QCD – from the vacuum to high temperature

an analytical approach

Functional Renormalization Group

from small to large scales

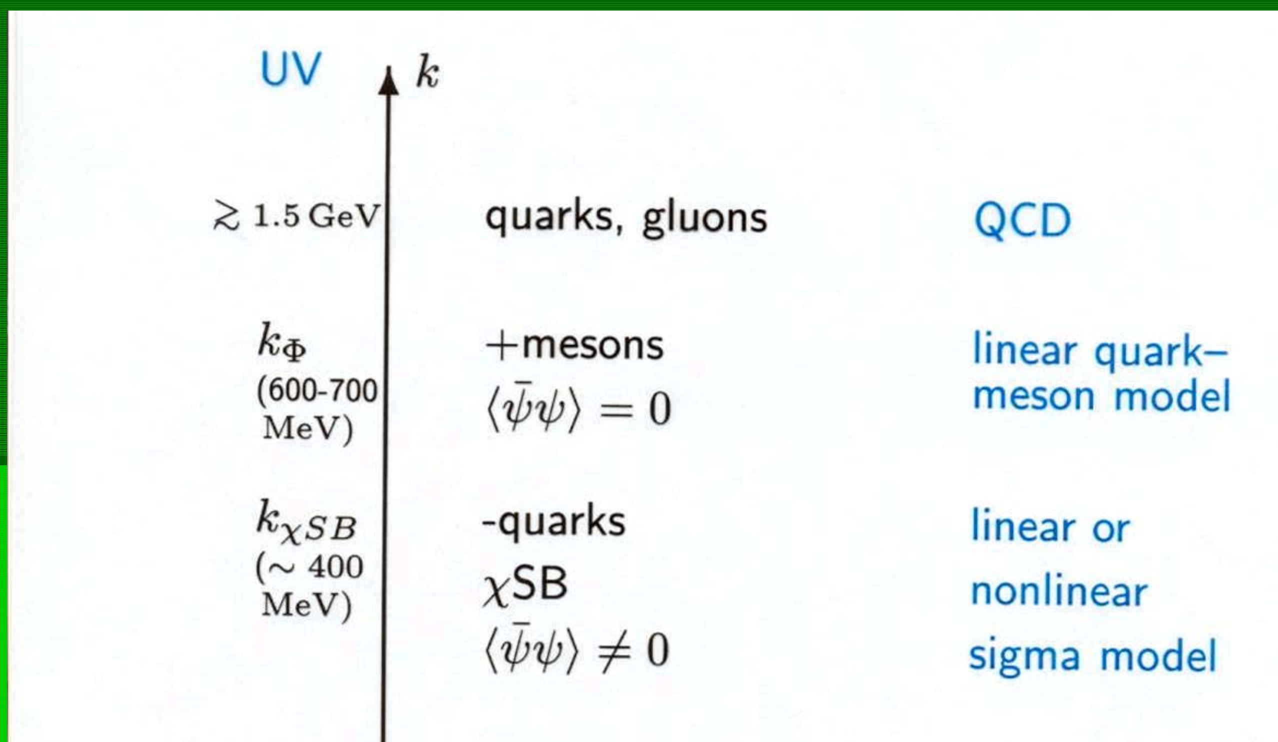
How to come from quarks and gluons to baryons and mesons ?

Find effective description where relevant degrees of freedom depend on momentum scale or resolution in space.

Microscope with variable resolution:

- High resolution , small piece of volume: quarks and gluons
- Low resolution, large volume : hadrons

Scales in strong interactions



From

Microscopic Laws

(Interactions, classical action)

to ↓ Fluctuations!

Macroscopic Observation

(Free energy functional, effective action)

Non-Perturbative

Renormalization Flow

in

Quantum Field Theory

and

Statistical Physics

- block spins

  Kadanoff, Wilson

- exact renormalization group equations

  Wilson, Kogut
  Wagner, Houghton
  Weinberg
  Polchinski
  Hasenfratz^2

- Lattice finite size scaling

  Lüscher,...

- coarse grained free energy/average action

Flow equations

(Exact renormalization group equations)

- interpolate from microphysics to large distances
- from simple laws to complexity
- **infrared cutoff k**
  only fluctuations with momenta $q^2 > k^2$ are included
- running couplings depend on k
- $k \to 0$: effective action:
  solution of (quantum)-field-theory

Average potential U_k

$\equiv$ **scale dependent effective potential**
$\equiv$ **coarse grained free energy**

Only fluctuations with momenta $q^2 > k^2$ included

k: infrared cutoff for fluctuations, "average scale"

Λ: characteristic scale for microphysics

$$U_\Lambda \approx S \to U_0 \equiv U$$

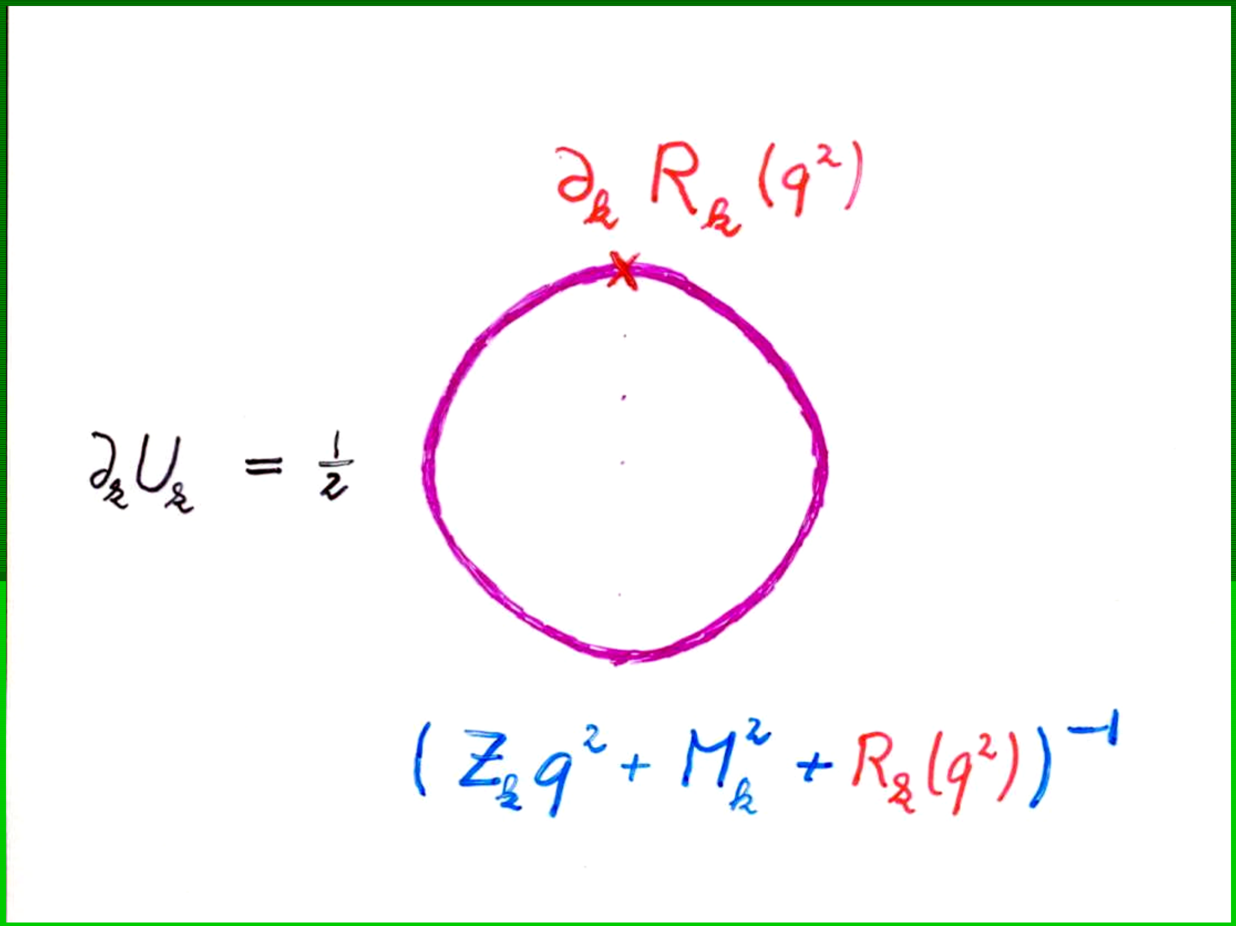
Flow equation for average potential

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}_{k,i}^2(\varphi)}$$

$$\bar{M}_{k,ab}^2 = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b} \quad : \quad \text{Mass matrix}$$

$$\bar{M}_{k,i}^2 \quad : \quad \text{Eigenvalues of mass matrix}$$

Simple one loop structure – nevertheless (almost) exact



Infrared cutoff

R_k : IR-cutoff

$$\text{e.g} \qquad R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$

$$\text{or} \qquad R_k = Z_k (k^2 - q^2)\Theta(k^2 - q^2) \qquad \text{(Litim)}$$

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Wave function renormalization and anomalous dimension

Z_k: wave function renormalization

$$k\partial_k Z_k = -\eta_k Z_K$$

η_k: anomalous dimension

$$t = \ln(k/\Lambda)$$

$$\partial_t \ln Z = -\eta$$

for $Z_k(\varphi, q^2)$: flow equation is **exact !**

approximations

On the exact level :

New flow equation for $Z_k(\varphi, q^2)$ needed !

Often approximative form of $Z_k(\varphi, q^2)$

is known or can be simply computed

e.g.small anomalous dimension

Flow equation for U_k

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^dq}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

'91

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

R_k : IR-cutoff

e.g $R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$

or $R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$ (Litim)

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Partial differential equation for function U(k,φ) depending on two (or more) variables

$Z_k = c\, k^{-\eta}$

Regularisation

For suitable R_k :

$$R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$
$$R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$$

- Momentum integral is ultraviolet and infrared finite
- Numerical integration possible
- Flow equation defines a regularization scheme (ERGE –regularization)

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Integration by momentum shells

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Momentum integral is dominated by $q^2 \sim k^2$.

Flow only sensitive to physics at scale k

Scalar field theory

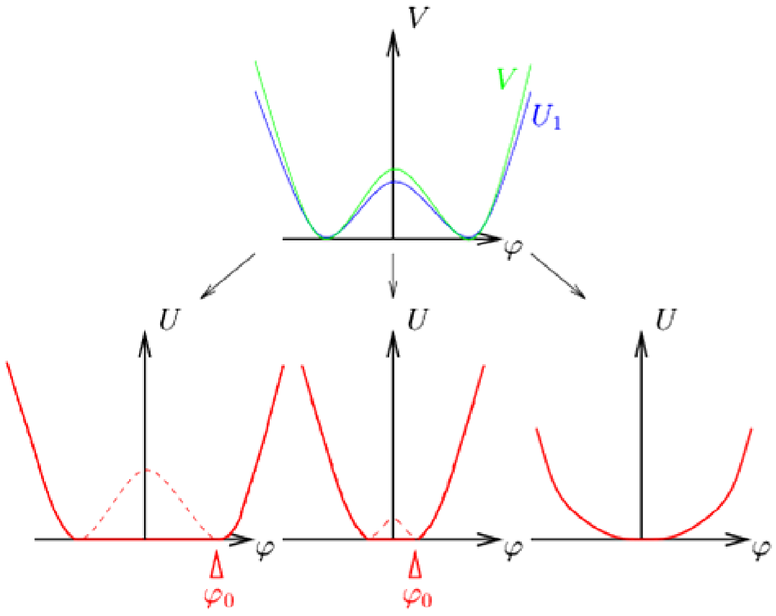
e.g. linear sigma-model for
chiral symmetry breaking in QCD

Scalar field theory

$\varphi_a(x)$: magnetization, density, chemical concentration, Higgs field, meson field, inflaton, cosmon

$O(N)$-symmetry:

$$S = \int d^dx \left\{ \frac{1}{2}\partial_\mu\varphi_a\partial_\mu\varphi_a + V(\rho) \right\}; \quad \rho = \frac{1}{2}\varphi_a\varphi_a$$



O(N) - model

- First order derivative expansion

$$\partial_k U_k(\rho) = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d} \cdot$$

$$\left\{ \frac{\partial_k R_k(q^2)}{\tilde{Z}_k(\rho)q^2 + R_k(q^2) + U'(\rho) + 2\rho U''(\rho)} + (N-1)\frac{\partial_k R_k(q^2)}{Z_k(\rho)q^2 + R_k(q^2) + U'(\rho)} \right\}$$

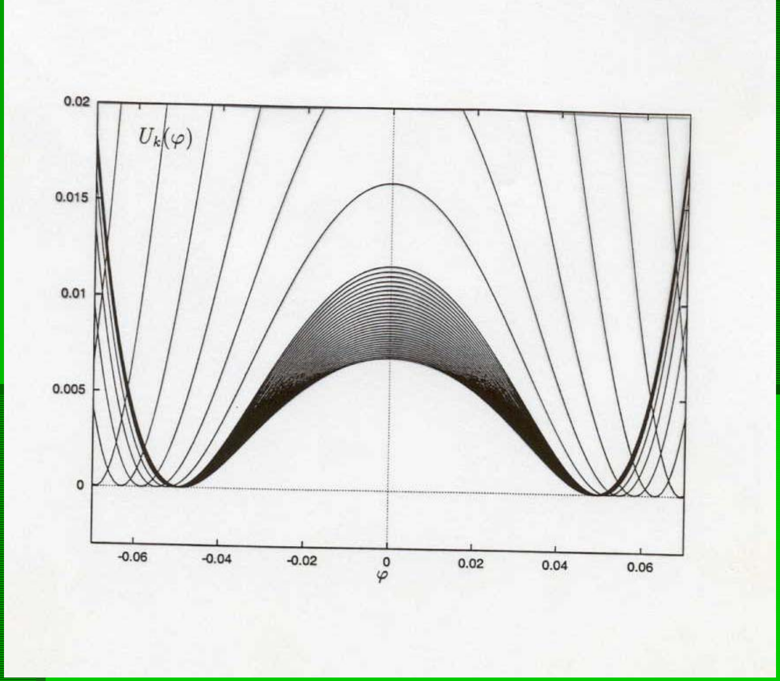
$\rho = \frac{1}{2}\varphi_a\varphi_a$, $U' = \frac{\partial U}{\partial \rho}$

Lowest order derivative expansion :

$Z_k(\rho)$, $\tilde{Z}_k(\rho) \rightarrow Z_k$

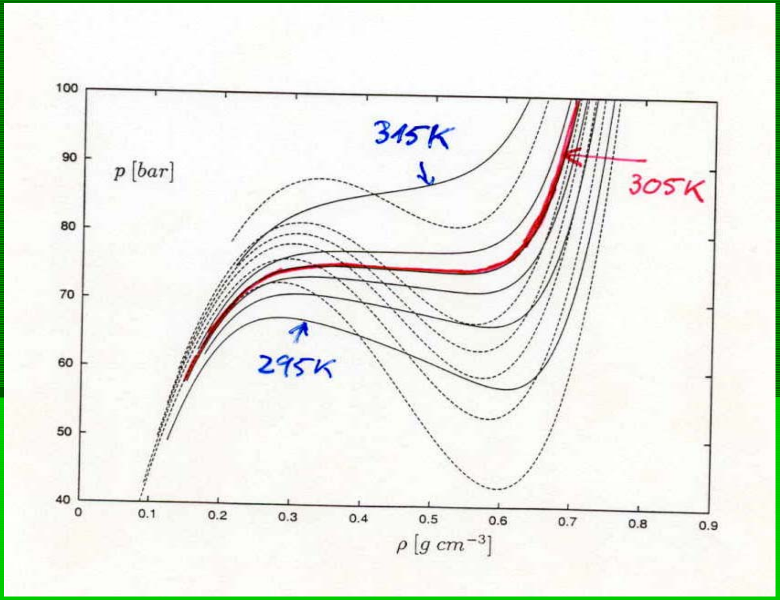
Flow of effective potential

Ising model



S.Seide …

CO_2



Experiment :

T_* =304.15 K
p_* =73.8.bar
ρ_* = 0.442 g cm-2

Critical behaviour

correlation length

$$\xi = m_R^{-1} \sim |T - T_c|^{-\nu}$$

$$m_R^2 = 2\rho_0 \frac{\partial^2 U}{\partial \rho^2}(\rho_0)\, Z^{-1}$$

anomalous dimension

decay of correlation fct. for $T = T_c$

$$\langle \varphi^*(q)\varphi(q')\rangle_c \sim \frac{1}{(q^2)^{1-\eta/2}}\,\delta(q-q')$$

$(\eta = -\partial_t \ln Z$ at fixed point$)$

Critical exponents

$d = 3$

Critical exponents ν and η

N	ν		η	
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
3	0.704	0.7102	0.049	0.0380
4	0.739	0.7474	0.047	0.0363
10	0.881	0.886	0.028	0.025
100	0.990	0.980	0.0030	0.003
		↑		↑

"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

Scaling form of evolution equation

$$u = \frac{U_k}{k^d}$$
$$\tilde{\rho} = Z_k k^{2-d} \rho$$
$$u' = \frac{\partial u}{\partial \tilde{\rho}} \quad \text{etc.}$$

$$\begin{aligned} \partial_t u|_{\tilde{\rho}} = & -du + (d-2+\eta)\tilde{\rho}u' \\ & +2v_d\{l_0^d(u'+2\tilde{\rho}u'';\eta) \\ & +(N-1)\, l_0^d(u';\eta)\} \end{aligned}$$

$$v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)$$

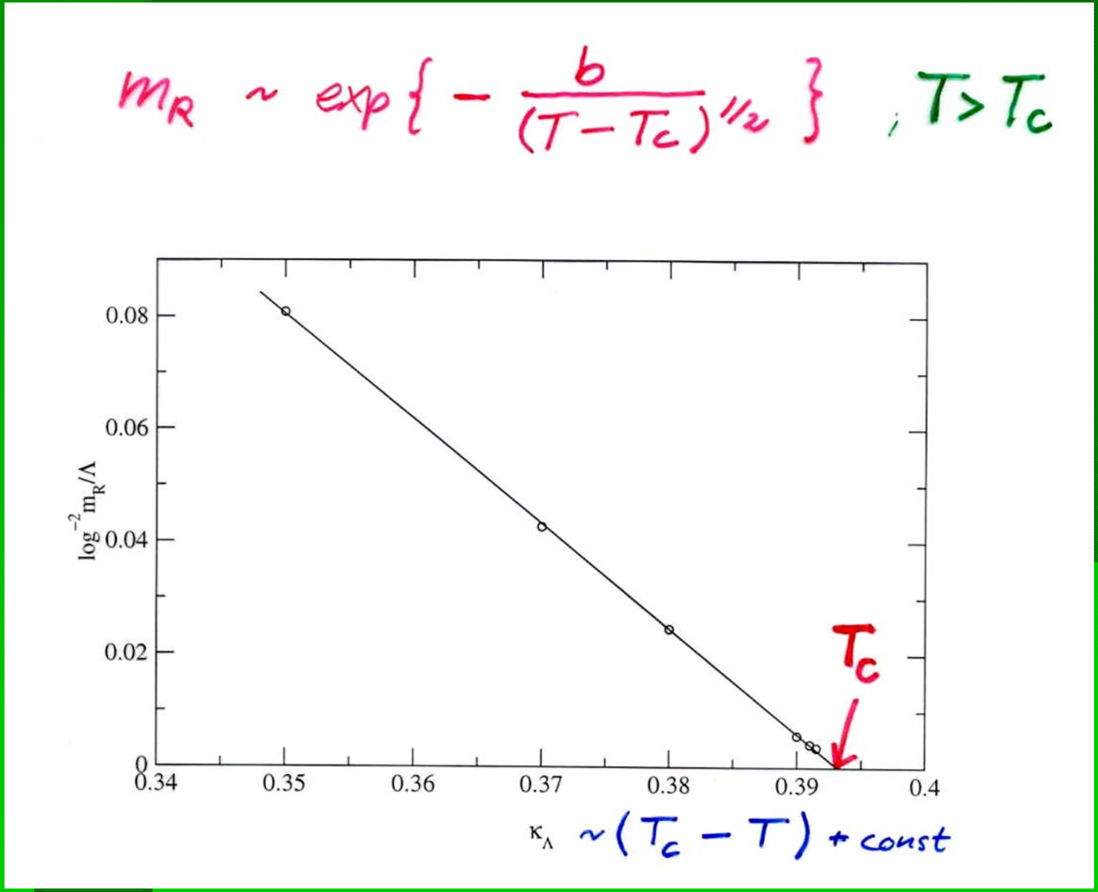
linear cutoff:

$$l_0^d(w;\eta) = \frac{2}{d}\left(1 - \frac{\eta}{d+2}\right)\frac{1}{1+w}$$

On r.h.s. :
neither the scale k
nor the wave function
renormalization Z
appear explicitly.
Fixed point corresponds
to second order
phase transition.

Tetradis …

Essential scaling : d=2,N=2



- Flow equation contains correctly the non-perturbative information !
- (essential scaling usually described by vortices)

Von Gersdorff …

Kosterlitz-Thouless phase transition (d=2,N=2)

Correct description of phase with

Goldstone boson

(infinite correlation length)

for $T<T_c$

Exact renormalization group equation

Exact flow equation

for scale dependence of average action

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1} \partial_k R_k\right\}$$

'92

$$\left(\Gamma_k^{(2)}\right)_{ab}(q,q') = \frac{\delta^2 \Gamma_k}{\delta\varphi_a(-q)\delta\varphi_b(q')}$$

$$\mathrm{Tr} : \sum_a \int \frac{d^d q}{(2\pi)^d}$$

(fermions : STr)

Generating functional

generating functional for connected Green's functions in presence of quadratic infrared cutoff

$$W_k[j] = \ln \int \mathcal{D}\chi \, \exp\left(-S[\chi] - \Delta_k S[\chi] + \int d^d x \, j_a \chi_a\right)$$

$$\Delta_k S = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q^2) \chi_a(-q) \chi_a(q)$$

$$\text{e.g.} \quad R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$

$$\lim_{k \to 0} R_k = 0$$

$$R_{k \to \infty} \to \infty$$

Effective average action

$$\Gamma_k[\varphi] = -W_k[j] + \int d^dx\, j_a \varphi_a - \Delta_k S[\varphi]$$

$$\varphi_a = \langle \chi_a \rangle = \frac{\delta W_k}{\delta j_a}$$

$\Gamma_0[\varphi]$: quantum effective action

generates 1PI vertices

free energy: $F = \Gamma T + \mu n V$

Γ_k includes all fluctuations (quantum, thermal) with $q^2 > k^2$

Γ_Λ specifies microphysics

Loop expansion :
perturbation theory
with
infrared cutoff
in propagator

Quantum effective action

for $k \to 0$
all fluctuations (quantum + thermal)
are included

knowledge of $\Gamma_{k \to 0} \hat{=}$ solution of model

Proof of exact flow equation

$$\partial_k \Gamma|_\phi = -\partial_k W|_j - \partial_k \Delta_k S[\varphi]$$
$$= \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k(\langle\phi\phi\rangle - \langle\phi\rangle\langle\phi\rangle)\right\}$$
$$= \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k W_k^{(2)}\right\}$$

$$W_k^{(2)}(\Gamma_k^{(2)} + R_k) = \mathbb{1}$$

$$(\Delta_k S^{(2)} \equiv R_k)$$

$$\Longrightarrow$$

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1}\right\}$$

Truncations

Functional differential equation –

cannot be solved exactly

Approximative solution by truncation of

most general form of effective action

derivative expansion

Tetradis,...; Morris

$O(N)$-model:

$$\Gamma_k = \int d^d x \{ U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \varphi_a \partial_\mu \varphi_a + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial_\mu \rho + \cdots \}$$

$$(N = 1 : \quad Y_k \equiv 0)$$

field expansion
(flow eq. for 1PI vertices)

Weinberg; Ellwanger,...

$$\Gamma_k = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{j=0}^{n} d^d x_j \, \Gamma_k^{(n)}(x_1, x_2, \ldots, x_n) \prod_{j=0}^{n} (\phi(x_j) - \phi_0)$$

error estimate?

Expansion in canonical dimension of couplings

Lowest order:

$$d = 4: \quad \rho_0, \bar{\lambda}, Z$$
$$d = 3: \quad \rho_0, \bar{\lambda}, \bar{\gamma}, Z$$
$$U = \frac{1}{2}\bar{\lambda}(\rho - \rho_0)^2 + \frac{1}{6}\bar{\gamma}(\rho - \rho_0)^3$$

works well for $O(N)$ models
Tetradis,...; Tsypin

polynomial expansion of potential converges
if expanded around ρ_0
Tetradis,...; Aoki et al.

Exact flow equation for effective potential

- Evaluate exact flow equation for homogeneous field φ .
- R.h.s. involves exact propagator in homogeneous background field φ.

Nambu Jona-Lasinio model

$$S = \int d^4x \left\{ i \bar{\psi}^i_a \gamma^\mu \partial_\mu \psi^i_a \right.$$

$$\left. + 2\lambda_G \left(\bar{\psi}^i_{Lb} \psi^i_{Ra} \right)\left(\bar{\psi}^j_{Ra} \psi^j_{Lb} \right) \right\}$$

$$\psi_{L,R} = \frac{1 \pm \gamma^5}{2} \psi$$

$i, j = 1 \dots N_c$ color ($N_c = 3$)

$a, b = 1 \dots N_F$ flavor ($N_F = 3, 2$)

chiral flavor symmetry :

$$SU_L(N_F) \times SU_R(N_F)$$

Critical temperature , $N_f = 2$

$\frac{m_\pi}{\text{MeV}}$	0	45	135	230
$\frac{T_{pc}}{\text{MeV}}$	100.7	$\simeq 110$	$\simeq 130$	$\simeq 150$

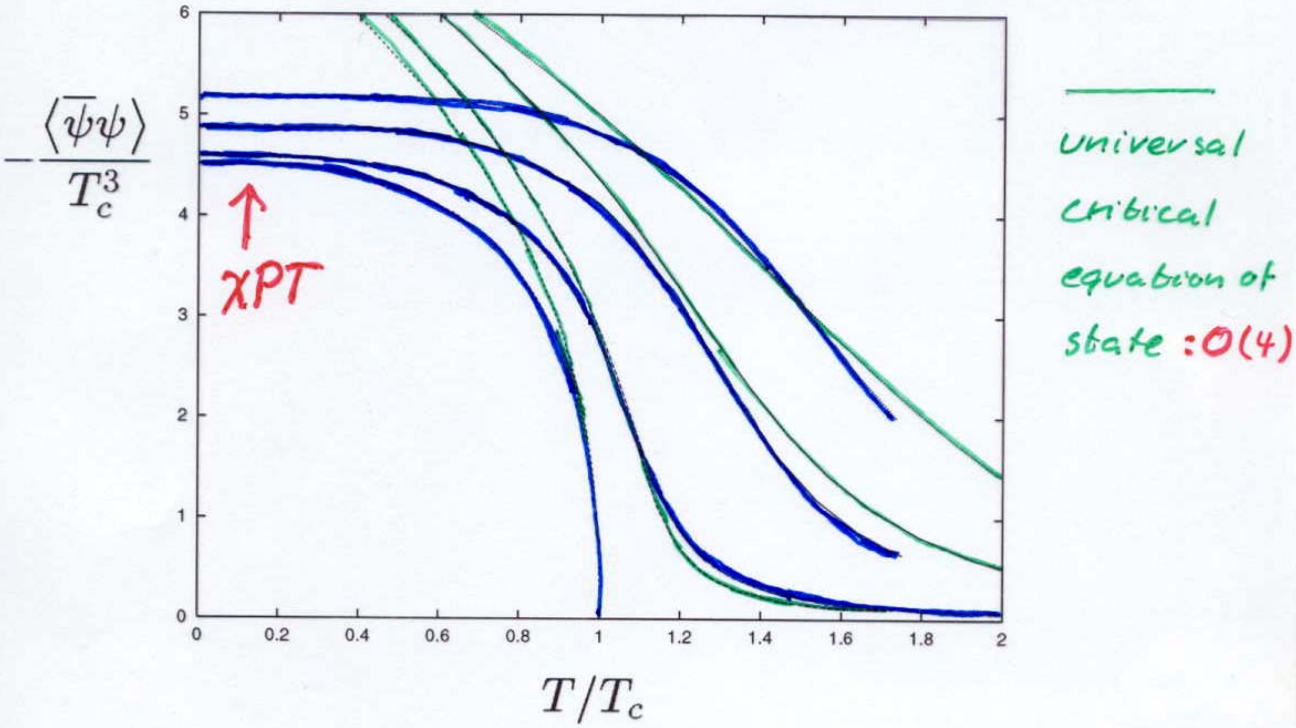
for $f_\pi = 93$ MeV

Lattice simulation

J.Berges,D.Jungnickel,…

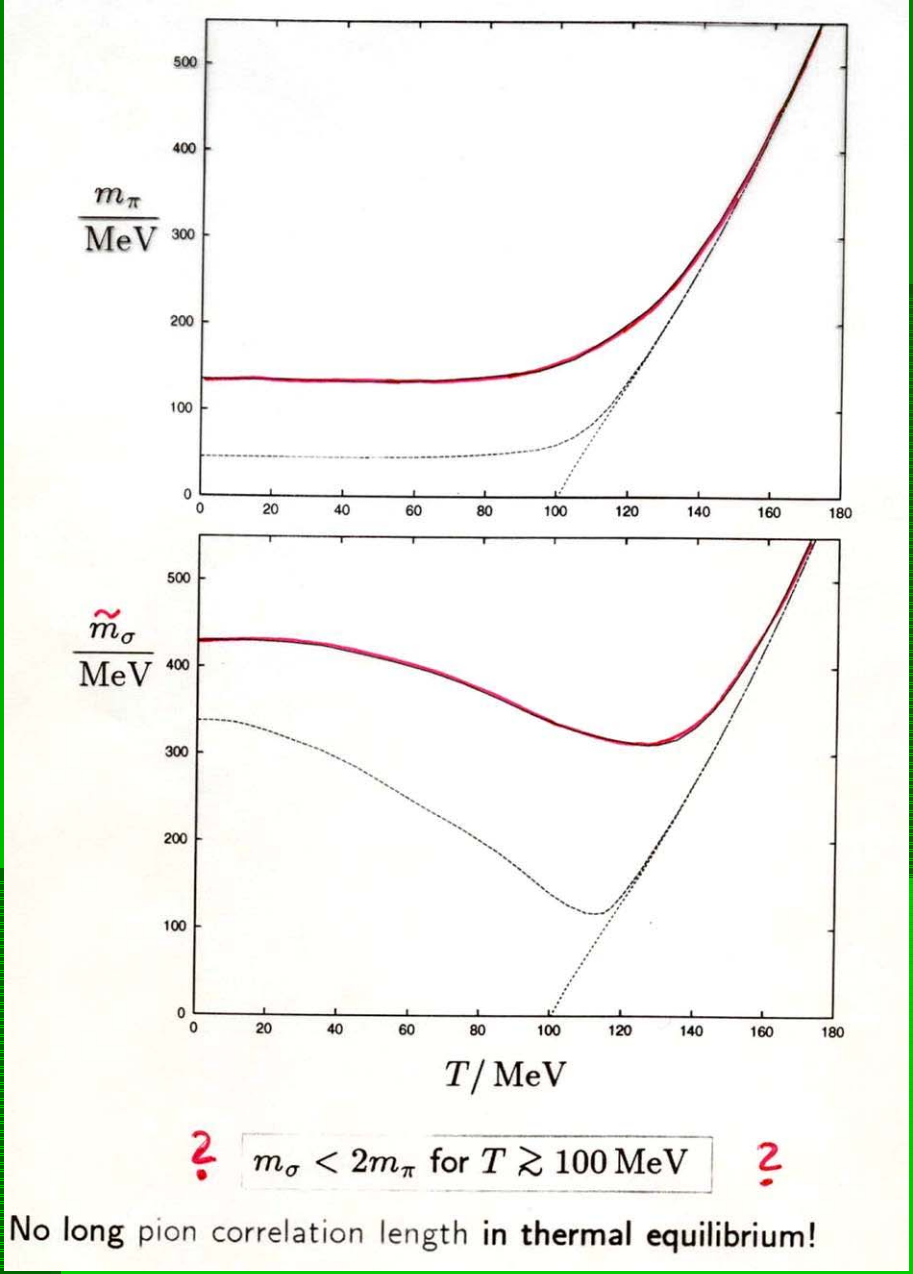
Chiral condensate

2nd order PT (expected for $O(4)$ Heisenberg model)



⟹ Explicit link between χPT domain of validity (4d) and critical (universal) domain near T_c (3d)

temperature dependent masses



pion mass

sigma mass

Critical equation of state

Critical behavior for second order phase transitions :

correlation length $\xi = m_R^{-1}$

only relevant length scale

φ_R : renormalized field variable

$U(\varphi_R)$ depends only on m_R

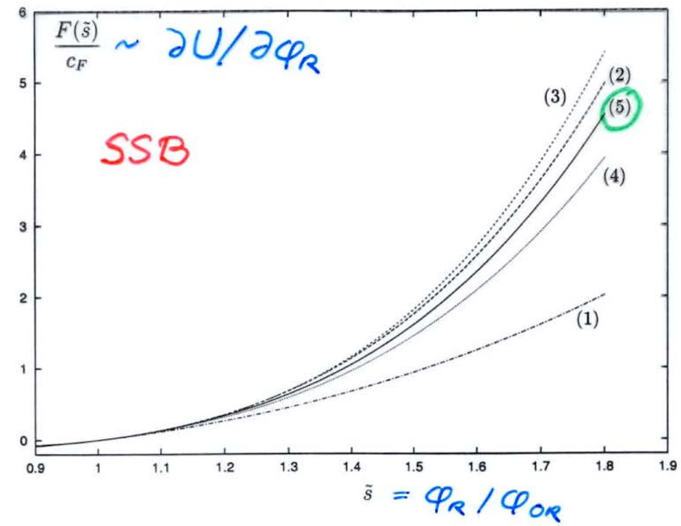
$$B \sim \frac{\partial U}{\partial \varphi_R} = F\left(\frac{\varphi_R}{\sqrt{m_R}}\right) m_R^{5/2}$$

↑

Widom scaling function

Scaling form of equation of state

Berges, Tetradis,…



critical equation of state

(2) ERGE, lowest order derivative exp. ; Berges, Tetradis, …

(5) ERGE, first order derivative exp. ; Seide, …

(1) mean field

(4) high-T-series, loop expansion, ε-expansion

(3) Monte Carlo

Universal critical equation of state
is valid near critical temperature
if the only light degrees of freedom
are pions + sigma with
O(4) – symmetry.

Not necessarily valid in QCD, even
for two flavors !

Chiral quark–meson model

Truncation of the effective average action:

$$\mathcal{L}_k^{(\mathrm{QCD})} = Z_{q,k}\bar{q}_a i\gamma^\mu \partial_\mu q^a + Z_{\Phi,k}\,\mathrm{tr}\left[\partial_\mu \Phi \partial^\mu \Phi\right] + U_k(\Phi, \Phi^\dagger) - \mathrm{tr}\left(\Phi \jmath\right) + \bar{h}_k \bar{q}^a \left(\frac{1+\gamma_5}{2}\Phi_{ab} - \frac{1-\gamma_5}{2}\Phi^\dagger_{ab}\right) q^b$$

The wave function renormalizations $Z_{q,k}$, $Z_{\Phi,k}$, the effective potential $U_k$and the Yukawa coupling $\bar{h}_k$ are scale– or k–dependent.

The initial values we use for $k = k_\Phi$ are NJL–motivated but more general:

$$Z_{q,k_\Phi} = \bar{h}_{k_\Phi} = 1$$
$$Z_{\Phi,k_\Phi} \ll 1$$
$$U_{k_\Phi} = \bar{m}^2_{k_\Phi}\,\mathrm{Tr}\,\Phi^2 + \ldots; \quad \bar{m}^2_{k_\Phi} > 0$$

J.Berges,
D.Jungnickel…

Effective low energy theory

- Imagine that for scale k=700 MeV all other fields except quarks have been integrated out and the result is an essentially pointlike four quark interaction
- Not obviously a valid approximation !

Connection with four quark interaction

- "large k" :

$$U_k = \bar{m}_k^2 \, Tr \, \phi^\dagger \phi \quad ; \quad Z_{\phi,k} \approx 0$$

≙ NJL-model !

solve scalar field equations :

$$\phi_{ab} = \frac{\bar{h}}{\bar{m}^2} \, \bar{\psi}_{bL} \, \psi_{aR}$$

reinsert into effective action

⇔

effective four quark interactions

In principle, m can be computed from four quark interaction in QCD

Meggiolaro,…

Chiral quark-meson model – three flavors

- Φ : $(\bar{3},3)$ of $SU(3)_L \times SU(3)_R$

$$\mathcal{L}_k^{(\mathrm{QCD})} = Z_{q,k}\bar{q}_a i\gamma^\mu \partial_\mu q^a + Z_{\Phi,k}\,\mathrm{tr}\left[\partial_\mu \Phi \partial^\mu \Phi\right] + U_k(\Phi,\Phi^\dagger) - \mathrm{tr}\left(\Phi \jmath\right) + \bar{h}_k \bar{q}^a \left(\frac{1+\gamma_5}{2}\Phi_{ab} - \frac{1-\gamma_5}{2}\Phi^\dagger_{ab}\right) q^b$$

- Effective potential depends on invariants

$$\rho = tr(\varphi^\dagger \varphi)\ , \quad \tau_2 = \frac{N}{N-1}\, tr(\varphi^\dagger\varphi)^2 - \frac{1}{N-1}\rho^2$$

$$\xi = \det\varphi + \det\varphi^\dagger\ , \quad \ldots$$

Spontaneous chiral symmetry breaking

$$\varphi_0 = \begin{pmatrix} \bar\sigma_0 & & \\ & \bar\sigma_0 & \\ & & \bar\sigma_0 \end{pmatrix}$$

$$\Rightarrow \quad \rho_0 = N\,|\bar\sigma_0|^2$$

$$\tau_{20} = 0$$

$$\xi_0 = 2\,\bar\sigma_0^{\,N}$$

$$U_k = \frac{1}{2}\bar\lambda_1(k)\big(\rho - \rho_0(k)\big)^2 + \frac{N-1}{4}\bar\lambda_2(k)\,\tau_2 - \frac{1}{2}\bar\nu(k)\,\xi + \frac{1}{2}\bar\nu(k)\left(\frac{\rho_0(k)}{N}\right)^{\frac{N-2}{2}}\rho$$

---

symmetric regime $(\bar\sigma_0 = 0)$

$$U_k = \bar m^2(k)\,\rho + \frac{1}{2}\bar\lambda_1(k)\,\rho^2 + \frac{N-1}{4}\bar\lambda_2(k)\,\tau_2 - \frac{1}{2}\bar\nu(k)\,\xi$$

$\bar\nu \neq 0$: $U_A(1)$ broken (anomaly!)

Limitations

- Confinement not included
- Pointlike interaction at scale k_φ not a very accurate description of the physics associated with gluons
- → substantial errors in nonuniversal quantities (e.g. T_c)

Conclusions

Non-perturbative flow equation (ERGE)

- Useful non-perturbative method
- Well tested in simple bosonic and fermionic systems
- Interesting generalizations to gauge theories , gravity

Flow equations for QCD

Much work in progress by various groups

- Gluodynamics (no quarks)
- Quark-meson models
- Description of bound states
- For realistic QCD : if Higgs picture correct, this would greatly enhance the chances of quantitatively reliable solutions

end

Fermionic Models

* Nambu-Jona-Lasinio model (NJL)

  (QCD)

* Hubbard model

* Gross-Neveu model (GN)

---

for low T, chemical potential $\mu \neq 0$

difficult region in momentum space :

Fermi surface

Choose $R_k^{(F)}$ such that momenta

near Fermi surface are cut off for $k > 0$!

e.g. $T \rightarrow (T^2 + k^2)^{1/2}$

Baier, Bick, ...

Gross-Neveu model

$$S = \int d^d x \left\{ i \bar{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{G}{2} (\bar{\psi}_a \psi_a)^2 \right\}$$

$a = 1 \dots N$

SO(d) – „Lorentz"-symmetry

$d = 3$:

two space dimensions, $T = 0$

Second order phase transition

($d = 3$, all N)

„ quantum critical point" , $T = 0$

order parameter : $\sigma_0 = iG < \bar{\psi}_a \psi_a >$

Critical exponents

N	ν	η_σ	η_ψ
1	0.62	0.31	0.11
2	0.93/1.00(4)	0.53/0.75(1)	0.07
4	1.02/1.02(8)	0.76/0.81(13)	0.03

Mass gap

$m_{\psi,R} = \Delta_\psi \rho_0$, $m_{\sigma,R} = \Delta_\sigma \rho_0$

$\rho_0 = \frac{1}{2} Z_\sigma G^2 < \bar{\psi} \psi >^2$

$N = 1$: $\Delta_\psi = 14.5$, $\Delta_\sigma = 16.8$

Hoefling, Nowak, … ; Vitale, Rosa, …