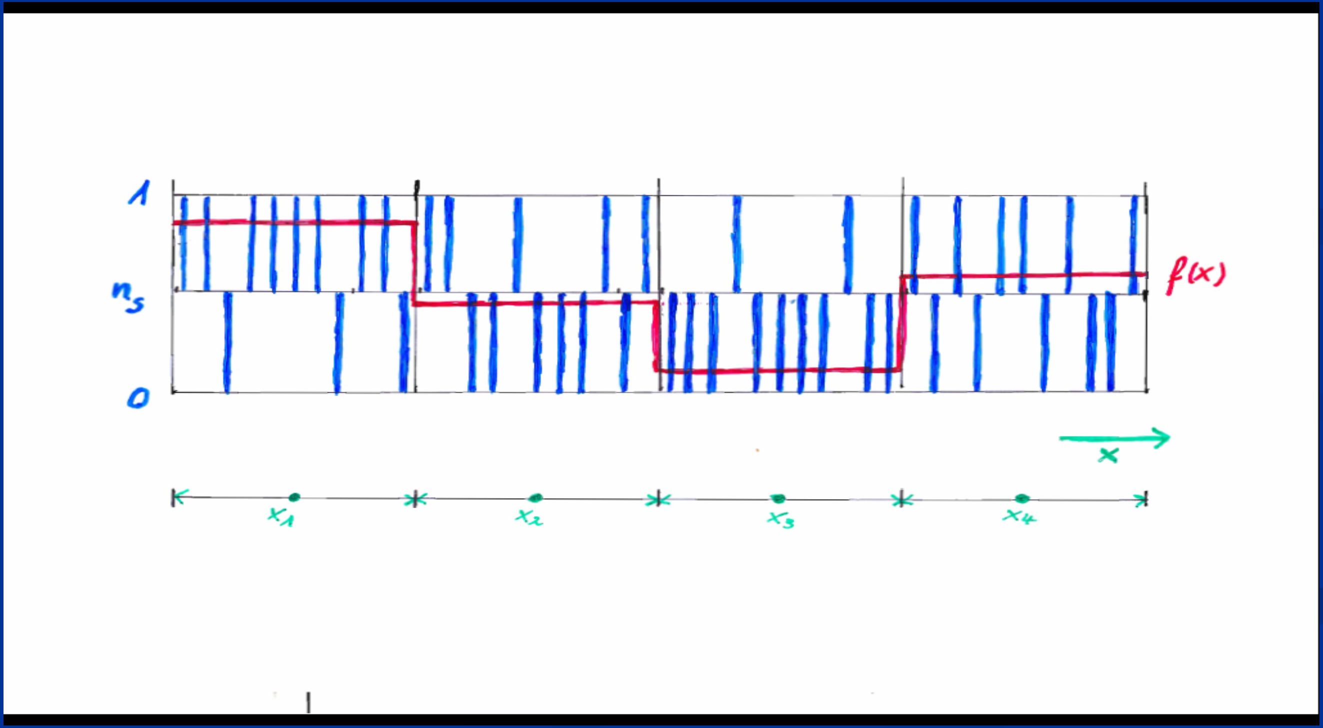
Fermions and non-commuting observables from classical probabilities



quantum mechanics can be described by classical statistics !

statistical picture of the world

- basic theory is not deterministic
- basic theory makes only statements about probabilities for sequences of events and establishes correlations
- probabilism is fundamental , not determinism !

quantum mechanics from classical statistics :
not a deterministic hidden variable theory

Probabilistic realism

Physical theories and laws only describe probabilities

Physics only describes probabilities

Gott würfelt

fermions from classical statistics

microphysical ensemble

- states τ
- labeled by sequences of occupation numbers or bits $n_s = 0$ or 1
- $\tau = [\, n_s \,] = [0,0,1,0,1,1,0,1,0,1,1,1,1,0,\ldots]$ etc.
- probabilities $p_\tau > 0$

Grassmann functional integral

action :

$$S = \sum_{t'} L(t')$$

$$S = \sum_{t} \left\{ \hat{\psi}(t)^T \left(\psi(t+\epsilon) - \psi(t) \right) + i\epsilon H \left[\hat{\psi}(t), \psi(t+\epsilon) \right] \right\}$$

partition function :

$$Z = \int \mathcal{D}\psi(t')\mathcal{D}\hat{\psi}(t')\hat{g}\left(\hat{\psi}(t_f)\right)\hat{T}\{e^{-S[\psi,\hat{\psi}]}\}g\left(\psi(t_{in})\right)$$

$\psi(t'),\ \hat{\psi}(t'),\ t_{in} \le t' \le t_f$

$$\int \mathcal{D}\psi(t')\mathcal{D}\hat{\psi}(t') = \prod_{t'} \int d\psi(t')d\hat{\psi}(t')$$

Grassmann wave function

$$S = S_< + S_> - \hat{\psi}(t)\psi(t)$$

$$S_< = \sum_{t'<t} L(t'),$$

$$S_> = \sum_{t'\geq t} L(t') + \hat{\psi}(t)\psi(t)$$

$$g\big(\psi(t)\big) = Z_<^{-1} \int \mathcal{D}\psi(t' < t)\mathcal{D}\hat{\psi}(t' < t)\hat{T}\{e^{-S_<}\}g_{in},$$

$$\hat{g}\big(\hat{\psi}(t)\big) = Z_>^{-1} \int \mathcal{D}\psi(t' > t)\mathcal{D}\hat{\psi}(t' > t)\hat{g}_f\hat{T}\{e^{-S_>}\}$$

$$g(t) = \int_{t'<t} \mathcal{D}\psi\mathcal{D}\hat{\psi}e^{-S_<}g_{in},$$

$$\hat{g}(t) = \int_{t'>t} \mathcal{D}\psi\mathcal{D}\hat{\psi}\hat{g}_f e^{-S_>},$$

observables

$$\langle A \rangle = \sum_\tau p_\tau A_\tau$$

$$(\hat{A}q)_\tau = \sum_\rho A_{\tau\rho} q_\rho , \qquad A_{\tau\rho} = A_\tau \delta_{\tau\rho}.$$

$$\langle A \rangle = \langle q \hat{A} q \rangle = \sum_{\tau,\rho} q_\tau A_{\tau\rho} q_\rho$$
$$= \sum_\tau q_\tau^2 A_\tau = \sum_\tau p_\tau A_\tau.$$

representation as functional integral

$$\langle A \rangle = \int \mathcal{D}\psi \mathcal{D}\hat{\psi} A[\hat{\psi}, \psi] G[\psi, \hat{\psi}]$$

particle numbers

$$\langle N(t)\rangle = Z^{-1}\int \mathcal{D}\psi(t')\mathcal{D}\hat{\psi}(t')\hat{g}_f N(t) \times \hat{T}\{\exp\left(-S[\psi(t'),\hat{\psi}(t')]\right)\}g_{in},$$

$$N(t) = \hat{\psi}(t)\psi(t)$$

$$\langle N(t)\rangle = \int d\psi(t)d\hat{\psi}(t)N(t)e^{\hat{\psi}(t)\psi(t)}\hat{g}\left(\hat{\psi}(t)\right)g\left(\psi(t)\right)$$

$$\langle N(t)\rangle = \int D\psi\hat{g}(t)N(t)g(t)$$

time evolution

$$\partial_t g(t) = -i\mathcal{H}\left[\frac{\partial}{\partial\psi}, \psi\right] g(t)$$

d=2 quantum field theory

$$S = \sum_{t,x} \{ \hat{\psi}_+(t,x) \left(\psi_+(t+\epsilon, x-\epsilon) - \psi_+(t,x) \right)$$
$$+ \hat{\psi}_-(t, x-\epsilon) \left(\psi_-(t+\epsilon, x) - \psi_-(t, x-\epsilon) \right) \}$$

$$S = \int_{t,x} \{ \hat{\psi}_+ \partial_t \psi_+ + \hat{\psi}_- \partial_t \psi_- - \hat{\psi}_+ \partial_x \psi_+ + \hat{\psi}_- \partial_x \psi_- \}$$

$$= \int_{t,x} \psi^\dagger \partial_t \psi + i \int_t H,$$

$$H = i \int_x \{ \hat{\psi}_+ \partial_x \psi_+ - \hat{\psi}_- \partial_x \psi_- \} = i \int_x \psi^\dagger \tau_3 \partial_x \psi$$

time evolution of Grassmann wave function

$$\partial_t g = -i\mathcal{H}g,$$
$$\mathcal{H} = i\int_x \{\frac{\partial}{\partial\psi_+}\partial_x\psi_+ - \frac{\partial}{\partial\psi_-}\partial_x\psi_-\}$$

Lorentz invariance

$$\bar{\psi} = (-\hat{\psi}_-, \hat{\psi}_+) = \psi^\dagger \gamma^0$$

$$\gamma^0 = i\tau_2, \qquad \gamma_1 = \tau_1$$

$$S = -\int_{t,x} \bar{\psi}\gamma^\mu \partial_\mu \psi.$$

what is an atom ?

- quantum mechanics : isolated object
- quantum field theory : excitation of complicated vacuum
- classical statistics : sub-system of ensemble with infinitely many degrees of freedom

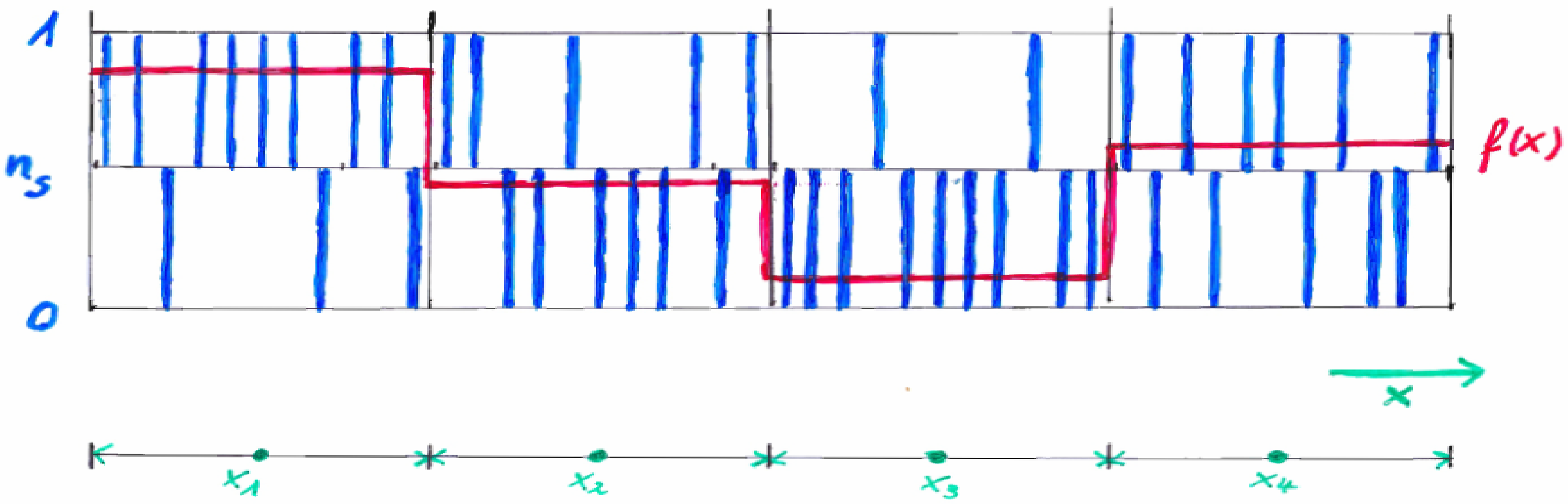
one - particle wave function
from coarse graining
of microphysical
classical statistical ensemble

non – commutativity in classical statistics

microphysical ensemble

- states τ
- labeled by sequences of occupation numbers or bits $n_s = 0$ or 1
- $\tau = [\, n_s \,] = [0,0,1,0,1,1,0,1,0,1,1,1,1,0,\ldots]$ etc.
- probabilities $p_\tau > 0$

function observable



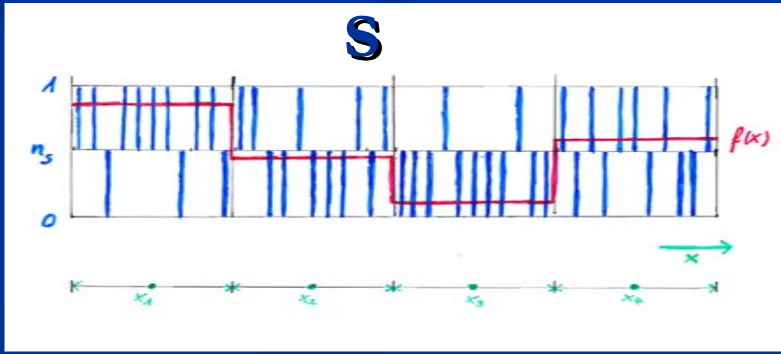
function observable

$$f_\mathcal{T}(x_i) = \mathcal{N}^{-\frac{1}{2}} \sum_{s \in I(x_i)} (2n_s - 1)$$

normalized difference between occupied and empty bits in interval

$$\int dx\, f_\mathcal{T}^2(x) = \sum_i f_\mathcal{T}^2(x_i) = 1$$

$$\mathcal{N} = \sum_{x_i} \Big(\sum_{s \in I(x_i)} (2n_s - 1) \Big)^2$$



$I(x_1)$ $I(x_2)$ $I(x_3)$ $I(x_4)$

generalized function observable

normalization

$$\int dx\, f_\tau^2(x) = 1$$

classical expectation value

$$\langle f(x) \rangle = \sum_\tau p_\tau f_\tau(x)$$

several species α

$$\sum_\alpha \int dx\, f_{\alpha,\tau}^2(x) = 1$$

position

$$X_\tau = \int dx\, x\, f_\tau^2(x)$$

classical observable :

fixed value for every state τ

momentum

- derivative observable

$$P_\tau = \int dx [f_{1,\tau}(x) \partial_x f_{2,\tau}(x) - f_{2,\tau}(x) \partial_x f_{1,\tau}(x)]$$

classical observable :
fixed value for every state τ

complex structure

$$f_\tau(x) = f_{1,\tau}(x) + i f_{2,\tau}(x)$$

$$\int dx\, f_\tau^*(x) f_\tau(x) = 1$$

$$P_\tau = \int dx\, f_\tau^*(x)(-i\partial_x) f_\tau(x)$$

$$X_\tau = \int dx\, f_\tau^*(x) x f_\tau(x)$$

$$P_\tau = \int dx [f_{1,\tau}(x)\partial_x f_{2,\tau}(x) - f_{2,\tau}(x)\partial_x f_{1,\tau}(x)]$$

classical product of position and momentum observables

$$\langle X \cdot P \rangle_{cl} = \langle P \cdot X \rangle_{cl} = \sum_{\tau} p_{\tau} X_{\tau} P_{\tau}$$

commutes !

different products of observables

$$(X^2)_\tau = \int dx f_\tau^*(x) x^2 f_\tau(x)$$

$$\langle X^2 \rangle = \sum_\tau p_\tau (X^2)_\tau$$

differs from classical product

$$\begin{aligned}\langle X \cdot X \rangle &= \sum_\tau p_\tau X_\tau^2 \\ &= \sum_\tau p_\tau \left(\int dx f_\tau^*(x) x f_\tau(x) \right)^2\end{aligned}$$

Which product describes correlations of measurements ?

coarse graining of information for subsystems

density matrix from coarse graining

- position and momentum observables use only small part of the information contained in p_τ ,

- relevant part can be described by density matrix

$$\rho(x,x') = \sum_\tau p_\tau f_\tau(x) f_\tau^*(x')$$

- subsystem described only by information which is contained in density matrix
- coarse graining of information

quantum density matrix

density matrix has the properties of a quantum density matrix

$$\mathrm{Tr}\rho = \int dx \rho(x,x) = 1\,,\ \rho^*(x,x') = \rho(x',x)$$

$$\rho(x,x') = \sum_{\tau} p_\tau f_\tau(x) f_\tau^*(x')$$

quantum operators

$$\hat{X}(x', x) = \delta(x' - x)x$$
$$\hat{P}(x', x) = -i\delta(x' - x)\frac{\partial}{\partial x}$$

$$\langle X \rangle = \sum_\tau p_\tau X_\tau = \mathrm{Tr}(\hat{X}\rho) = \int dx x \rho(x, x)$$

$$\langle P \rangle = \sum_\tau p_\tau P_\tau = \mathrm{Tr}(\hat{P}\rho)$$
$$= -i \int dx' dx \delta(x' - x) \partial_x \rho(x, x')$$

quantum product of observables

the product

$$(X^2)_\tau = \int dx f_\tau^*(x) x^2 f_\tau(x)$$

$$\langle X^2 \rangle = \sum_\tau p_\tau (X^2)_\tau$$

is compatible with the coarse graining

$$\langle X^2 \rangle = \int dx x^2 \rho(x, x)$$

and can be represented by operator product

incomplete statistics

classical product

$$\langle X \cdot X \rangle = \sum_\tau p_\tau X_\tau^2$$
$$= \sum_\tau p_\tau \left(\int dx \, f_\tau^*(x) x f_\tau(x) \right)^2$$

- is not computable from information which is available for subsystem !
- cannot be used for measurements in the subsystem !

classical and quantum dispersion

$$\Delta_x^2 = \langle X^2 \rangle - \langle X \rangle^2 \,,\; (\Delta_x^{(cl)})^2 = \langle X \cdot X \rangle - \langle X \rangle^2$$

$$\Delta_x^2 - (\Delta_x^{(cl)})^2 = \sum_\tau p_\tau \int dx\, f_\tau^*(x)(x - X_\tau)^2 f_\tau(x) \geq 0$$

$$\begin{aligned} \langle X \cdot X \rangle &= \sum_\tau p_\tau X_\tau^2 \\ &= \sum_\tau p_\tau \left(\int dx\, f_\tau^*(x) x f_\tau(x) \right)^2 \end{aligned}$$

$$(X^2)_\tau = \int dx\, f_\tau^*(x) x^2 f_\tau(x)$$

$$\langle X^2 \rangle = \sum_\tau p_\tau (X^2)_\tau$$

subsystem probabilities

$$w(x) = \rho(x,x) = \sum_\tau p_\tau |f_\tau(x)|^2$$

$$w(x) \geq 0 \,, \quad \int dx w(x) = 1$$

$$\langle X^n \rangle = \int dx x^n w(x)$$

in contrast :

$$\langle X \cdot X \rangle = \int dx dy \; xy \; w_{cl}(x,y)$$

$$w_{cl}(x,y) = \sum_\tau p_\tau |f_\tau|^2(x) |f_\tau|^2(y)$$

squared momentum

$$(P^2)_\tau = \int dx f_\tau^*(x)(-\partial_x^2) f_\tau(x) = \int dx |\partial_x f_\tau(x)|^2$$

$$\langle P^2 \rangle = \sum_\tau p_\tau (P^2)_\tau = \mathrm{tr}(\hat{P}^2 \rho) = \int dx dx' \delta(x'-x)(-\partial_x^2)\rho(x,x')$$

$$\langle P \cdot P \rangle = \sum_\tau p_\tau P_\tau^2 = -\sum_\tau p_\tau \left(\int dx f_\tau^*(x) \partial_x f_\tau(x) \right)^2$$

quantum product between classical observables : maps to product of quantum operators

non – commutativity in classical statistics

$$(XP)_\tau = \int dx f_\tau^*(x) x(-i\partial_x) f_\tau(x)$$

$$(PX)_\tau = \int dx f_\tau^*(x)(-i\partial_x) x f_\tau(x)$$

$$\langle XP \rangle = \mathrm{tr}(\hat{X}\hat{P}\rho) \ , \ \langle PX \rangle = \mathrm{tr}(\hat{P}\hat{X}\rho)$$

$$XP - PX = i$$

commutator depends on choice of product !

measurement correlation

- correlation between measurements of positon and momentum is given by quantum product
- this correlation is compatible with information contained in subsystem

$$\langle XP \rangle_m = \frac{1}{2}(\langle XP \rangle + \langle PX \rangle)$$

coarse graining

from fundamental fermions $p([n_s])$

at the Planck scale

to atoms at the Bohr scale $\rho(x\ ,\ x')$

conclusion

- quantum statistics emerges from classical statistics

  quantum state, superposition, interference, entanglement, probability amplitude
- unitary time evolution of quantum mechanics can be described by suitable time evolution of classical probabilities
- conditional correlations for measurements both in quantum and classical statistics

end