Quantum Mechanics from Classical Statistics

what is an atom ?

- quantum mechanics : isolated object
- quantum field theory : excitation of complicated vacuum
- classical statistics : sub-system of ensemble with infinitely many degrees of freedom

quantum mechanics can be described by classical statistics !

quantum mechanics from classical statistics

- probability amplitude
- entanglement
- interference
- superposition of states
- fermions and bosons
- unitary time evolution
- transition amplitude
- non-commuting operators
- violation of Bell's inequalities

statistical picture of the world

- basic theory is not deterministic
- basic theory makes only statements about probabilities for sequences of events and establishes correlations
- probabilism is fundamental , not determinism !

quantum mechanics from classical statistics :
not a deterministic hidden variable theory

essence of quantum mechanics

description of appropriate subsystems of classical statistical ensembles

1) equivalence classes of probabilistic observables
2) incomplete statistics
3) correlations between measurements based on conditional probabilities
4) unitary time evolution for isolated subsystems

classical statistical implementation of quantum computer

classical ensemble , discrete observable

- Classical ensemble with probabilities $\hat{p}_\tau$

$$\hat{p}_\tau \geq 0 \quad , \quad \sum_\tau \hat{p}_\tau = 1$$

- qubit :

  one discrete observable A , values +1 or -1

probabilities to find A=1 : w_+ and A=-1: w_-

$$\langle A \rangle = w_+ - w_-$$

classical ensemble for one qubit

- classical states labeled by

$$(\sigma_1, \sigma_2, \sigma_3) \qquad \sigma_j = \pm 1$$

eight states

- state of subsystem depends on three numbers

$$\rho_j = \sum_{\sigma_1, \sigma_2, \sigma_3} \sigma_j p(\sigma_1, \sigma_2, \sigma_3)$$

- expectation value of qubit

$$\langle A \rangle = \rho_3 \ , \ w_+ = \frac{1}{2}(1 + \rho_3)$$

classical probability distribution

$$p(\sigma_1, \sigma_2, \sigma_3) = p_s(\sigma_1, \sigma_2, \sigma_3) + \delta p_e(\sigma_1, \sigma_2, \sigma_3)$$

$$p_s(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{8}(1 + \sigma_1\rho_1)(1 + \sigma_2\rho_2)(1 + \sigma_3\rho_3)$$

characterizes subsystem

$$\sum_{\sigma_1,\sigma_2,\sigma_3} \delta p_e(\sigma_1, \sigma_2, \sigma_3) = 0 \ , \quad \sum_{\sigma_1,\sigma_2,\sigma_3} \sigma_j \delta p_e(\sigma_1, \sigma_2, \sigma_3) = 0$$

different δp_e characterize environment

state of system independent of environment

- ρ_j does not depend on precise choice of δp_e

$$\rho_j = \sum_{\sigma_1,\sigma_2,\sigma_3} \sigma_j p(\sigma_1, \sigma_2, \sigma_3)$$

$$\sum_{\sigma_1,\sigma_2,\sigma_3} \delta p_e(\sigma_1, \sigma_2, \sigma_3) = 0 \, , \quad \sum_{\sigma_1,\sigma_2,\sigma_3} \sigma_j \delta p_e(\sigma_1, \sigma_2, \sigma_3) = 0$$

time evolution

rotations of ϱ_k

$$\rho_k(t,t') = \hat{S}_{kl}(t,t')\rho_l(t')\ ,\ \hat{S}\hat{S}^T = 1$$

$$\frac{\partial}{\partial t}\rho_k = T_{kl}\rho_l\ ,\ (T)^T = -T$$

example :

$$\hat{S} = \begin{pmatrix} \cos^2\varphi & , & \sqrt{2}\sin\varphi\cos\varphi & , & \sin^2\varphi \\ -\sqrt{2}\sin\varphi\cos\varphi & , & 1-2\sin^2\varphi & , & \sqrt{2}\sin\varphi\cos\varphi \\ \sin^2\varphi & , & -\sqrt{2}\sin\varphi\cos\varphi & , & \cos^2\varphi \end{pmatrix}$$

time evolution of classical probability

- evolution of p_s according to evolution of ϱ_k
- evolution of δp_e arbitrary , consistent with constraints

state after finite rotation

$$\varphi(t=\Delta)=\frac{\pi}{2}$$

$$\hat{S}=\begin{pmatrix} \cos^2\varphi & , & \sqrt{2}\sin\varphi\cos\varphi & , & \sin^2\varphi \\ -\sqrt{2}\sin\varphi\cos\varphi & , & 1-2\sin^2\varphi & , & \sqrt{2}\sin\varphi\cos\varphi \\ \sin^2\varphi & , & -\sqrt{2}\sin\varphi\cos\varphi & , & \cos^2\varphi \end{pmatrix}$$

$$\rho_3(t)=\rho_{1,0}\ ,\ \rho_1(t)=\rho_{3,0}\ ,\ \rho_2(t)=-\rho_{2,0}$$

$$\begin{aligned} p_s(\sigma_1,\sigma_2,\sigma_3;t) &= p_s(\sigma_3,\sigma_2,\sigma_1;0), \\ p_s(\sigma_1,\sigma_2,\sigma_3;t) &= p_s(\sigma_3,-\sigma_2,\sigma_1;0) \end{aligned}$$

this realizes Hadamard gate

purity

$$P = \rho_k \rho_k$$

consider ensembles with P ≤ 1

purity conserved by time evolution

density matrix

- define hermitean 2x2 matrix :

$$\rho = \frac{1}{2}(1 + \rho_k \tau_k)$$

- properties of density matrix

$$tr\rho = 1$$

$$\rho_{\alpha\alpha} \geq 1$$

$$tr\rho^2 \leq 1$$

operators

if observable $A(e_k)$ obeys $\langle A(e_k)\rangle =: \rho_k e_k$

associate hermitean operators $\hat{A}(e_k) = e_k \tau_k$

$$\begin{aligned}\langle A(e_k)\rangle &= tr(\hat{A}(e_k)\rho) \\ &= \frac{1}{2}\rho_k e_\ell \{\tau_k, \tau_\ell\} = \rho_k e_k\end{aligned}$$

in our case : $e_3=1$, $e_1=e_2=0$

quantum law for expectation values

$$\langle A \rangle = tr(\hat{A}\rho)$$

pure state

$$P = 1 \;\longrightarrow\; \varrho^2 = \varrho$$

wave function

$$\rho_{\alpha\beta} = \psi_\alpha \psi^*_\beta \; , \; \psi^*_\alpha \psi_\alpha = 1$$

$$\langle A \rangle = \psi^*_\alpha (\tau_3)_{\alpha\beta} \psi_\beta = \langle \psi | \hat{A} | \psi \rangle$$

unitary time evolution

$$\psi_\alpha(t) = U_{\alpha\beta}(t) \psi_\beta(0)$$

Hadamard gate

$$\rho_3(t) = \rho_{1,0}\ ,\ \ \rho_1(t) = \rho_{3,0}\ ,\ \ \rho_2(t) = -\rho_{2,0}$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, & 1 \\ 1, & -1 \end{pmatrix}$$

CNOT gate

$$U = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0 \end{pmatrix}$$

Four state quantum system
- two qubits -

k=1, …,15   P ≤ 3

$$\rho = \frac{1}{4}(1 + \rho_k L_k)\ ,\ \mathrm{tr}(L_k L_l) = 4\delta_{kl}$$

normalized SU(4) – generators :

$$\begin{aligned}
L_1 &= \tau_3 \otimes 1\ ,\ L_2 = 1 \otimes \tau_3\ ,\ L_3 = \tau_3 \otimes \tau_3,\\
L_4 &= 1 \otimes \tau_1\ ,\ L_5 = 1 \otimes \tau_2\ ,\ L_6 = \tau_3 \otimes \tau_1,\\
L_7 &= \tau_3 \otimes \tau_2\ ,\ L_8 = \tau_1 \otimes 1\ ,\ L_9 = \tau_2 \otimes 1,\\
L_{10} &= \tau_1 \otimes \tau_3\ ,\ L_{11} = \tau_2 \otimes \tau_3\ ,\ L_{12} = \tau_1 \otimes \tau_1,\\
L_{13} &= \tau_1 \otimes \tau_2,\ L_{14} = -\tau_2 \otimes \tau_2\ ,\ L_{15} = \tau_2 \otimes \tau_1
\end{aligned}$$

four – state quantum system

$$P = \rho_k \rho_k$$

$$\hat{A} = e_k L_k \ , \ \langle A \rangle = \rho_k e_k = \mathrm{tr}(\rho \hat{A})$$

$P \leq 3$

pure state : $P = 3$ and

copurity $C = tr[(\rho^2 - \rho)^2]$ must vanish

$$\rho_{\alpha\beta} = \psi_\alpha \psi_\beta^* \ , \ \psi_\alpha = U_{\alpha\beta}(\hat{\psi}_m)_\beta$$

$$(\hat{\psi}_m)_\beta = \delta_{m\beta} \ , \ \langle A \rangle = \psi^\dagger \hat{A} \psi$$

suitable rotation of ϱ_k

$$\rho_2 \leftrightarrow \rho_3 \ , \ \rho_5 \leftrightarrow \rho_7 \ , \ \rho_8 \leftrightarrow \rho_{12},$$
$$\rho_9 \leftrightarrow \rho_{15} \ , \ \rho_{10} \leftrightarrow \rho_{14} \ , \ \rho_{11} \leftrightarrow \rho_{13}$$

yields transformation of the density matrix

$$\rho_{13} \leftrightarrow \rho_{14} \ , \ \rho_{23} \leftrightarrow \rho_{24} \ , \ \rho_{31} \leftrightarrow \rho_{41} \ , \ \rho_{32} \leftrightarrow \rho_{42},$$
$$\rho_{33} \leftrightarrow \rho_{44} \ , \ \rho_{34} \leftrightarrow \rho_{43}$$

and realizes CNOT gate

$$U = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0 \end{pmatrix}$$

classical probability distribution for 2^{15} classical states

$$p_s(\{\sigma_k\}) = 2^{-15} \prod_k (1 + \sigma_k \rho_k)$$

$$\sum_{\{\sigma_k\}} \delta p_e(\{\sigma_k\}) = 0 \, , \; \sum_{\{\sigma_k\}} \sigma_j \delta p_e(\{\sigma_k\}) = 0$$

$$\rho_j = \sum_{\{\sigma_k\}} \sigma_j p(\{\sigma_k\})$$

probabilistic observables

for a given state of the subsystem , specified by $\{\varrho_k\}$:

The possible measurement values +1 and -1
of the discrete two - level observables are found
with probabilities $w_+(\varrho_k)$ and $w_-(\varrho_k)$.

In a quantum state the observables have a probabilistic distribution of values , rather than a fixed value as for classical states .

probabilistic quantum observable

spectrum $\{ \gamma_\alpha \}$
probability that γ_α is measured : w_α
can be computed from state of subsystem

$$\langle A \rangle = \sum_\alpha w_\alpha(\rho_k)\gamma_\alpha$$

$$w_\alpha(\rho_k) = \rho'_{\alpha\alpha} = (U_A \rho U_A^\dagger)_{\alpha\alpha}$$

non – commuting quantum operators

for two qubits :

- all L_k represent two – level observables
- they do not commute

$$\langle A \rangle = tr(\hat{A}\rho)$$

- the laws of quantum mechanics for expectation values are realized
- uncertainty relation etc.

incomplete statistics

joint probabilities depend on environment
and are not available for subsystem !

$$C_{12} = \sum_{\{\sigma_k\}} \sigma_1 \sigma_2 p(\{\sigma_k\}) = p_{++} + p_{--} - p_{+-} - p_{-+}$$

$$C_{ij} = \sum_{\{\sigma_k\}} \sigma_i \sigma_j p(\{\sigma_k\})$$

$p = p_s + \delta p_e$

$$p_s(\{\sigma_k\}) = 2^{-15} \prod_k (1 + \sigma_k \rho_k)$$

$$\sum_{\{\sigma_k\}} \delta p_e(\{\sigma_k\}) = 0 \ , \ \sum_{\{\sigma_k\}} \sigma_j \delta p_e(\{\sigma_k\}) = 0$$

quantum mechanics from classical statistics

- probability amplitude ☺
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conditional correlations

classical correlation

- point wise multiplication of classical observables on the level of classical states
- classical correlation depends on probability distribution for the atom and its environment

$$C_{ij} = \sum_{\{\sigma_k\}} \sigma_i \sigma_j p(\{\sigma_k\})$$

- not available on level of probabilistic observables
- definition depends on details of classical observables , while many different classical observables correspond to the same probabilistic observable

needed : correlation that can be formulated in terms of probabilistic observables and density matrix !

conditional probability

$w^{AB}_{+,\alpha}$ probability to find value +1 for product of measurements of A and B

$$
\begin{aligned}
w^{AB}_{+,\alpha} &= (w^A_+)^B_+ w^B_{+,\alpha} + (w^A_-)^B_- w^B_{-,\alpha} \\
w^{AB}_{-,\alpha} &= (w^A_+)^B_- w^B_{-,\alpha} + (w^A_-)^B_+ w^B_{+,\alpha}
\end{aligned}
$$

$(w^A_+)^B_+$ probability to find A=1 after measurement of B=1

… can be expressed in terms of expectation value of A in eigenstate of B

$$
\begin{aligned}
(w^A_\pm)^B_+ &= \frac{1}{2}(1 \pm \langle A \rangle_{+B}) \\
(w^A_\pm)^B_- &= \frac{1}{2}(1 \pm \langle A \rangle_{-B})
\end{aligned}
$$

measurement correlation

$$\langle BA \rangle_m = (w_+^B)_+^A w_{+,s}^A - (w_-^B)_+^A w_{+,s}^A - (w_+^B)_-^A w_{-,s}^A + (w_-^B)_-^A w_{-,s}^A$$

After measurement A=+1 the system must be in eigenstate with this eigenvalue. Otherwise repetition of measurement could give a different result ! ρ_{A+}

$$(w_+^B)_+^A - (w_-^B)_+^A = \mathrm{tr}(\hat{B}\rho_{A+})$$

measurement changes state
in all statistical systems !

quantum and classical

eliminates possibilities that are not realized

physics makes statements
about possible
sequences of events
and their probabilities

unique eigenstates for M=2

$\mathrm{M} = 2:$

$$\rho_{A+} = \frac{1}{2}(1 + \hat{A})$$

$$(w_{\pm}^{B})_{+}^{A} = \frac{1}{2} \pm \frac{1}{4}\mathrm{tr}(\hat{B}\hat{A}) \ , \ (w_{\pm}^{B})_{-}^{A} = \frac{1}{2} \mp \frac{1}{4}\mathrm{tr}(\hat{B}\hat{A})$$

eigenstates with A = 1

$$\rho_{A+} = \frac{1}{M}(1 + \hat{A} + X)\ ,\ \text{tr}(\hat{A}X) = 0\ ,\ \text{tr}X = 0$$

$$P = M\text{tr}(\rho_{A+}^2) = 1 + \frac{1}{M}\text{tr}X^2$$

$$\rho_{A+}^2 - \rho_{A+} = \frac{1}{M^2}(X^2 + \{\hat{A}, X\}) - \left(1 - \frac{2}{M}\right)\rho_{A+}$$

measurement preserves pure states if projection

$$\rho_{A+} = \frac{1}{2(1 + \langle A \rangle)}(1 + \hat{A})\rho(1 + \hat{A})$$

measurement correlation equals quantum correlation

$$\langle BA \rangle_m = \frac{1}{2}\mathrm{tr}(\{\hat{A}, \hat{B}\}\rho)$$

probability to measure A=1 and B=1 :

$$w_{++} = \frac{1}{4}(1 + \langle A \rangle + \langle B \rangle + \langle AB \rangle_m)$$

$$w_{++} = \frac{1}{4}\Big(1 + e_k^{(A)} e_k^{(B)} + \rho_k [e_k^{(A)} + e_k^{(B)} + d_{mlk} e_m^{(A)} e_l^{(B)}]\Big)$$

probability that A and B have both the value +1 in classical ensemble

$$p_{++} = \frac{1}{4}(1 + \langle A \rangle + \langle B \rangle + \langle A \cdot B \rangle)$$

$$\langle A \cdot B \rangle = \sum_{\tau} p_\tau A_\tau B_\tau$$

not a property of the subsystem

probability to measure A and B both +1

$$w_{++} = \frac{1}{4}(1 + \langle A \rangle + \langle B \rangle + \langle AB \rangle_m)$$

$$w_{++} = \frac{1}{4}\left(1 + e_k^{(A)} e_k^{(B)} + \rho_k [e_k^{(A)} + e_k^{(B)} + d_{mlk} e_m^{(A)} e_l^{(B)}]\right)$$

can be computed from the subsystem

sequence of three measurements and quantum commutator

$$\begin{aligned}
\langle ABC\rangle_m - \langle ACB\rangle_m &= \frac{1}{4}\mathrm{tr}\Big([\hat{A},[\hat{B},\hat{C}]]\,\rho\Big),\\
\langle ABC\rangle_m - \langle CBA\rangle_m &= \frac{1}{4}\mathrm{tr}\Big([\hat{B},[\hat{A},\hat{C}]]\,\rho\Big),\\
\langle ABC\rangle_m - \langle BAC\rangle_m &= 0
\end{aligned}$$

two measurements commute , not three

conclusion

- quantum statistics arises from classical statistics

  states, superposition , interference , entanglement , probability amplitudes
- quantum evolution embedded in classical evolution
- conditional correlations describe measurements both in quantum theory and classical statistics

quantum particle from classical statistics

- quantum and classical particles can be described within the same classical statistical setting
- different time evolution , corresponding to different Hamiltonians
- continuous interpolation between quantum and classical particle possible !

end ?

time evolution

transition probability

time evolution of probabilities

$$\partial_t p_\sigma = F_\sigma(p_{\sigma'})$$

(fixed observables)

induces transition probability matrix

$$p_\sigma(t) = \tilde{S}_{\sigma\tau}(t, t')p_\tau(t')$$

reduced transition probability

- induced evolution

$$\partial_t \rho_k = \sum_\sigma \partial_t p_\sigma \overline{A}_\sigma^{(k)} = \sum_\sigma F_\sigma(p_{\sigma'}) \overline{A}_\sigma^{(k)}$$

- reduced transition probability matrix

$$\rho_k(t) = S_{k\ell}(t,t')\rho_\ell(t')$$

$$S_{k\ell}(t,t') = \frac{\sum_{\sigma\tau\rho} \tilde{S}_{\sigma\tau}(t,t') p_\tau(t') p_\rho(t') \overline{A}_\sigma^{(k)} \overline{A}_\rho^{(\ell)}}{\rho_m(t')\rho_m(t')}$$

evolution of elements of density matrix in two – state quantum system

- infinitesimal time variation

$$\partial_t \rho_k(t) = \partial_t S_{k\ell}(t,t') S^{-1}_{\ell m}(t,t') \rho_m(t)$$

- scaling + rotation

$$S_{k\ell} = \hat{S}_{k\ell}\, d \qquad \hat{S}^{-1}_{k\ell} = \hat{S}_{\ell k}$$

$$\partial_t S S^{-1} = \partial_t \hat{S} \hat{S}^T + \partial_t \ln d$$

time evolution of density matrix

- Hamilton operator and scaling factor

$$\hat{H} = -\frac{1}{4}(\partial_t \hat{S}\hat{S}^T)_{\ell m}\, \varepsilon_{\ell m k} \tau_k$$

$$\lambda = \partial_t \ln d$$

- Quantum evolution and the rest ?

$$\partial_t \rho = -i[\hat{H}, \rho] + \lambda(\rho - \frac{1}{2})$$

λ=0 and pure state :

$$i\partial_t \psi = \hat{H}\psi$$

quantum time evolution

It is easy to construct explicit ensembles where

$$\lambda = 0$$

→ quantum time evolution

evolution of purity

change of purity

$$\partial_t P = \partial_t(\rho_k \rho_k) = \partial_t(2tr\,\rho^2 - 1)$$
$$\partial_t P = 2\lambda P$$

$$P = \rho_k \rho_k$$

attraction to randomness :
decoherence

$$\lambda < 0 \quad : \quad P \to 0$$

attraction to purity :
syncoherence

$$\lambda > 0 \quad : \quad P \to 1$$

classical statistics can describe
decoherence and syncoherence !
unitary quantum evolution : special case

pure state fixed point

pure states are special :

“ no state can be purer than pure “

fixed point of evolution for $P = 1 \quad , \quad \lambda = 0$

approach to fixed point $\partial_t \lambda = \beta_\lambda(\lambda, P, \rho_k/\sqrt{P}, \ldots)$

$$\beta_\lambda = -a\lambda + b(1 - P)$$

approach to pure state fixed point

solution : $1 - P = x_1 e^{-\varepsilon_1 t} + x_2 e^{-\varepsilon_2 t}$ $\lambda = \varepsilon_1 x_1 e^{-\varepsilon_1 t} + \varepsilon_2 x_2 e^{-\varepsilon_2 t}$

$$\varepsilon_{1,2} = \frac{1}{2}(a \pm \sqrt{a^2 - 4b}$$

syncoherence describes exponential approach to pure state if

$$a > 0 \quad , \quad a < b < \frac{1}{4}a^2$$

decay of mixed atom state to ground state *purity conserving evolution : subsystem is well isolated*

two bit system and entanglement

ensembles with P=3

non-commuting operators

15 spin observables labeled by $e_k \quad , \quad k = 1 \ldots 15$

$$\rho_k = \sum_\sigma p_\sigma \overline{A}_\sigma^{(k)} \quad , \quad \langle A(e_k) \rangle = \sum_k \rho_k e_k \quad , \quad -1 \leq \rho_k \leq 1$$

density matrix $\rho = \frac{1}{4}(1 + \rho_k L_k)$

$$L_k^2 = 1 \quad , \quad tr L_k = 0 \quad , \quad tr(L_k L_\ell) = 4\delta_{k\ell}$$

SU(4) - generators

$$L_k^2 = 1\ ,\ \mathrm{tr} L_k = 0\ ,\ \mathrm{tr}(L_k L_l) = 4\delta_{kl}$$

$$L_1 = \mathrm{diag}(1,1,-1,-1)\ ,\ L_2 = \mathrm{diag}(1,-1,1,-1)$$

$$L_3 = \mathrm{diag}(1,-1,-1,1)$$

$$L_4 = \begin{pmatrix} \tau_1, & 0 \\ 0, & \tau_1 \end{pmatrix} \quad L_5 = \begin{pmatrix} \tau_2, & 0 \\ 0, & \tau_2 \end{pmatrix}$$

$$L_6 = \begin{pmatrix} \tau_1, & 0 \\ 0, & -\tau_1 \end{pmatrix}, L_7 = \begin{pmatrix} \tau_2, & 0 \\ 0, & -\tau_2 \end{pmatrix}$$

density matrix

- pure states : P=3

$$tr\rho^2 = \frac{1}{4}(1 + \rho_k\rho_k) = \frac{1}{4}(1 + P)$$

$$P \leq 3 \quad : \quad tr\rho^2 \leq 1$$

$$\hat{A}(e_k) = e_k L_k \quad , \quad e_k e_k = 1 \quad \text{for} \quad \hat{A}^2(e_k) = 1$$

entanglement

- three commuting observables

$$L_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad , \quad L_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \quad , \quad L_3 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

L_1 : bit 1 , L_2 : bit 2 L_3 : product of two bits

- expectation values of associated observables related to probabilities to measure the combinations (++) , etc.

$$\begin{aligned} \langle T_1 \rangle &= W_{++} + W_{+-} - W_{-+} - W_{--} \\ \langle T_2 \rangle &= W_{++} - W_{+-} + W_{-+} - W_{--} \\ \langle T_3 \rangle &= W_{++} - W_{+-} - W_{-+} + W_{--} \end{aligned}$$

“classical” entangled state

- pure state with maximal anti-correlation of two bits

$$W_{++} = W_{--} = 0 \quad , \quad W_{+-} = W_{-+} = \frac{1}{2}$$

- bit 1: random , bit 2: random
- **if bit 1 = 1 necessarily bit 2 = -1 , and vice versa**

$$\langle L_1 \rangle = \langle L_2 \rangle = 0 \quad , \quad \langle L_3 \rangle = -1$$

classical state described by entangled density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 1, & \pm 1, & 0 \\ 0, & \pm 1, & 1, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix} \quad , \quad tr\rho^2 = 1$$

$$\rho = \frac{1}{4}(1 - L_3 \pm (L_{12} - L_{14}))$$

$$\rho_1 = \rho_2 = 0 \quad \Rightarrow \quad \langle T_1 \rangle = \langle T_2 \rangle = 0$$

$$\rho_3 = -1 \quad \Rightarrow \quad \langle T_3 \rangle = -1$$

entangled quantum state

$$\psi_\pm = \frac{1}{\sqrt{2}}(\psi_2 \pm \psi_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm 1 \\ 0 \end{pmatrix}$$

end

pure state density matrix

- elements ϱ_k are vectors on unit sphere
- can be obtained by unitary transformations

$$\rho = U\hat{\rho}_1 U^\dagger \quad , \quad UU^\dagger = U^\dagger U = 1$$

$$\hat{\rho}_1 = \begin{pmatrix} 1 & ; & 0 \\ 0 & ; & 0 \end{pmatrix}$$

- SO(3) equivalent to SU(2)

wave function

- “root of pure state density matrix “

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\hat{\psi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \psi = U\hat{\psi}_1$$

$$\rho_{\alpha\beta} = \psi_\alpha \psi_\beta^*$$

$$tr(\hat{A}\rho) = \hat{A}_{\alpha\beta}\rho_{\beta\alpha} = \hat{A}_{\alpha\beta}\psi_\beta\psi_\alpha^*$$

- quantum law for expectation values

$$\langle A \rangle = \psi^\dagger \hat{A} \psi$$