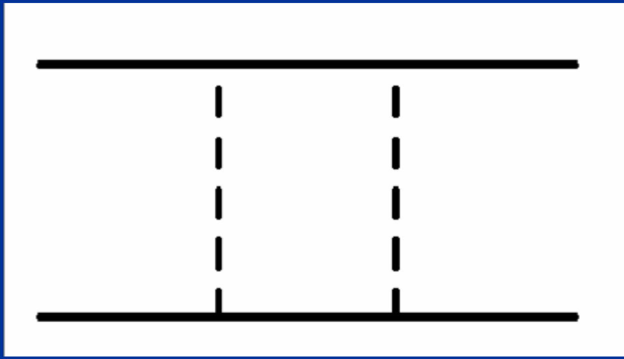




mean field theory

Rebosonization

- adapt bosonization to every scale k such that



is translated to bosonic interaction

H.Gies , …

$$
\begin{aligned}
\Gamma_k[\psi,\psi^*,\phi] \;=\; & \sum_Q \psi^*(Q)P_{\psi,k}\psi(Q) \\
& +\frac{1}{2}\sum_Q \phi(-Q)P_{\phi,k}(Q)\phi(Q) \\
& -\sum_Q h_k(Q)\phi(Q)\tilde{\phi}(-Q) \\
& +\sum_Q \lambda_{\psi,k}(Q)\tilde{\phi}(Q)\tilde{\phi}(-Q)
\end{aligned}
$$

k-dependent field redefinition

$$\phi_k(Q) = \phi_{\bar{k}}(Q) + \Delta\alpha_k(Q)\tilde{\phi}(Q)$$

$$\partial_k\phi_k(Q) = -\,\partial_k\alpha_k(Q)\tilde{\phi}(Q)$$

absorbs four-fermion coupling

Modification of evolution of couplings …

Evolution with k-dependent field variables

$$\partial_k \Gamma_k[\psi,\psi^*,\phi_k] = \partial_k \Gamma_k[\psi,\psi^*,\phi_k]|_{\phi_k} + \sum_Q \left(\frac{\delta}{\delta\phi_k}\Gamma[\psi,\psi^*,\phi_k]\right)\partial_k\phi_k$$
$$= \partial_k \Gamma_k[\psi,\psi^*,\phi_k]|_{\phi_k} + \sum_Q \big(-\partial_k\alpha_k(Q)P_{\phi,k}(Q)\phi_k(Q)\tilde{\phi}(-Q) + h_k(Q)\partial_k\alpha_k(Q)\tilde{\phi}(Q)\tilde{\phi}(-Q)\big)$$

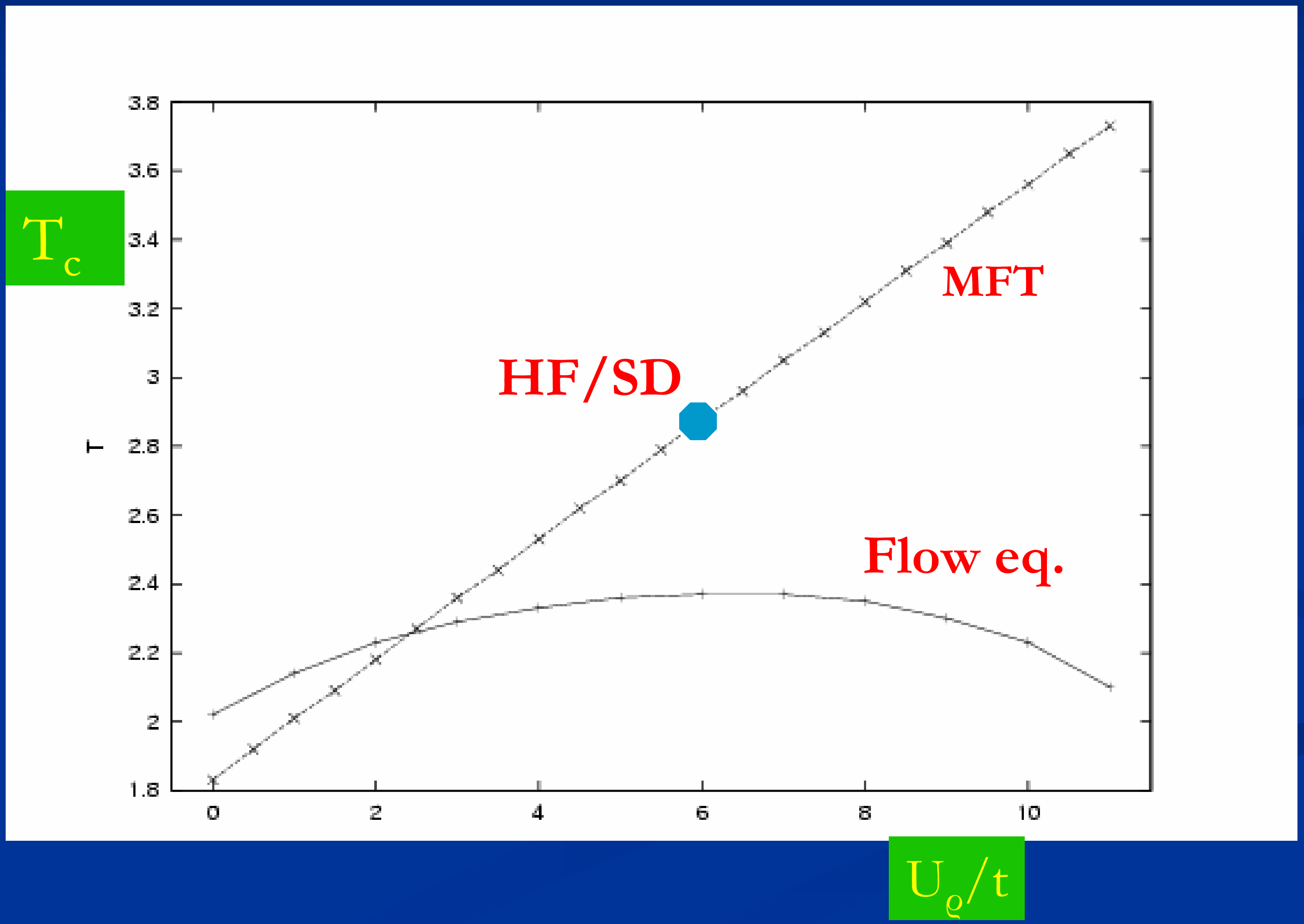
Rebosonisation

$$\begin{aligned}\partial_k h_k(Q) &= \partial_k h_k(Q)|_{\phi_k} + \partial_k\alpha_k(Q)P_{\phi,k}(Q),\\ \partial_k\lambda_{\psi,k}(Q) &= \partial_k\lambda_{\psi,k}(Q)|_{\phi_k} + h_k(Q)\,\partial_k\alpha_k(Q).\end{aligned}$$

Choose α_k such that no four fermion coupling is generated ⟹

$$\partial_k h_k(Q) = \partial_k h_k(Q)|_{\phi_k} - \frac{P_{\phi,k}(Q)}{h_k(Q)}\partial_k\lambda_{\psi,k}(Q)|_{\phi_k}$$

…cures mean field ambiguity



conclusions

Flow equation for effective average action:

- Does it work?
- Why does it work?
- When does it work?
- How accurately does it work?

end

Flow equation for the Hubbard model

T.Baier , E.Bick , …

Truncation

Concentrate on antiferromagnetism

$$\vec{a}(Q) = \vec{m}(Q + \Pi)$$

Potential U depends only on $\alpha = a^2$

$$\Gamma_{\psi,k}[\psi, \psi^*] = \sum_Q \psi^\dagger(Q) P_F(Q) \psi(Q),$$
$$P_F(Q) = i\omega_F + \epsilon - \mu, \quad \epsilon(\boldsymbol{q}) = -2t(\cos q_x + \cos q_y),$$

$$\Gamma_{Y,k}[\psi, \psi^*, \vec{a}] = -\bar{h}_{a,k} \sum_{KQQ'} \vec{a}(K) \psi^*(Q) \vec{\sigma} \psi(Q') \times \delta(K - Q + Q' + \Pi)$$

$$\Gamma_{a,k}[\vec{a}] = \frac{1}{2} \sum_Q \vec{a}(-Q) P_a(Q) \vec{a}(Q) + \sum_X U[\vec{a}(X)]$$

$$\begin{aligned}
\mathrm{SYM}: \sum_X U[\vec{a}] &= \sum_K \bar{m}_a^2 \alpha(-K, K) + \\
&+ \frac{1}{2} \sum_{K_1 \ldots K_4} \bar{\lambda}_a \delta(K_1 + K_2 + K_3 + K_4) \\
&\quad \times \alpha(K_1, K_2) \alpha(K_3, K_4), \\
\mathrm{SSB}: \sum_X U[\vec{a}] &= \frac{1}{2} \sum_{K_1 \ldots K_4} \bar{\lambda}_a \delta(K_1 + K_2 + K_3 + K_4) \\
&\quad \times (\alpha(K_1, K_2) - \alpha_0 \delta(K_1) \delta(K_2)) \\
&\quad \times (\alpha(K_3, K_4) - \alpha_0 \delta(K_3) \delta(K_4))
\end{aligned}$$

$$\alpha(K, K') = \frac{1}{2} \vec{a}(K) \vec{a}(K')$$

scale evolution of effective potential for antiferromagnetic order parameter

$$
\begin{aligned}
\partial_k U \;(\alpha) &= \partial_k U^B(\alpha) + \partial_k U^F(\alpha) \\
&= \frac{1}{2} \sum_{Q,i} \tilde{\partial}_k \ln[P_a(Q) + \hat{M}_i^2(\alpha) + R_k^a(Q)] \\
&\quad -2T \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2} \tilde{\partial}_k \ln \cosh y(\alpha).
\end{aligned}
$$

boson contribution

fermion contribution

$$
\begin{aligned}
&\hat{M}^2_{1,2,3}(\alpha) = \\
&\quad = \begin{cases} (\bar{m}_a^2 + 3\bar{\lambda}_a \alpha, \bar{m}_a^2 + \bar{\lambda}_a \alpha, \bar{m}_a^2 + \bar{\lambda}_a \alpha) & \text{SYM} \\ (\bar{\lambda}_a (3\alpha - \alpha_0), \bar{\lambda}_a(\alpha - \alpha_0), \bar{\lambda}_a(\alpha - \alpha_0)) & \text{SSB} \end{cases}
\end{aligned}
$$

$$
y(\alpha) = \frac{1}{2T_k} \sqrt{\epsilon^2(\boldsymbol{q}) + 2\bar{h}_a^2 \alpha}.
$$

effective masses depend on α !

gap for fermions ~α

running couplings

SYM:

$$\partial_k \bar{m}_a^2 = \frac{\partial}{\partial \alpha}(\partial_k U(\alpha))|_{\alpha=0},$$

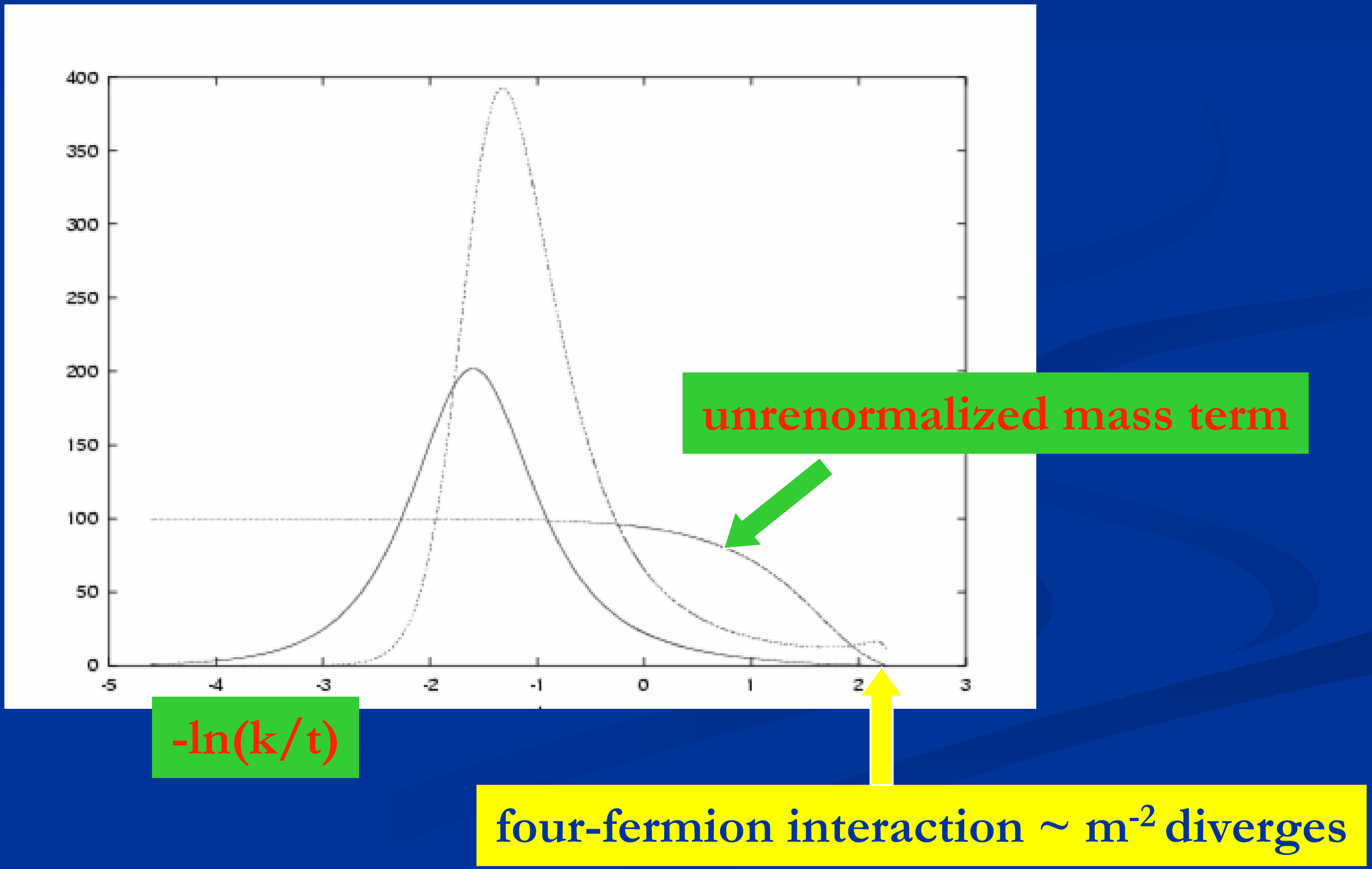
$$\partial_k \bar{\lambda}_a = \frac{\partial^2}{\partial \alpha^2}(\partial_k U(\alpha))|_{\alpha=0},$$

SSB:

$$\partial_k \alpha_0 = -\frac{1}{\bar{\lambda}_a}\frac{\partial}{\partial \alpha}(\partial_k U(\alpha))|_{\alpha=\alpha_0},$$

$$\partial_k \bar{\lambda}_a = \frac{\partial^2}{\partial \alpha^2}(\partial_k U(\alpha))|_{\alpha=\alpha_0}.$$

Running mass term



dimensionless quantities

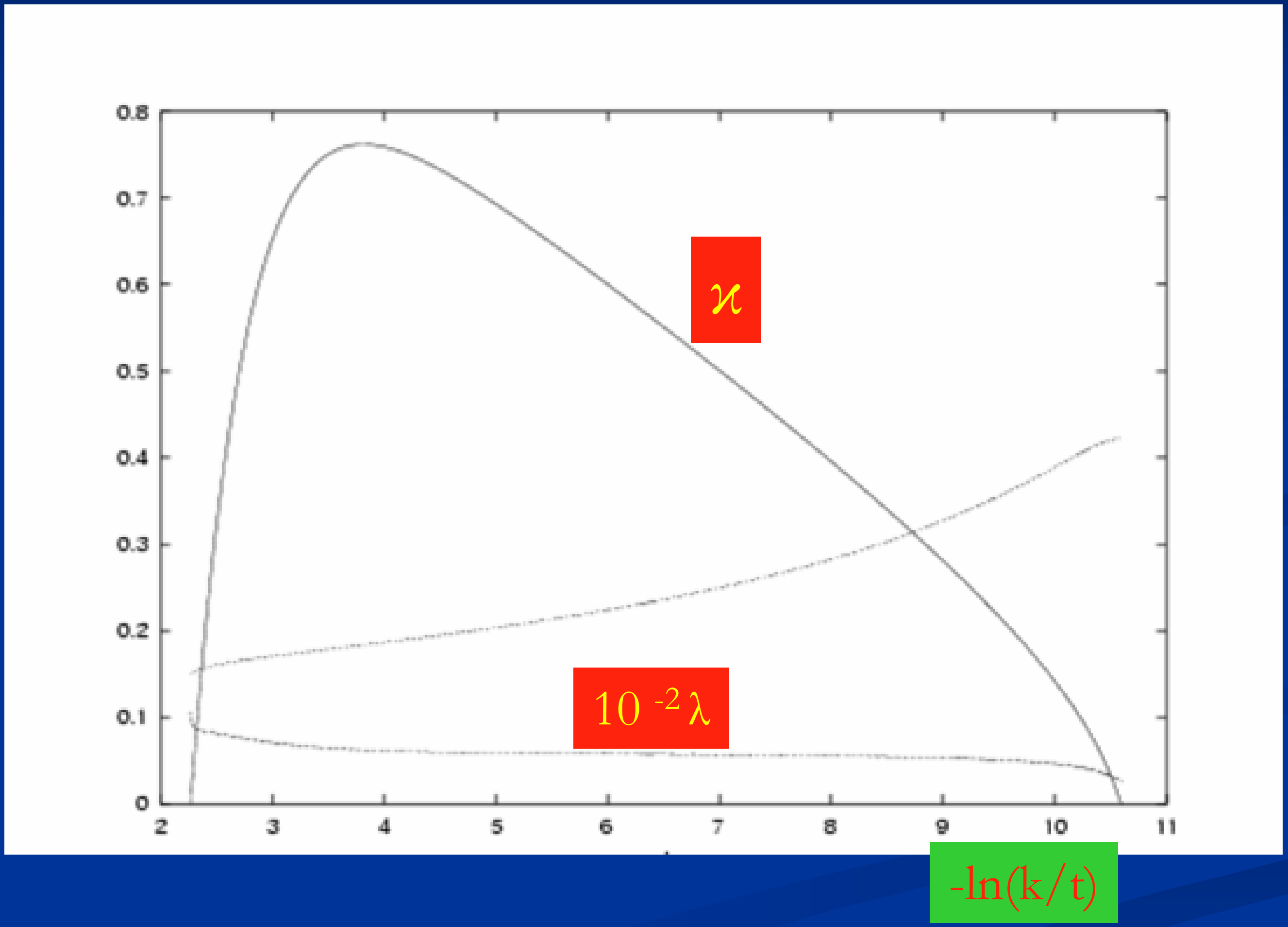
$$u = \frac{Ut^2}{Tk^2}, \quad \tilde{\alpha} = \frac{Z_a t^2 \alpha}{T}$$

$$m_a^2 = \frac{\bar{m}_a^2}{Z_a k^2} = \frac{\partial u}{\partial \tilde{\alpha}}, \quad \kappa_a = \frac{Z_a t^2}{T} \alpha_0,$$

$$\lambda_a = \frac{T}{Z_a^2 t^2 k^2} \bar{\lambda}_a = \frac{\partial^2 u}{\partial \tilde{\alpha}^2}, \quad h_a^2 = \frac{T}{Z_a t^4} \bar{h}_a^2$$

renormalized antiferromagnetic order parameter ϰ

evolution of potential minimum

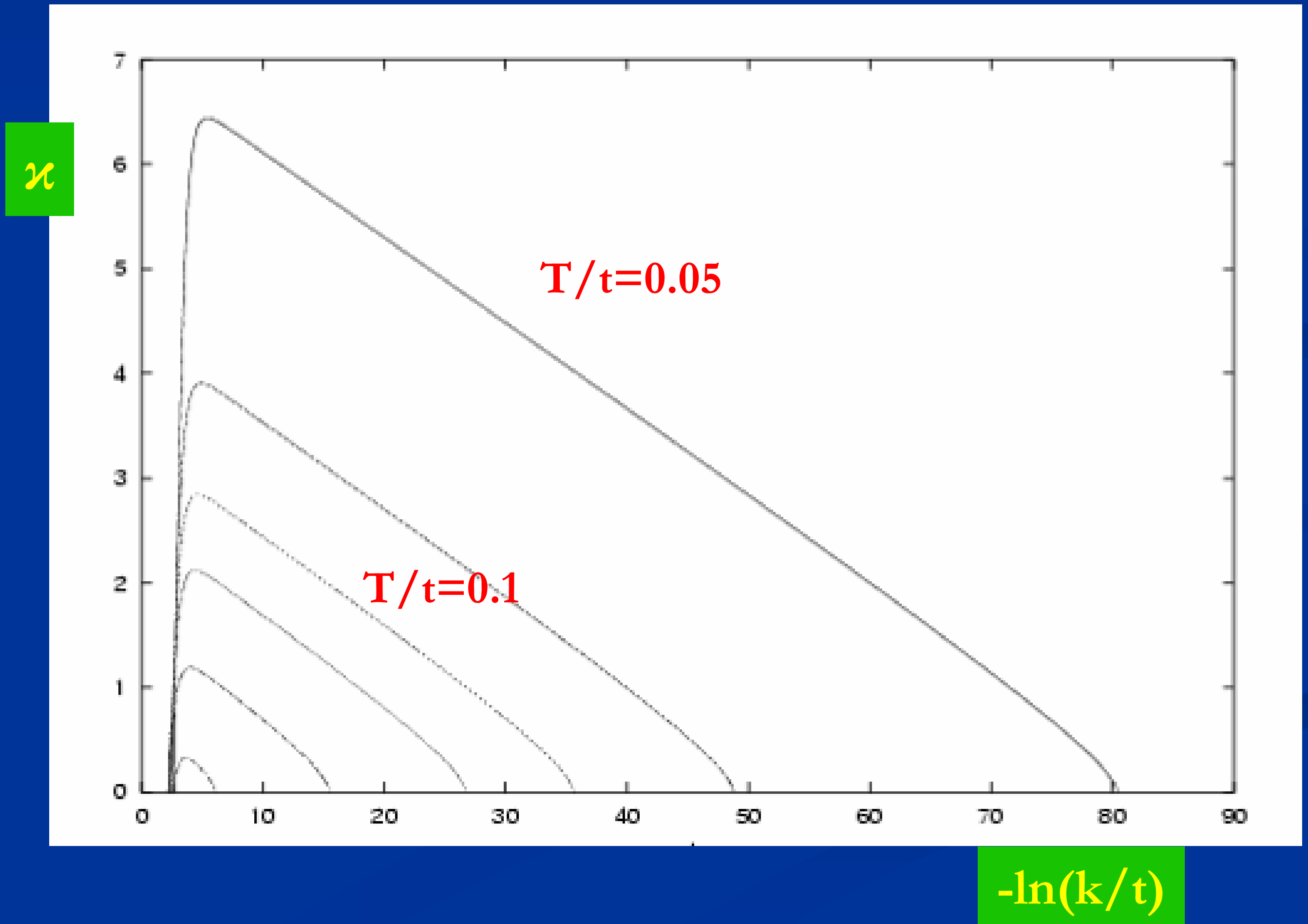


U/t = 3 , T/t = 0.15

Critical temperature

For $T<T_c$: $\varkappa$ remains positive for $k/t > 10^{-9}$

size of probe > 1 cm

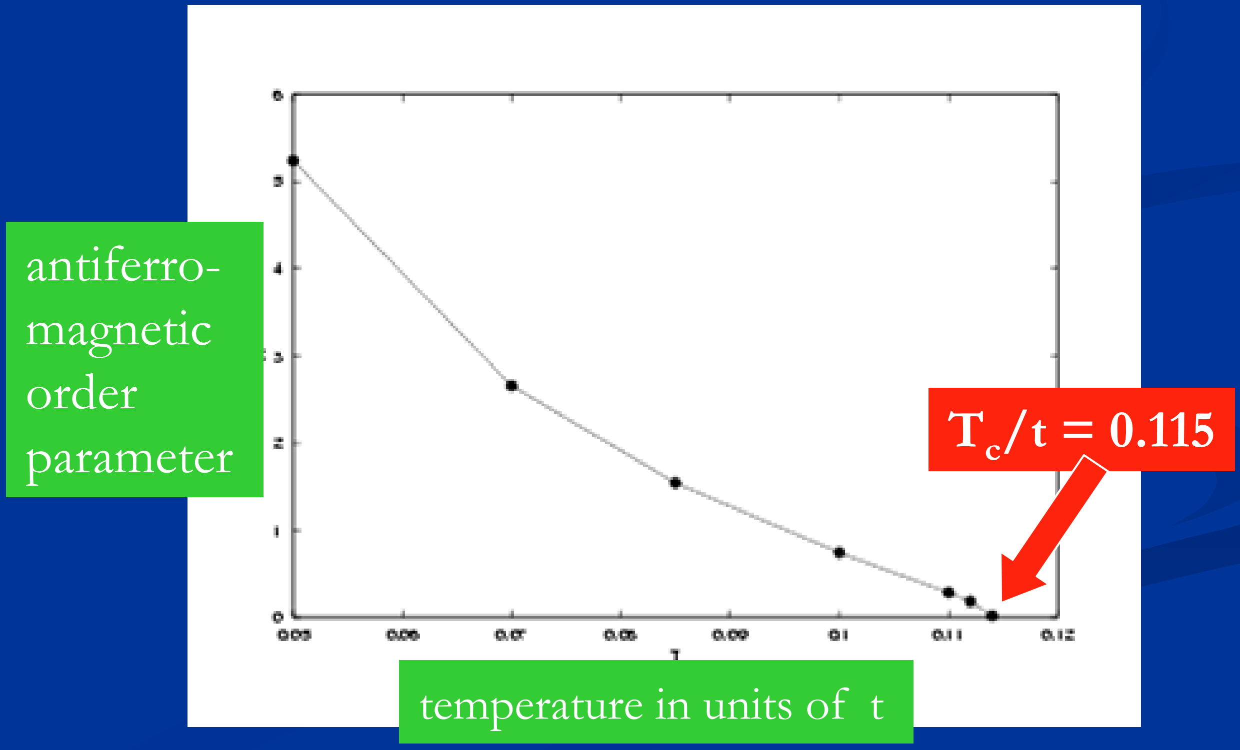


$T_c=0.115$

Below the critical temperature :

Infinite-volume-correlation-length becomes larger than sample size

finite sample ≈ finite k   :   order remains effectively



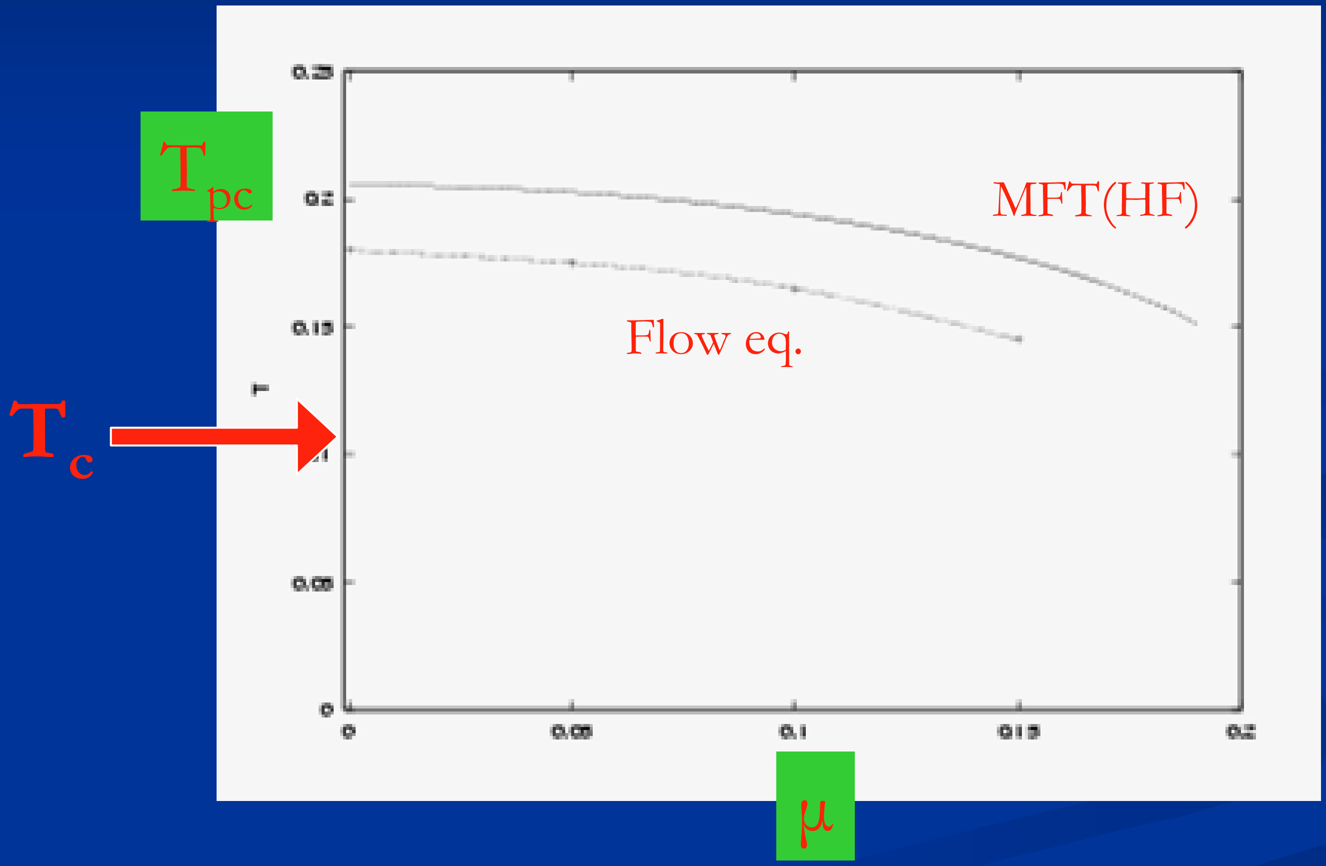
Pseudocritical temperature T_{pc}

- Limiting temperature at which bosonic mass term vanishes ($\varkappa$ becomes nonvanishing)

- It corresponds to a diverging four-fermion coupling

- This is the “critical temperature” computed in MFT !

Pseudocritical temperature



Below the pseudocritical temperature

the reign of the goldstone bosons

effective nonlinear O(3) – σ - model

critical behavior

for interval $T_c < T < T_{pc}$
evolution as for classical Heisenberg model

cf. Chakravarty,Halperin,Nelson

$$k\partial_k\kappa = \frac{1}{4\pi} + \frac{1}{16\pi^2\kappa} + 0(\kappa^{-2})$$

$$\kappa(k) = \kappa_m(T) - \frac{1}{4\pi}\ln\frac{k_m(T)}{k}$$

critical correlation length

$$\xi t = c(T)\exp\left\{20.7\beta(T)\frac{T_c}{T}\right\}$$

c,β : slowly varying functions

exponential growth of correlation length
compatible with observation !

at T_c : correlation length reaches sample size !

$$\begin{aligned}\beta(T) &= \frac{\hat{\alpha}_0(T)\hat{Z}_a(T)}{\hat{\alpha}_0(T_c)\hat{Z}_a(T_c)},\\ c(T) &= C_{\mathrm{SR}}\frac{k_m(T_c)}{k_m(T)}\left(\frac{k_m(T_c)}{t}\right)^{\delta(T)},\\ \delta(T) &= \beta(T)\frac{T_c}{T}-1\end{aligned}$$

$$\xi = \frac{C_{\mathrm{SR}}}{k_m(T)}\exp\left(4\pi\kappa_m(T)\right)$$

$$\xi = \tilde{C}\exp\left(\frac{\gamma}{T}\right)$$

$$\gamma = 4\pi\hat{\alpha}_0(T)\hat{Z}_a(T)t^2.$$

$$T_c(k) = \frac{\gamma(T_c)}{\ln\left(k_m(T_c)/k\right)}$$

critical behavior for order parameter and correlation function

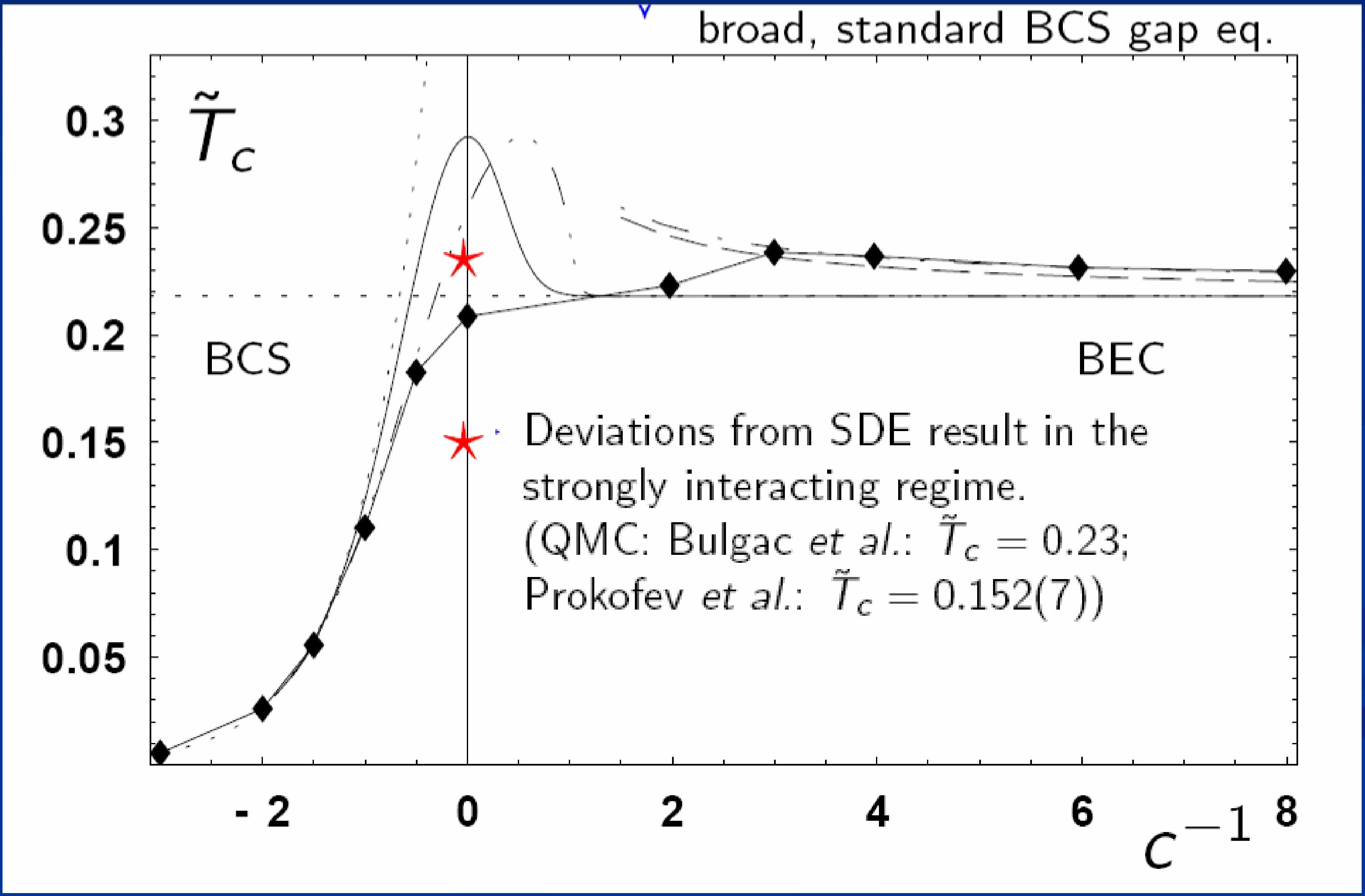
$$\kappa_a(T) = \left(\frac{\gamma(T)T_c}{T} - 1\right)\kappa_m(T_c) + \frac{1}{4\pi}\ln\frac{k_m(T_c)}{k_m(T)}.$$

$$G(q^2) = (Z_a(k = \sqrt{q^2})q^2)^{-1} \sim (q^2)^{-1+\eta_a/2}$$

Mermin-Wagner theorem ?

No spontaneous symmetry breaking of continuous symmetry in $d=2$!

crossover phase diagram



shift of BEC critical temperature