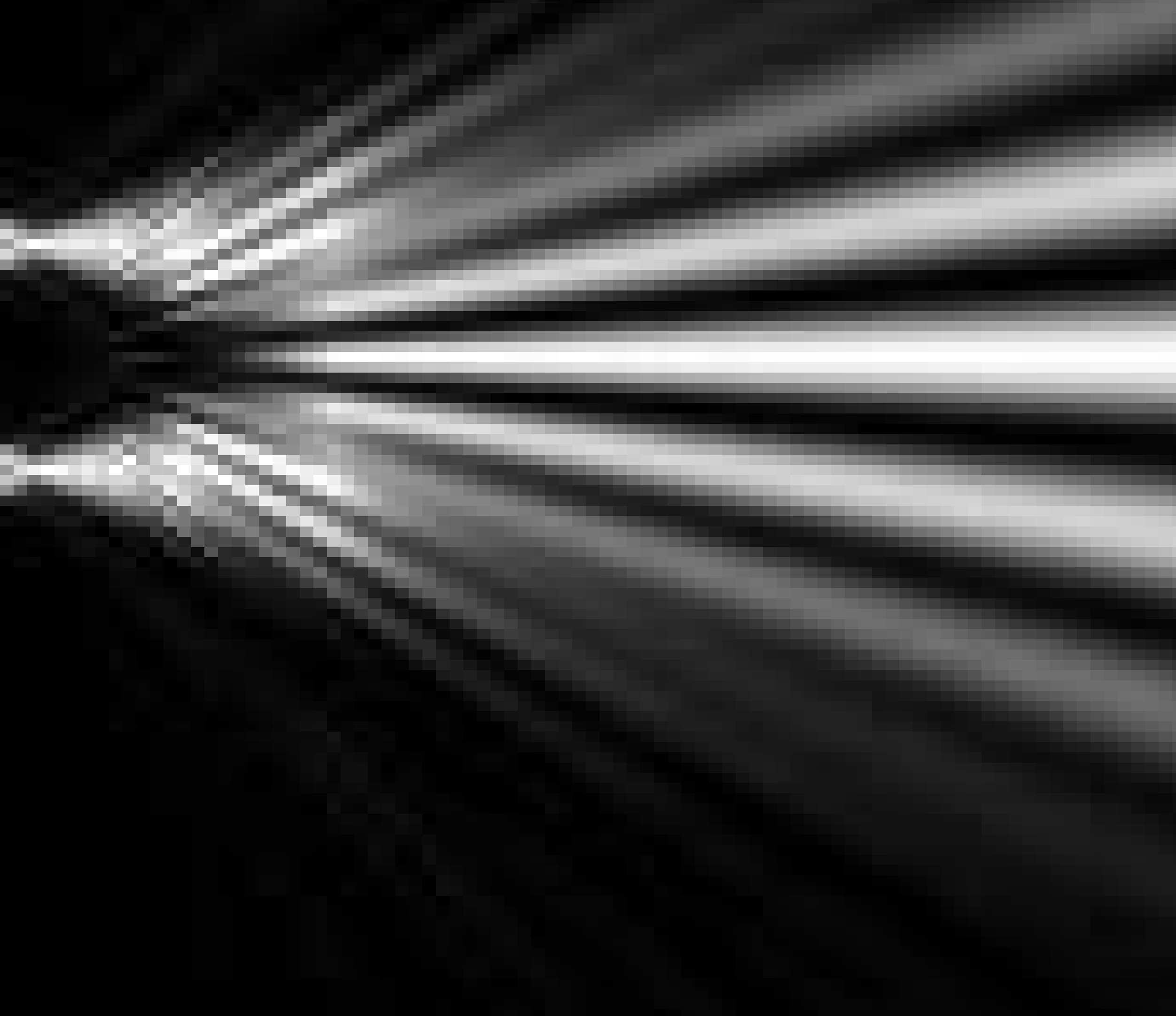
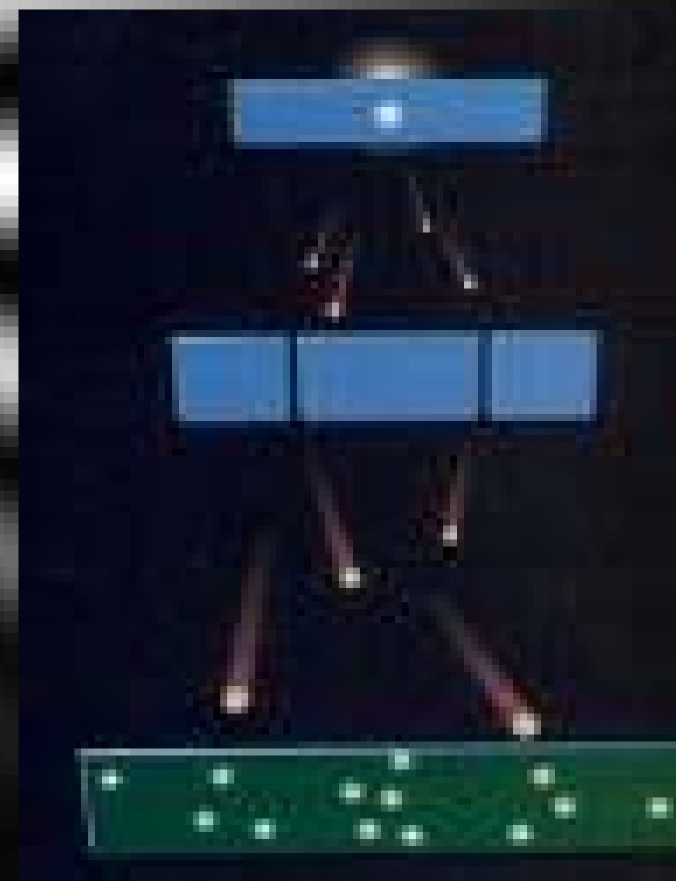
Quantum fermions from classical statistics

quantum mechanics can be described by classical statistics !

quantum particle from classical probabilities





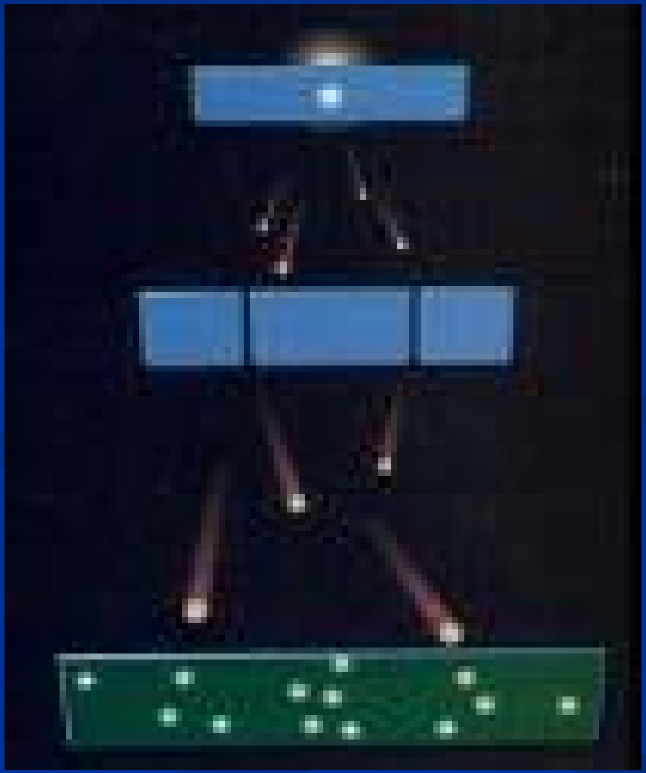
Double slit experiment

Is there a classical probability density w(x,t) describing interference ?

Or hidden parameters w(x,α,t) ?
or w(x,p,t) ?

Suitable time evolution law :
local , causal ?   Yes !

Bell's inequalities ?
Kochen-Specker Theorem ?



statistical picture of the world

- basic theory is not deterministic
- basic theory makes only statements about probabilities for sequences of events and establishes correlations
- probabilism is fundamental , not determinism !

quantum mechanics from classical statistics :
not a deterministic hidden variable theory

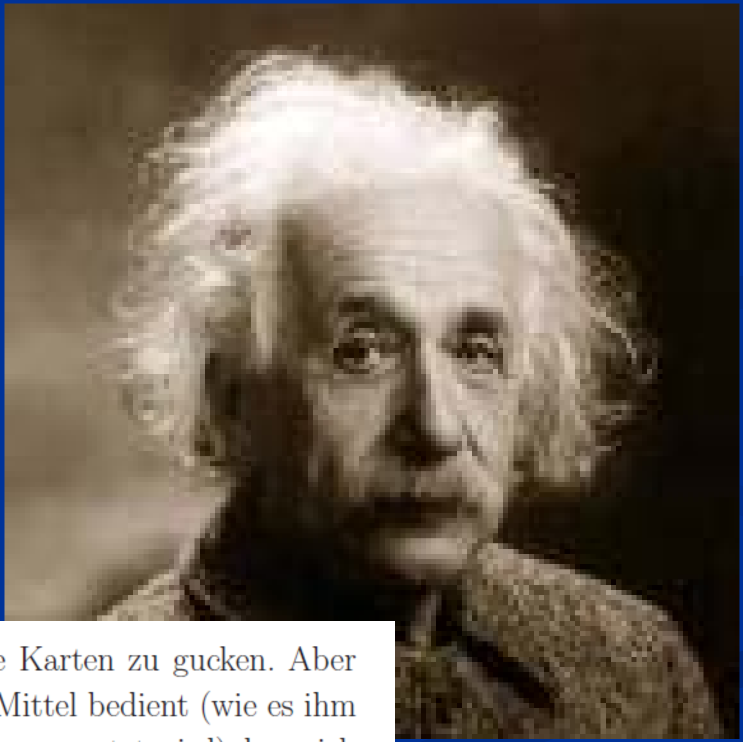
Probabilistic realism

Physical theories and laws only describe probabilities

Physics only describes probabilities

Gott würfelt

Gott würfelt nicht



"Es scheint hart, dem Herrgott in die Karten zu gucken. Aber dass er würfelt und sich telepatischer Mittel bedient (wie es ihm von der gegenwärtigen Quantentheorie zugemutet wird), kann ich keinen Augenblick glauben.."

Einstein: Brief an Cornelius Lanczos am 21. März 1942

Physics only describes probabilities

Gott würfelt

but nothing beyond classical statistics is needed for quantum mechanics !

fermions from classical statistics

Classical probabilities for two interfering Majorana spinors

$$p(x,1,y,1) = \frac{1}{4}L^{-6}\Big[\cos^2\{p_3(t+x_3)\}\cos^2\{p_1(t+y_1)\}$$
$$+\cos^2\{p_1(t+x_1)\}\cos^2\{p_3(t+y_3)\}$$
$$-2\cos\{p_3(t+x_3)\}\cos\{p_1(t+x_1)\}$$
$$\times\cos\{p_3(t+y_3)\}\cos\{p_1(t+y_1)\}\Big].$$

Interference terms

microphysical ensemble

- states τ
- labeled by sequences of occupation numbers or bits $n_s = 0$ or 1
- $\tau = [\, n_s \,] = [0,0,1,0,1,1,0,1,0,1,1,1,1,0,\ldots]$ etc.
- probabilities $p_\tau > 0$

Classical wave function

$$p_\tau(t) \geq 0\ ,\ \sum_\tau p_\tau(t) = 1$$

$$q_\tau(t) = s_\tau(t)\sqrt{p_\tau(t)}\ ,\ p_\tau(t) = q_\tau^2(t)\ ,\ s_\tau(t) = \pm 1.$$

Classical wave function q is real , not necessarily positive
Positivity of probabilities automatic.

$$\sum_\tau p_\tau = \sum_\tau q_\tau^2 = 1$$

Time evolution

$$q_\tau(t') = \sum_\rho R_{\tau\rho}(t', t) q_\rho(t) \ , \ R^T R = 1.$$

Rotation preserves normalization of probabilities

$$\sum_\tau p_\tau = \sum_\tau q_\tau^2 = 1$$

Evolution equation specifies dynamics

simple evolution : R independent of q

(infinitely) many degrees of freedom

$s = (x, \gamma)$

x : lattice points , γ : different species

number of values of s : B

number of states τ : 2^B

Grassmann wave function

Map between classical states and basis elements of Grassmann algebra

$$g_\tau = \psi_{\gamma_1}(x_1)\psi_{\gamma_2}(x_2)\ldots$$

$s = (x, \gamma)$

For every $n_s = 0$: g_τ contains factor ψ_s

Grassmann wave function :

$$g = \sum_\tau q_\tau g_\tau$$

Functional integral

Grassmann wave function depends on t ,
since classical wave function q depends on t

(fixed basis elements of Grassmann algebra)

$$g = \sum_{\tau} q_{\tau} g_{\tau}$$

Evolution equation for g(t)

⇕

Functional integral

Wave function from functional integral

$$Z = \int \mathcal{D}\psi \bar{g}_f \left[\psi(t_f)\right] e^{-S} g_{in} \left[\psi(t_{in})\right]$$

$$\int \mathcal{D}\psi = \prod_{t,x} \int \left(d\psi_4(t,x) \ldots d\psi_1(t,x)\right)$$

$$S = \sum_{t=t_{in}}^{t_f - \epsilon} L(t),$$

L(t) depends only on ψ(t) and ψ(t+ε)

$$\begin{aligned} S &= S_< + S_>, \\ S_< &= \sum_{t'<t} L(t') \ , \ S_> = \sum_{t' \geq t} L(t') \end{aligned}$$

$$g(t) = \int \mathcal{D}\psi(t' < t) e^{-S_<} g_{in}.$$

$$g(t) = \sum_\tau q_\tau(t) g_\tau \left[\psi(t)\right]$$

Evolution equation

- Evolution equation for classical wave function , and therefore also for classical probability distribution , is specified by action S

- Real Grassmann algebra needed , since classical wave function is real

Massless Majorana spinors in four dimensions

$$S = \int_{t,x} \{\psi_\gamma \partial_t \psi_\gamma - \psi_\gamma (T_k)_{\gamma\delta} \partial_k \psi_\delta\}.$$

$$T_1 = \begin{pmatrix} 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \end{pmatrix} , \; T_2 = \begin{pmatrix} 0, & 0, & 0, & 1 \\ 0, & 0, & -1, & 0 \\ 0, & -1, & 0, & 0 \\ 1, & 0, & 0, & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix} ,$$

$$\gamma^0 = \begin{pmatrix} 0, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0 \\ 0, & -1, & 0, & 0 \\ -1, & 0, & 0, & 0 \end{pmatrix} , \; \gamma^k = -\gamma^0 T_k$$

$$S = -\int_{t,x} \bar{\psi} \gamma^\mu \partial_\mu \psi \; , \; \bar{\psi} = \psi^T \gamma^0$$

Time evolution

$$S = \sum_{t=t_{in}}^{t_f - \epsilon} L(t),$$

$$g(t+\epsilon) = \int \mathcal{D}\psi(t) e^{-L(t)} g(t).$$

$$\partial_t g = \mathcal{K} g.$$

$$\mathcal{K} = \sum_x \frac{\partial}{\partial \psi_\gamma(x)} (T_k)_{\gamma\delta} \partial_k \psi_\delta(x).$$

$$\partial_t q_\tau(t) = \sum_\rho K_{\tau\rho} q_\rho(t)$$

$$K_{\rho\tau} = -K_{\tau\rho}.$$

linear in q , non-linear in p

$$\partial_t p_\tau = 2 \sum_\rho K_{\tau\rho} s_\tau s_\rho \sqrt{p_\tau p_\rho}.$$

One particle states

$$a_\gamma^\dagger(x)g = \frac{\partial}{\partial\psi_\gamma(x)}g \ , \ a_\gamma(x)g = \psi_\gamma(x)g$$

$$\{a_\gamma^\dagger(x),\ a_\epsilon(y)\} \ = \ \delta_{\gamma\epsilon}\delta(x-y) \ , \ \mathcal{N} = \int_x a_\gamma^\dagger(x)a_\gamma(x)$$

$$g_1(t) = \int_x q_\gamma(t,x)a_\gamma^\dagger(x)g_0$$

g_0 : arbitrary static "vacuum" state

One –particle wave function obeys
Dirac equation

$$\gamma^\mu\partial_\mu q = 0$$

Dirac spinor in electromagnetic field

$$S = \int_{t,x} \{\psi_1(\partial_t - T_k\partial_k + m\gamma^0\tilde{I})\psi_1 + \psi_2(\partial_t - T_k\partial_k + m\gamma^0\tilde{I})\psi_2\},$$

$$\tilde{I} = \begin{pmatrix} 0, & -1, & 0, & 0 \\ 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & -1 \\ 0, & 0, & 1, & 0 \end{pmatrix} = T_1T_2T_3.$$

$$\Delta S = -e\int_{t,x} \{\psi_1(A_0 - A_kT_k)\psi_2 - \psi_2(A_0 - A_kT_k)\psi_1\}.$$

one particle state obeys Dirac equation

complex Dirac equation in electromagnetic field

$$\Psi_D = q_1 + iq_2 \ , \ \gamma^\mu(\partial_\mu + ieA_\mu)\Psi_D = 0.$$

Schrödinger equation

Non – relativistic approximation :

- Time-evolution of particle in a potential described by standard Schrödinger equation.
- Time evolution of probabilities in classical statistical Ising-type model generates all quantum features of particle in a potential , as interference (double slit) or tunneling. This holds if initial distribution corresponds to one-particle state.

quantum particle from classical probabilities

ν

what is an atom ?

- quantum mechanics : isolated object
- quantum field theory : excitation of complicated vacuum
- classical statistics : sub-system of ensemble with infinitely many degrees of freedom

i

Phases and complex structure

$$S = \int_{t,x} \{\psi_1(\partial_t - T_k\partial_k + m\gamma^0\tilde{I})\psi_1 + \psi_2(\partial_t - T_k\partial_k + m\gamma^0\tilde{I})\psi_2\},$$

$$\Delta S = -e\int_{t,x} \{\psi_1(A_0 - A_kT_k)\psi_2 - \psi_2(A_0 - A_kT_k)\psi_1\}.$$

introduce complex spinors : $\psi_D = \psi_1 + i\psi_2$ $\bar{\psi}_D = \psi_D^\dagger\gamma^0.$

$$S = \int_{t,x} \psi_D^\dagger(\partial_t - T_k\partial_k + m\gamma^0\tilde{I})\psi_D = -\int_{t,x} \bar{\psi}_D(\gamma^\mu\partial_\mu - m\tilde{I})\psi_D,$$

$$\Delta S = -ie\int_{t,x} \bar{\psi}_D\gamma^\mu A_\mu\psi_D$$

complex wave function : $\Psi_D = q_1 + iq_2 \ , \ \gamma^\mu(\partial_\mu + ieA_\mu)\Psi_D = 0.$

h

Simple conversion factor for units

unitary time evolution

ν

fermions and bosons

ν

$$[A,B]=C$$

non-commuting observables

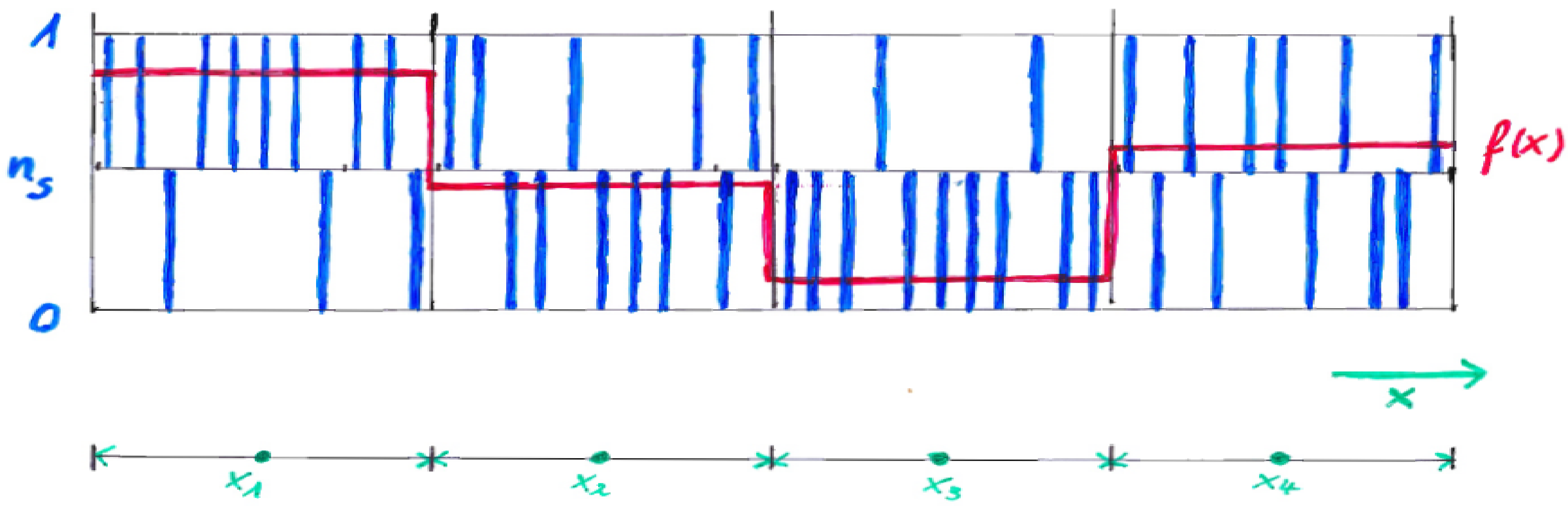
- classical statistical systems admit many product structures of observables
- many different definitions of correlation functions possible , not only classical correlation !
- type of measurement determines correct selection of correlation function !
- example 1 : euclidean lattice gauge theories
- example 2 : function observables

function observables

microphysical ensemble

- states τ
- labeled by sequences of occupation numbers or bits $n_s = 0$ or 1
- $\tau = [\, n_s \,] = [0,0,1,0,1,1,0,1,0,1,1,1,1,0,\ldots]$ etc.
- probabilities $p_\tau > 0$

function observable



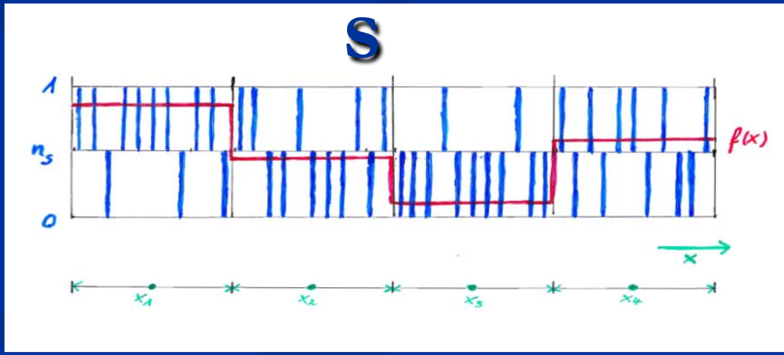
function observable

$$f_\tau(x_i) = \mathcal{N}^{-\frac{1}{2}} \sum_{s \in I(x_i)} (2n_s - 1)$$

normalized difference between occupied and empty bits in interval

$$\int dx\, f_\tau^2(x) = \sum_i f_\tau^2(x_i) = 1$$

$$\mathcal{N} = \sum_{x_i} \Big(\sum_{s \in I(x_i)} (2n_s - 1) \Big)^2$$



$I(x_1)$ $I(x_2)$ $I(x_3)$ $I(x_4)$

generalized function observable

normalization

$$\int dx\, f_\tau^2(x) = 1$$

classical expectation value

$$\langle f(x) \rangle = \sum_\tau p_\tau f_\tau(x)$$

several species α

$$\sum_\alpha \int dx\, f_{\alpha,\tau}^2(x) = 1$$

position

$$X_\tau = \int dx\, x\, f_\tau^2(x)$$

classical observable :
fixed value for every state τ

momentum

- derivative observable

$$P_\tau = \int dx[f_{1,\tau}(x)\partial_x f_{2,\tau}(x) - f_{2,\tau}(x)\partial_x f_{1,\tau}(x)]$$

classical observable :
fixed value for every state τ

complex structure

$$f_\tau(x) = f_{1,\tau}(x) + i f_{2,\tau}(x)$$

$$\int dx\, f_\tau^*(x) f_\tau(x) = 1$$

$$P_\tau = \int dx\, f_\tau^*(x)(-i\partial_x) f_\tau(x)$$

$$X_\tau = \int dx\, f_\tau^*(x) x f_\tau(x)$$

$$P_\tau = \int dx [f_{1,\tau}(x)\partial_x f_{2,\tau}(x) - f_{2,\tau}(x)\partial_x f_{1,\tau}(x)]$$

classical product of position and momentum observables

$$\langle X \cdot P \rangle_{cl} = \langle P \cdot X \rangle_{cl} = \sum_{\tau} p_{\tau} X_{\tau} P_{\tau}$$

commutes !

different products of observables

$$(X^2)_\tau = \int dx f_\tau^*(x) x^2 f_\tau(x)$$

$$\langle X^2 \rangle = \sum_\tau p_\tau (X^2)_\tau$$

differs from classical product

$$\begin{aligned}\langle X \cdot X \rangle &= \sum_\tau p_\tau X_\tau^2 \\ &= \sum_\tau p_\tau \left(\int dx f_\tau^*(x) x f_\tau(x) \right)^2\end{aligned}$$

Which product describes correlations of measurements ?

coarse graining of information for subsystems

density matrix from coarse graining

- position and momentum observables use only small part of the information contained in p_τ ,

- relevant part can be described by density matrix

$$\rho(x, x') = \sum_\tau p_\tau f_\tau(x) f_\tau^*(x')$$

- subsystem described only by information which is contained in density matrix
- coarse graining of information

quantum density matrix

density matrix has the properties of a quantum density matrix

$$\mathrm{Tr}\rho = \int dx \rho(x, x) = 1 \, , \; \rho^*(x, x') = \rho(x', x)$$

$$\rho(x, x') = \sum_\tau p_\tau f_\tau(x) f_\tau^*(x')$$

quantum operators

$$\hat{X}(x',x) = \delta(x'-x)x$$
$$\hat{P}(x',x) = -i\delta(x'-x)\frac{\partial}{\partial x}$$

$$\langle X\rangle = \sum_\tau p_\tau X_\tau = \mathrm{Tr}(\hat{X}\rho) = \int dx x\rho(x,x)$$

$$\langle P\rangle = \sum_\tau p_\tau P_\tau = \mathrm{Tr}(\hat{P}\rho)$$
$$= -i\int dx' dx \delta(x'-x)\partial_x \rho(x,x')$$

quantum product of observables

the product

$$(X^2)_\tau = \int dx f_\tau^*(x) x^2 f_\tau(x)$$

$$\langle X^2 \rangle = \sum_\tau p_\tau (X^2)_\tau$$

is compatible with the coarse graining

$$\langle X^2 \rangle = \int dx x^2 \rho(x,x)$$

and can be represented by operator product

incomplete statistics

classical product

$$\langle X \cdot X \rangle = \sum_\tau p_\tau X_\tau^2$$
$$= \sum_\tau p_\tau \left(\int dx f_\tau^*(x) x f_\tau(x) \right)^2$$

- is not computable from information which is available for subsystem !
- cannot be used for measurements in the subsystem !

classical and quantum dispersion

$$\Delta_x^2 = \langle X^2\rangle - \langle X\rangle^2 \, , \, (\Delta_x^{(cl)})^2 = \langle X \cdot X\rangle - \langle X\rangle^2$$

$$\Delta_x^2 - (\Delta_x^{(cl)})^2 = \sum_\tau p_\tau \int dx \, f_\tau^*(x)(x - X_\tau)^2 f_\tau(x) \geq 0$$

$$\begin{aligned} \langle X \cdot X\rangle &= \sum_\tau p_\tau X_\tau^2 \\ &= \sum_\tau p_\tau \Big(\int dx \, f_\tau^*(x) x f_\tau(x) \Big)^2 \end{aligned}$$

$$(X^2)_\tau = \int dx \, f_\tau^*(x) x^2 f_\tau(x)$$

$$\langle X^2\rangle = \sum_\tau p_\tau (X^2)_\tau$$

subsystem probabilities

$$w(x) = \rho(x,x) = \sum_\tau p_\tau |f_\tau(x)|^2$$

$$w(x) \geq 0\,, \quad \int dx w(x) = 1$$

$$\langle X^n \rangle = \int dx x^n w(x)$$

in contrast :

$$\langle X \cdot X \rangle = \int dxdy\; xy\; w_{cl}(x,y)$$

$$w_{cl}(x,y) = \sum_\tau p_\tau |f_\tau|^2(x) |f_\tau|^2(y)$$

squared momentum

$$(P^2)_\tau = \int dx f_\tau^*(x)(-\partial_x^2) f_\tau(x)$$
$$= \int dx |\partial_x f_\tau(x)|^2$$

$$\langle P^2 \rangle = \sum_\tau p_\tau (P^2)_\tau = \mathrm{tr}(\hat{P}^2 \rho)$$
$$= \int dx dx' \delta(x' - x)(-\partial_x^2)\rho(x, x')$$

$$\langle P \cdot P \rangle = \sum_\tau p_\tau P_\tau^2$$
$$= -\sum_\tau p_\tau \left(\int dx f_\tau^*(x) \partial_x f_\tau(x) \right)^2$$

quantum product between classical observables :
maps to product of quantum operators

non – commutativity in classical statistics

$$(XP)_\tau = \int dx f_\tau^*(x) x(-i\partial_x) f_\tau(x)$$

$$(PX)_\tau = \int dx f_\tau^*(x)(-i\partial_x) x f_\tau(x)$$

$$\langle XP \rangle = \mathrm{tr}(\hat{X}\hat{P}\rho) \; , \; \langle PX \rangle = \mathrm{tr}(\hat{P}\hat{X}\rho)$$

$$XP - PX = i$$

commutator depends on choice of product !

measurement correlation

- correlation between measurements of positon and momentum is given by quantum product
- this correlation is compatible with information contained in subsystem

$$\langle XP \rangle_m = \frac{1}{2}(\langle XP \rangle + \langle PX \rangle)$$

coarse graining

from fundamental fermions at the Planck scale to atoms at the Bohr scale

$p([n_s])$

$\varrho(x\,,x')$

quantum mechanics from classical statistics

- probability amplitude
- entanglement
- interference
- superposition of states
- fermions and bosons
- unitary time evolution
- transition amplitude
- non-commuting operators
- violation of Bell's inequalities

conclusion

- quantum statistics emerges from classical statistics

  quantum state, superposition, interference, entanglement, probability amplitude
- unitary time evolution of quantum mechanics can be described by suitable time evolution of classical probabilities
- conditional correlations for measurements both in quantum and classical statistics

zwitters

zwitters

- no different concepts for classical and quantum particles
- continuous interpolation between quantum particles and classical particles is possible

(not only in classical limit)

quantum particle	classical particle
particle-wave duality	particles
uncertainty	sharp position and momentum
no trajectories	classical trajectories
tunneling	maximal energy limits motion
interference for double slit	only through one slit

quantum particle from classical probabilities in phase space

- probability distribution in phase space for

one particle

w(x,p)

$$w(x,p) \geq 0$$

$$\int_{x,p} w(x,p) = 1$$

as for classical particle !

- observables different from classical observables

- time evolution of probability distribution different from the one for classical particle

wave function for classical particle

classical probability distribution in phase space

$$w(x, p; t)$$

wave function for classical particle

$$\psi_C(x, p; t)$$

depends on position and momentum !

$$w = \psi_C^2$$

modification of evolution for classical probability distribution

$$i\hbar\frac{\partial}{\partial t}\psi_C = H_L\psi_C$$

$$H_L = -i\hbar L = -i\hbar\frac{p}{m}\frac{\partial}{\partial x} + i\hbar\frac{\partial V}{\partial x}\frac{\partial}{\partial p}$$

$$H_L \longrightarrow H_W$$

$$H_W = -i\hbar\frac{p}{m}\frac{\partial}{\partial x} + V\left(x + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) - V\left(x - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)$$

zwitter

difference between quantum and classical particles only through different time evolution

$$H_L = -i\hbar L = -i\hbar \frac{p}{m}\frac{\partial}{\partial x} + i\hbar \frac{\partial V}{\partial x}\frac{\partial}{\partial p}$$

CL

↕ continuous interpolation

$$H_W = -i\hbar \frac{p}{m}\frac{\partial}{\partial x} + V\left(x + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) - V\left(x - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)$$

QM

zwitter - Hamiltonian

$$H_\gamma = \cos^2\gamma H_W + \sin^2\gamma H_L$$

- $\gamma=0$: quantum – particle
- $\gamma=\pi/2$: classical particle

other interpolating Hamiltonians possible !

How good is quantum mechanics ?

small parameter γ can be tested experimentally

$$H_\gamma = \cos^2\gamma H_W + \sin^2\gamma H_L$$

zwitter : no conserved microscopic energy

static state : $H_\gamma \psi_C = 0$ or $[H_\gamma, \rho_Q] = 0$

experiments for determination or limits on zwitter – parameter γ ?

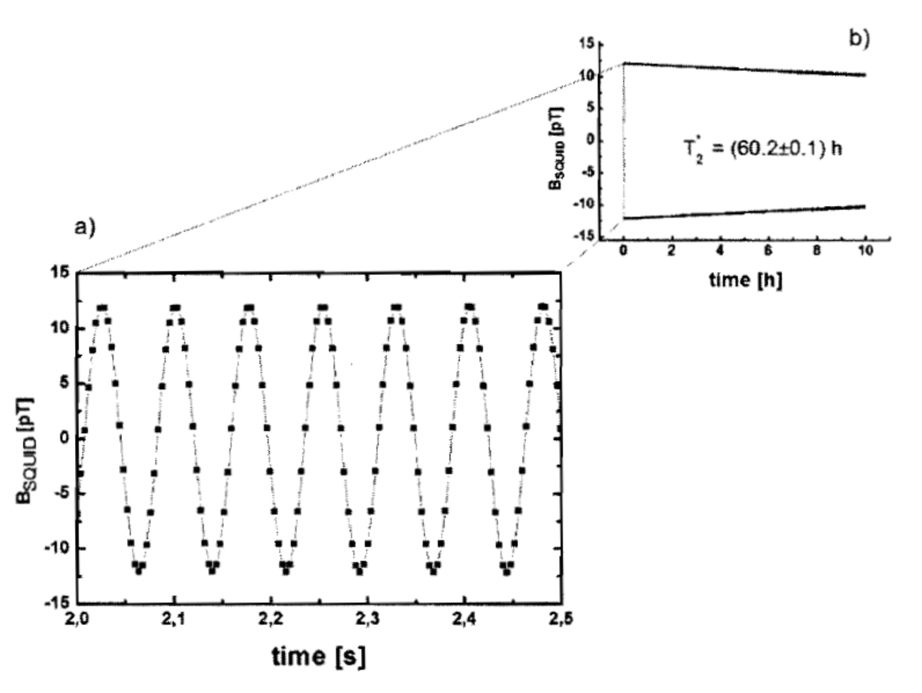


Fig.3 : a) Free spin-precession signal of a polarized ^3He sample cell recorded by means of a low-T_c SQUID (sampling rate: 250 Hz). b) Envelope of the decaying signal amplitude. From an exponential fit to the data, a transverse relaxation time of $T_2^* = (60.2 \pm 0.1)[h]$ can be deduced.

lifetime of nuclear spin states > 60 h (Heil et al.) : $\gamma < 10^{-14}$

Can quantum physics be described by classical probabilities ?

“ No go “ theorems

Bell , Clauser , Horne , Shimony , Holt

implicit assumption : use of classical correlation function for correlation between measurements

Kochen , Specker

assumption : unique map from quantum operators to classical observables