Quantum Mechanics
from Classical Statistics
what is an atom?

- quantum mechanics: isolated object
- quantum field theory: excitation of complicated vacuum
- classical statistics: sub-system of ensemble with infinitely many degrees of freedom
quantum mechanics can be described by classical statistics!
quantum mechanics from classical statistics

- probability amplitude
- entanglement
- interference
- superposition of states
- fermions and bosons
- unitary time evolution
- transition amplitude
- non-commuting operators
- violation of Bell’s inequalities
statistical picture of the world

- basic theory is not deterministic
- basic theory makes only statements about probabilities for sequences of events and establishes correlations
- probabilism is fundamental, not determinism!

quantum mechanics from classical statistics:
not a deterministic hidden variable theory
essence of quantum mechanics

description of appropriate subsystems of classical statistical ensembles

1) equivalence classes of probabilistic observables
2) incomplete statistics
3) correlations between measurements based on conditional probabilities
4) unitary time evolution for isolated subsystems
classical statistical implementation of quantum computer
Classical ensemble, discrete observable

- Classical ensemble with probabilities $\hat{p}_\tau$

$$\hat{p}_\tau \geq 0, \quad \sum_\tau \hat{p}_\tau = 1$$

- qubit:

  one discrete observable $A$, values +1 or -1

  probabilities to find $A=1$: $w_+$ and $A=-1$: $w_-$

$$\langle A \rangle = w_+ - w_-$$
classical ensemble for one qubit

- classical states labeled by
  \[ (\sigma_1, \sigma_2, \sigma_3) \quad \sigma_j = \pm 1 \]

- state of subsystem depends on three numbers
  \[ \rho_j = \sum_{\sigma_1,\sigma_2,\sigma_3} \sigma_j p(\sigma_1, \sigma_2, \sigma_3) \]

- expectation value of qubit
  \[ \langle A \rangle = \rho_3, \quad w_+ = \frac{1}{2} (1 + \rho_3) \]
classical probability distribution

\[ p(\sigma_1, \sigma_2, \sigma_3) = p_s(\sigma_1, \sigma_2, \sigma_3) + \delta p_e(\sigma_1, \sigma_2, \sigma_3) \]

\[ p_s(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{8}(1 + \sigma_1 \rho_1)(1 + \sigma_2 \rho_2)(1 + \sigma_3 \rho_3) \]

characterizes subsystem

\[ \sum_{\sigma_1, \sigma_2, \sigma_3} \delta p_e(\sigma_1, \sigma_2, \sigma_3) = 0, \quad \sum_{\sigma_1, \sigma_2, \sigma_3} \sigma_j \delta p_e(\sigma_1, \sigma_2, \sigma_3) = 0 \]

different \( \delta p_e \) characterize environment
state of system independent of environment

- \( q_j \) does not depend on precise choice of \( \delta p_e \)

\[
\rho_j = \sum_{\sigma_1, \sigma_2, \sigma_3} \sigma_j p(\sigma_1, \sigma_2, \sigma_3)
\]

\[
\sum_{\sigma_1, \sigma_2, \sigma_3} \delta p_e(\sigma_1, \sigma_2, \sigma_3) = 0, \\
\sum_{\sigma_1, \sigma_2, \sigma_3} \sigma_j \delta p_e(\sigma_1, \sigma_2, \sigma_3) = 0
\]
time evolution

rotations of $q_k$

example:

$$\rho_k(t, t') = \hat{S}_{kl}(t, t') \rho_l(t')$$

$$\hat{S}\hat{S}^T = 1$$

$$\frac{\partial}{\partial t} \rho_k = T_{kl} \rho_l$$

$$\begin{pmatrix} \cos^2 \varphi & \sqrt{2} \sin \varphi \cos \varphi & \sin^2 \varphi \\ -\sqrt{2} \sin \varphi \cos \varphi & 1 - 2 \sin^2 \varphi & \sqrt{2} \sin \varphi \cos \varphi \\ \sin^2 \varphi & -\sqrt{2} \sin \varphi \cos \varphi & \cos^2 \varphi \end{pmatrix}$$
time evolution of classical probability

- evolution of $p_s$ according to evolution of $q_k$

- evolution of $\delta p_e$ arbitrary, consistent with constraints
state after finite rotation

\[ \varphi(t = \Delta) = \frac{\pi}{2} \]

\[
\hat{S} = \\
\begin{pmatrix}
\cos^2 \varphi & \sqrt{2} \sin \varphi \cos \varphi & -\sqrt{2} \sin^2 \varphi \\
-\sqrt{2} \sin \varphi \cos \varphi & 1 - 2 \sin^2 \varphi & \sqrt{2} \sin \varphi \cos \varphi \\
\sin^2 \varphi & -\sqrt{2} \sin \varphi \cos \varphi & \cos^2 \varphi \\
\end{pmatrix}
\]

\[ \rho_3(t) = \rho_{1,0}, \ \rho_1(t) = \rho_{3,0}, \ \rho_2(t) = -\rho_{2,0} \]

\[
p_s(\sigma_1, \sigma_2, \sigma_3; t) = p_s(\sigma_3, \sigma_2, \sigma_1; 0), \\
p_s(\sigma_1, \sigma_2, \sigma_3; t) = p_s(\sigma_3, -\sigma_2, \sigma_1; 0)
\]
this realizes Hadamard gate
purity

\[ P = \rho_k \rho_k \]

consider ensembles with \( P \leq 1 \)

purity conserved by time evolution
density matrix

- define hermitean 2x2 matrix:

\[ \rho = \frac{1}{2} (1 + \rho_k \tau_k) \]

- properties of density matrix

\[ tr \rho = 1 \quad \rho_{\alpha \alpha} \geq 1 \quad tr \rho^2 \leq 1 \]
operators

If observable $A(e_k)$ obeys

$$\langle A(e_k) \rangle = : \rho_k e_k$$

associate hermitean operators

$$\hat{A}(e_k) = e_k \tau_k$$

$$\langle A(e_k) \rangle = tr(\hat{A}(e_k) \rho)$$

$$= \frac{1}{2} \rho_k e_\ell \{ \tau_k, \tau_\ell \} = \rho_k e_k$$

In our case: $e_3 = 1$, $e_1 = e_2 = 0$
quantum law for expectation values

\[ \langle A \rangle = tr(\hat{A}\rho) \]
pure state

\[ P = 1 \quad \rightarrow \quad \rho^2 = \rho \]

wave function

unitary time evolution

\[ \rho_{\alpha\beta} = \psi_{\alpha} \psi_{\beta}^*, \quad \psi_{\alpha}^* \psi_{\alpha} = 1 \]

\[ \langle A \rangle = \psi_{\alpha}^*(\tau_3)_{\alpha\beta} \psi_{\beta} = \langle \psi | \hat{A} | \psi \rangle \]

\[ \psi_{\alpha}(t) = U_{\alpha\beta}(t) \psi_{\beta}(0) \]
Hadamard gate

\[ \rho_3(t) = \rho_{1,0}, \quad \rho_1(t) = \rho_{3,0}, \quad \rho_2(t) = -\rho_{2,0} \]

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]
CNOT gate

\[ U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \]
Four state quantum system
- two qubits -

\[ k=1, \ldots, 15 \quad P \leq 3 \]

\[
\rho = \frac{1}{4} (1 + \rho_k L_k) \quad , \quad \text{tr}(L_k L_l) = 4 \delta_{kl}
\]

normalized SU(4) – generators:

\[
\begin{align*}
L_1 &= \tau_3 \otimes 1 ,
L_2 &= 1 \otimes \tau_3 ,
L_3 &= \tau_3 \otimes \tau_3 ,
L_4 &= 1 \otimes \tau_1 ,
L_5 &= 1 \otimes \tau_2 ,
L_6 &= \tau_3 \otimes \tau_1 ,
L_7 &= \tau_3 \otimes \tau_2 ,
L_8 &= \tau_1 \otimes 1 ,
L_9 &= \tau_2 \otimes 1 ,
L_{10} &= \tau_1 \otimes \tau_3 ,
L_{11} &= \tau_2 \otimes \tau_3 ,
L_{12} &= \tau_1 \otimes \tau_1 ,
L_{13} &= \tau_1 \otimes \tau_2 ,
L_{14} &= -\tau_2 \otimes \tau_2 ,
L_{15} &= \tau_2 \otimes \tau_1
\end{align*}
\]
four - state quantum system

pure state: \( P = 3 \) and

\[ c = \text{tr}[(\rho^2 - \rho)] \]

\[ \rho_{\alpha\beta} = |\psi_{\alpha}\rangle \langle \psi_{\beta}|, \langle \psi_{\alpha} = U_{\alpha\beta} |\phi_{m}\rangle_\beta \]

\[ A = e_i L_k, \langle A \rangle = \rho_{kk} = \text{tr}(\rho A) \]

\[ P = \rho_{kk} \]

copurity

\[ \rho = \frac{1}{4}, \langle \phi \rangle = \psi \hat{A} \psi \]

must vanish

\[ P \leq 3 \]
suitable rotation of $Q_k$

\[
\begin{align*}
\rho_2 & \leftrightarrow \rho_3, \quad \rho_5 \leftrightarrow \rho_7, \quad \rho_8 \leftrightarrow \rho_{12}, \\
\rho_9 & \leftrightarrow \rho_{15}, \quad \rho_{10} \leftrightarrow \rho_{14}, \quad \rho_{11} \leftrightarrow \rho_{13}
\end{align*}
\]

yields transformation of the density matrix

\[
\begin{align*}
\rho_{13} & \leftrightarrow \rho_{14}, \quad \rho_{23} \leftrightarrow \rho_{24}, \quad \rho_{31} \leftrightarrow \rho_{41}, \quad \rho_{32} \leftrightarrow \rho_{42}, \\
\rho_{33} & \leftrightarrow \rho_{44}, \quad \rho_{34} \leftrightarrow \rho_{43}
\end{align*}
\]

and realizes CNOT gate

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
classical probability distribution for $2^{15}$ classical states

\[
p_s(\{\sigma_k\}) = 2^{-15} \prod_k (1 + \sigma_k \rho_k)
\]

\[
\sum_{\{\sigma_k\}} \delta p_e(\{\sigma_k\}) = 0, \quad \sum_{\{\sigma_k\}} \sigma_j \delta p_e(\{\sigma_k\}) = 0
\]

\[
\rho_j = \sum_{\{\sigma_k\}} \sigma_j p(\{\sigma_k\})
\]
probabilistic observables

for a given state of the subsystem, specified by \( \{ \rho_k \} \):

The possible measurement values +1 and -1 of the discrete two-level observables are found with probabilities \( w_+(\rho_k) \) and \( w_- (\rho_k) \).

In a quantum state the observables have a probabilistic distribution of values, rather than a fixed value as for classical states.
probabilistic quantum observable

spectrum \{ \gamma_\alpha \}

probability that \gamma_\alpha is measured: w_\alpha

can be computed from state of subsystem

\[ \langle A \rangle = \sum_\alpha w_\alpha(\rho_k) \gamma_\alpha \]

\[ w_\alpha(\rho_k) = \rho'_{\alpha\alpha} = (U_A \rho U_A^\dagger)_{\alpha\alpha} \]
non – commuting quantum operators

for two qubits:
- all $L_k$ represent two – level observables
- they do not commute

$$\langle A \rangle = tr(\hat{A}\rho)$$

- the laws of quantum mechanics for expectation values are realized
- uncertainty relation etc.
incomplete statistics

joint probabilities depend on environment and are not available for subsystem!

\[ C_{12} = \sum_{\{\sigma_k\}} \sigma_1 \sigma_2 p(\{\sigma_k\}) = p_{++} + p_{--} - p_{+-} - p_{-+} \]

\[ C_{ij} = \sum_{\{\sigma_k\}} \sigma_i \sigma_j p(\{\sigma_k\}) \]

\[ p = p_s + \delta p_e \]

\[ p_s(\{\sigma_k\}) = 2^{-15} \prod_{k} (1 + \sigma_k \rho_k) \]

\[ \sum_{\{\sigma_k\}} \delta p_e(\{\sigma_k\}) = 0, \quad \sum_{\{\sigma_k\}} \sigma_j \delta p_e(\{\sigma_k\}) = 0 \]
quantum mechanics from classical statistics

- probability amplitude ☻
- entanglement
- interference
- superposition of states
- fermions and bosons
- unitary time evolution ☻
- transition amplitude
- non-commuting operators ☻
- violation of Bell’s inequalities
conditional correlations
classical correlation

- point wise multiplication of classical observables on the level of classical states
- classical correlation depends on probability distribution for the atom and its environment
- not available on level of probabilistic observables
- definition depends on details of classical observables, while many different classical observables correspond to the same probabilistic observable

needed: correlation that can be formulated in terms of probabilistic observables and density matrix!
conditional probability

\[ w_{+, \alpha}^{AB} = (w_+^A)^B w_+^{B, \alpha} + (w_-^A)^B w_-^{B, \alpha} \]
\[ w_{-, \alpha}^{AB} = (w_+^A)^B w_-^{B, \alpha} + (w_-^A)^B w_+^{B, \alpha} \]

probability to find value +1 for product of measurements of A and B

... can be expressed in terms of expectation value of A in eigenstate of B

\[ (w_+^A)^B = \frac{1}{2} (1 \pm \langle A \rangle_{+B}) \]
\[ (w_-^A)^B = \frac{1}{2} (1 \pm \langle A \rangle_{-B}) \]
measurement correlation

After measurement $A=+1$ the system must be in eigenstate with this eigenvalue. Otherwise repetition of measurement could give a different result!

$$\langle BA \rangle_m = (w^B_+)_+ w^A_+,s - (w^B_-)_+ w^A_+,s$$
$$- (w^B_+_+ w^A_-)_+,s + (w^B_-)_+ w^A_-)_+,s$$

$$\rho_{A+}$$

$$(w^B_+)_+ - (w^B_-)_+ = \text{tr}(\hat{B}\rho_{A+})$$
measurement changes state in all statistical systems!

quantum and classical eliminates possibilities that are not realized
physics makes statements about possible sequences of events and their probabilities
unique eigenstates for $M=2$

$M = 2$: 

$$\rho_{A+} = \frac{1}{2} (1 + \hat{A})$$

$$(w^B)^A_+ = \frac{1}{2} \pm \frac{1}{4} \text{tr}(\hat{B} \hat{A}) , \quad (w^B)^A_- = \frac{1}{2} \mp \frac{1}{4} \text{tr}(\hat{B} \hat{A})$$
eigenstates with $A = 1$

$$\rho_{A+} = \frac{1}{M}(1 + \hat{A} + X), \text{ tr}(\hat{A}X) = 0, \text{ tr}X = 0$$

$$P = M\text{ tr}(\rho_{A+}^2) = 1 + \frac{1}{M}\text{ tr}X^2$$

$$\rho_{A+}^2 - \rho_{A+} = \frac{1}{M^2}(X^2 + \{\hat{A},X\}) - \left(1 - \frac{2}{M}\right)\rho_{A+}$$

measurement preserves pure states if projection

$$\rho_{A+} = \frac{1}{2(1 + \langle A \rangle)}(1 + \hat{A})\rho(1 + \hat{A})$$
measurement correlation equals quantum correlation

\[
\langle BA \rangle_m = \frac{1}{2} \text{tr}(\{\hat{A}, \hat{B}\} \rho)
\]

probability to measure \(A=1\) and \(B=1\):

\[
w_{++} = \frac{1}{4} \left( 1 + \langle A \rangle + \langle B \rangle + \langle AB \rangle_m \right)
\]

\[
w_{++} = \frac{1}{4} \left( 1 + e_k^{(A)} e_k^{(B)} + \rho_k [e_k^{(A)} + e_k^{(B)} + d_{mlk} e_m^{(A)} e_l^{(B)}] \right)
\]
probability that A and B have both the value +1 in classical ensemble

\[ p_{++} = \frac{1}{4} (1 + \langle A \rangle + \langle B \rangle + \langle A \cdot B \rangle) \]

\[ \langle A \cdot B \rangle = \sum_{\tau} p_{\tau} A_{\tau} B_{\tau} \]

not a property of the subsystem

probability to measure A and B both +1

\[ w_{++} = \frac{1}{4} (1 + \langle A \rangle + \langle B \rangle + \langle AB \rangle_m) \]

\[ w_{++} = \frac{1}{4} \left(1 + e_k^{(A)} e_k^{(B)} + \rho_k [e_k^{(A)} + e_k^{(B)} + d_{mlk} e_m^{(A)} e_l^{(B)}] \right) \]

can be computed from the subsystem
sequence of three measurements and quantum commutator

\[
\begin{align*}
\langle ABC \rangle_m - \langle ACB \rangle_m &= \frac{1}{4} \text{tr} \left( [\hat{A}, [\hat{B}, \hat{C}]] \rho \right), \\
\langle ABC \rangle_m - \langle CBA \rangle_m &= \frac{1}{4} \text{tr} \left( [\hat{B}, [\hat{A}, \hat{C}]] \rho \right), \\
\langle ABC \rangle_m - \langle BAC \rangle_m &= 0
\end{align*}
\]

two measurements commute, not three
conclusion

- quantum statistics arises from classical statistics
  states, superposition, interference, entanglement, probability amplitudes
- quantum evolution embedded in classical evolution
- conditional correlations describe measurements both in quantum theory and classical statistics
quantum particle from classical statistics

- quantum and classical particles can be described within the same classical statistical setting
- different time evolution, corresponding to different Hamiltonians
- continuous interpolation between quantum and classical particle possible!
time evolution
transition probability

time evolution of probabilities

$$\partial_t p_\sigma = F_\sigma(p_{\sigma'})$$

(fixed observables)

induces transition probability matrix

$$p_\sigma(t) = \tilde{S}_{\sigma\tau}(t, t') p_{\tau}(t')$$
reduced transition probability

- induced evolution

\[ \partial_t \rho_k = \sum_{\sigma} \partial_t p_{\sigma} \bar{A}_{\sigma}^{(k)} = \sum_{\sigma} F_{\sigma}(p_{\sigma'}) \bar{A}_{\sigma}^{(k)} \]

- reduced transition probability matrix

\[ \rho_k(t) = S_{k\ell}(t, t') \rho_\ell(t') \]

\[ S_{k\ell}(t, t') = \sum_{\sigma\tau\rho} \tilde{S}_{\sigma\tau}(t, t') p_{\tau}(t') p_{\rho}(t') \bar{A}_{\sigma}^{(k)} \bar{A}_{\rho}^{(\ell)} \rho_m(t') \rho_m(t') \]
evolution of elements of density matrix in two-state quantum system

- infinitesimal time variation

\[ \partial_t \rho_k(t) = \partial_t S_{k\ell}(t, t') S_{\ell m}^{-1}(t, t') \rho_m(t) \]

- scaling + rotation

\[ S_{k\ell} = \hat{S}_{k\ell} d \quad \hat{S}_{k\ell}^{-1} = \hat{S}_{\ell k} \]

\[ \partial_t SS^{-1} = \partial_t \hat{S} \hat{S}^T + \partial_t \ln d \]
**time evolution of density matrix**

- **Hamilton operator and scaling factor**
  \[
  \hat{H} = -\frac{1}{4} (\partial_t \hat{S} \hat{S}^T)_{\ell m} \varepsilon_{\ell m k} \tau_k
  \]
  \[
  \lambda = \partial_t \ln d
  \]

- **Quantum evolution and the rest?**
  \[
  \partial_t \rho = -i[\hat{H}, \rho] + \lambda (\rho - \frac{1}{2})
  \]

\[\lambda = 0 \text{ and pure state: } i\partial_t \psi = \hat{H}\psi\]
It is easy to construct explicit ensembles where

\[ \lambda = 0 \]
evolution of purity

change of purity

\[ \partial_t P = \partial_t (\rho_k \rho_k) = \partial_t (2tr \rho^2 - 1) \]
\[ \partial_t P = 2\lambda P \]

attraction to randomness: decoherence

\[ \lambda < 0 : \quad P \to 0 \]

attraction to purity: syncoherence

\[ \lambda > 0 : \quad P \to 1 \]

\[ P = \rho_k \rho_k \]
classical statistics can describe decoherence and syncoherence! 

unitary quantum evolution : special case
pure state fixed point

pure states are special:

“no state can be purer than pure“

fixed point of evolution for

\[ P = 1, \quad \lambda = 0 \]

approach to fixed point

\[ \partial_t \lambda = \beta_\lambda(\lambda, P, \rho_k/\sqrt{P}, \ldots) \]

\[ \beta_\lambda = -a\lambda + b(1 - P) \]
approach to pure state fixed point

solution:

\[ 1 - P = x_1 e^{-\varepsilon_1 t} + x_2 e^{-\varepsilon_2 t} \]

\[ \lambda = \varepsilon_1 x_1 e^{-\varepsilon_1 t} + \varepsilon_2 x_2 e^{-\varepsilon_2 t} \]

\[ \varepsilon_{1,2} = \frac{1}{2} \left( a \pm \sqrt{a^2 - 4b} \right) \]

syncoherence describes exponential approach to pure state if

\[ a > 0 , \quad a < b < \frac{1}{4} a^2 \]

decay of mixed atom state to ground state
purity conserving evolution:
subsystem is well isolated
two bit system and entanglement ensembles with \( P=3 \)
non-commuting operators

15 spin observables labeled by \( e_k \), \( k = 1 \ldots 15 \)

\[
\rho_k = \sum_\sigma p_\sigma \overline{A}_\sigma^{(k)} , \quad \langle A(e_k) \rangle = \sum_k \rho_k e_k , \quad -1 \leq \rho_k \leq 1
\]

density matrix

\[
\rho = \frac{1}{4} (1 + \rho_k L_k)
\]

\[
L_k^2 = 1 \quad , \quad tr L_k = 0 \quad , \quad tr (L_k L_\ell) = 4 \delta_{k\ell}
\]
\[ L_k^2 = 1 \quad \text{tr} L_k = 0 \quad \text{tr}(L_k L_l) = 4 \delta_{kl} \]

\[ L_1 = \text{diag}(1, 1, -1, -1) \quad L_2 = \text{diag}(1, -1, 1, -1) \]

\[ L_3 = \text{diag}(1, -1, -1, 1) \]

\[ L_4 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix} \quad L_5 = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix} \]

\[ L_6 = \begin{pmatrix} \tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix} \quad L_7 = \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} \]
density matrix

- pure states: $P=3$

$$
tr \rho^2 = \frac{1}{4} (1 + \rho_k \rho_k) = \frac{1}{4} (1 + P)
$$

$$
P \leq 3 : 
tr \rho^2 \leq 1
$$

$$
\hat{A}(e_k) = e_k L_k, \quad e_k e_k = 1 \quad \text{for} \quad \hat{A}^2(e_k) = 1
$$
entanglement

- three commuting observables

\[ L_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]

\( L_1 : \text{bit 1} \), \( L_2 : \text{bit 2} \) \( L_3 : \text{product of two bits} \)

- expectation values of associated observables related to probabilities to measure the combinations \((+++)\), etc.

\[
\begin{align*}
\langle T_1 \rangle &= W_{++} + W_{+-} - W_{-+} - W_{--} \\
\langle T_2 \rangle &= W_{++} - W_{+-} + W_{-+} - W_{--} \\
\langle T_3 \rangle &= W_{++} - W_{+-} - W_{-+} + W_{--}
\end{align*}
\]
“classical” entangled state

- pure state with maximal anti-correlation of two bits

\[
W_{++} = W_{--} = 0 \quad , \quad W_{+-} = W_{-+} = \frac{1}{2}
\]

- bit 1: random, bit 2: random
- if bit 1 = 1 necessarily bit 2 = -1, and vice versa

\[
\langle L_1 \rangle = \langle L_2 \rangle = 0 \quad , \quad \langle L_3 \rangle = -1
\]
classical state described by entangled density matrix

\[ \rho = \frac{1}{2} \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 1, & \pm 1, & 0 \\ 0, & \pm 1, & 1, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix} , \quad tr\rho^2 = 1 \]

\[ \rho = \frac{1}{4} (1 - L_3 \pm (L_{12} - L_{14})) \]

\[ \rho_1 = \rho_2 = 0 \quad \Rightarrow \quad \langle T_1 \rangle = \langle T_2 \rangle = 0 \]

\[ \rho_3 = -1 \quad \Rightarrow \quad \langle T_3 \rangle = -1 \]
entangled quantum state

\[ \psi_{\pm} = \frac{1}{\sqrt{2}} (\psi_2 \pm \psi_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm 1 \\ 0 \end{pmatrix} \]
end
pure state density matrix

- elements $q_k$ are vectors on unit sphere
- can be obtained by unitary transformations
- $SO(3)$ equivalent to $SU(2)$

\[
\rho = U \hat{\rho}_1 U^\dagger, \quad UU^\dagger = U^\dagger U = 1
\]

\[
\hat{\rho}_1 = \begin{pmatrix} 1 & \ 0 \\ 0 & 0 \end{pmatrix}
\]
wave function

“root of pure state density matrix “

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi = U \psi_1 \quad \rho_{\alpha \beta} = \psi_\alpha \psi_\beta^* \\
\]

\[
tr(\hat{A} \rho) = \hat{A}_{\alpha \beta} \rho_{\beta \alpha} = \hat{A}_{\alpha \beta} \psi_\beta \psi_\alpha^* \\
\]

quantum law for expectation values

\[
\langle A \rangle = \psi^\dagger \hat{A} \psi \\
\]