Lattice QCD near the light-cone

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Near light-cone coordinates

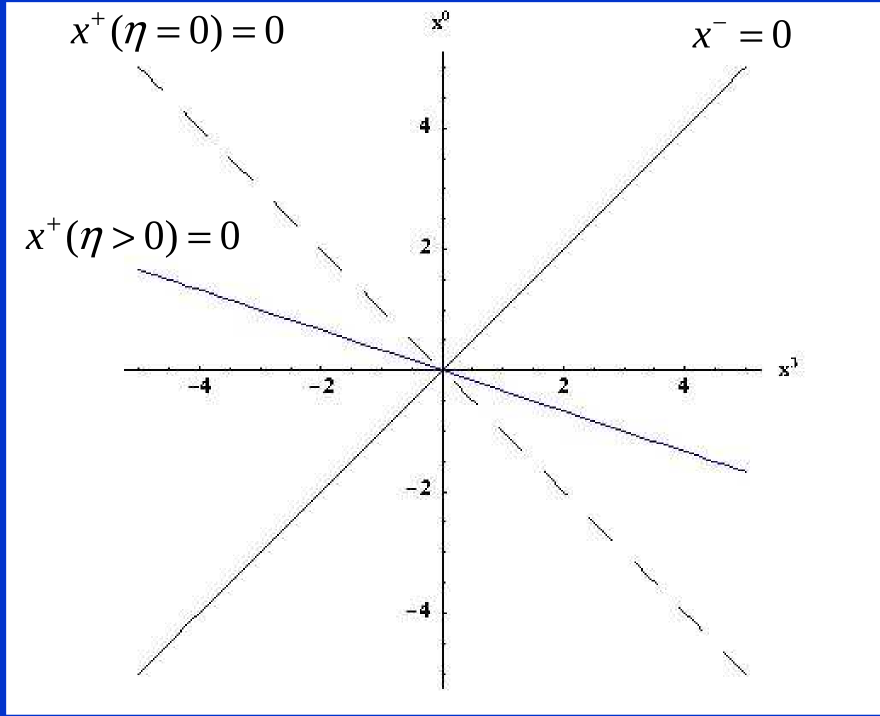
$$x^t = x^+ = \frac{1}{\sqrt{2}}\left\{\left(1+\frac{\eta^2}{2}\right)x^0 + \left(1-\frac{\eta^2}{2}\right)x^3\right\}$$

$$x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^3\right) .$$

x^1, x^2 are unchanged

- $\Leftrightarrow$ Boost with $\beta = \dfrac{1-\eta^2/2}{1+\eta^2/2}$

  \+ linear transformation



Nachtmann (**Eur.Phys.J.C7:459**): Meson-meson scattering amplitude governed by the correlation of two Wegner-Wilson loops near the light cone

$\Longrightarrow$ Near light cone coordinates are a promising tool in order to investigate high-energy scattering on the lattice

- Intention: Measure the correlation function of two Wegner-Wilson loops on the lattice
- Normal way of doing: Go over to Euclidean time
  Write the action in terms of links and plaquettes
  Sample the path integral by a Monte-Carlo
- Euclidean gluonic Lagrange density

$$x^+ = -\mathrm{i}\, x_E^+ \qquad S = \mathrm{i}\int d^4x_E \mathcal{L}_E \equiv \mathrm{i}S_E \qquad Z = \int DAe^{-S_E}$$

$$\mathcal{L}_E \equiv \frac{1}{2}F^a_{+-}F^a_{+-} + \sum_k \left(\frac{\eta^2}{2}F^a_{+k}F^a_{+k} - \mathrm{i}F^a_{+k}F^a_{-k}\right) + \frac{1}{2}F^a_{12}F^a_{12}$$

- a complex action remains $\Longrightarrow$ Ordinary MC sampling of the eucledian path integral is ruled out

$$\langle O\rangle = \left\langle e^{-i\,\mathrm{Im}(S_E)}O\right\rangle_{\mathrm{Re}(S_E)} \Big/ \left\langle e^{-i\,\mathrm{Im}(S_E)}\right\rangle_{\mathrm{Re}(S_E)}$$

- Possible way out: Hamiltonian formulation
  $\Longrightarrow$ Sampling of the ground state wavefunctional with guided diffusion quantum Monte-Carlo
- Derivation of the lattice Hamiltonian from the path integral formulation with the transfer matrix-method

transfer matrix method (Creutz Phys. Rev. D 15, 1128)

- Example: 1-d harmonic oscillator

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \qquad S = a\sum_i \left[\frac{1}{2}\left(\frac{x_{i+1}-x_i}{a}\right)^2 - \frac{\omega^2}{2}x_i^2\right] \qquad Z = \int \mathcal{D}x\, e^{iS}$$

- Observation: Path integral factorizes

$$Z = \int \prod_i \left(dx_i\, T_{x_{i+1},x_i}\right) \qquad T_{x',x} = exp\left\{\frac{\mathrm{i}}{2}\left[\frac{1}{a}(x'-x)^2 - a\omega^2 x^2\right]\right\}$$

Physically $T_{x',x}$ translates the system from one time-step to the other

- Construct: Hilbert-space

$$\langle\Psi'|\,|\Psi\rangle = \int dx \Psi'^*(x)\Psi(x)$$

$$\begin{aligned} \widehat{x}\,|x\rangle &= x\,|x\rangle \\ e^{-\mathrm{i}\widehat{p}r}\,|x\rangle &= |x+r\rangle \Rightarrow [\widehat{p},\widehat{x}] = -\mathrm{i} \end{aligned}$$

$$\begin{aligned} |\Psi\rangle &= \int dx \Psi(x)\,|x\rangle \\ \langle x'|\,|x\rangle &= \delta(x'-x) \\ \mathbb{1} &= \int dx\,|x\rangle\langle x| \end{aligned}$$

- Define the time translation operator to be:

$$\langle x'|\,\widehat{T}\,|x\rangle = T_{x',x}$$

- Rewrite as integral over all possible translations:

$$\begin{aligned}\widehat{T} &= \int d\Delta e^{1/(2a)\mathrm{i}\Delta^2} e^{-\mathrm{i}\widehat{p}\Delta} e^{-\mathrm{i}a\omega^2\widehat{x}^2/2} \\ &\propto e^{-\mathrm{i}a\widehat{p}^2/2} e^{-\mathrm{i}a\omega^2\widehat{x}^2/2} \\ &= e^{-\mathrm{i}a\widehat{H}+\mathcal{O}(a^2)}\end{aligned}$$

- The Hamiltonian is given by:

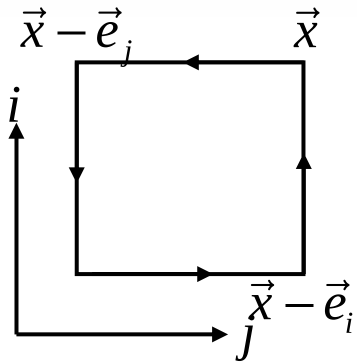
$$\widehat{H} = \lim_{a\to 0} \frac{1}{-\mathrm{i}a} \log\left(\widehat{T}\right)$$

- Similar considerations for the SU(2) near light-cone action

$$\begin{aligned}S_{lat} = \frac{2}{g^2}\sum_x \Bigg\{ &-\frac{a_\perp^2}{a_+a_-} Tr\left[Re\left(U_-(x-e_+)U_-^\dagger(x)\right)\right] \\ &+\frac{a_+a_-}{a_\perp^2} Tr\left[Re\left(U_{12}(x)\right)\right] \\ &+\sum_k Tr\left[Im\left(U_k(x-e_+)U_k^\dagger(x)\right) Im\left(U_{-k}(x)\right)\right] \\ &-\frac{a_-}{a_+}\eta^2 \sum_k Tr\left[Re\left(U_k(x-e_+)U_k^\dagger(x)\right)\right]\Bigg\} \\ &+\frac{2}{g^2}\sum_x Tr\left[2\eta^2\frac{a_-}{a_+} + \frac{a_\perp^2}{a_+a_-} - \frac{a_+a_-}{a_\perp^2}\right]\end{aligned}$$

$$U_i(x) \equiv \mathcal{P}\exp\left(\mathrm{i}g\int_{x-\widehat{e}_i}^{x} dy^\mu A_\mu^a(y)\lambda_a\right)$$

$$U_{ij}(x) = U_i(x)U_j(x-\widehat{e}_i)U_i^\dagger(x-\widehat{e}_j)U_j^\dagger(x)$$



$$
\begin{aligned}
&T\left(\left\{U^{(x_+-a_+)}\right\},\left\{U^{(x_+)}\right\}\right) = \\
&\quad \exp\left[\!\!\left[\mathrm{i}\frac{2}{g^2}\frac{a_\perp^2}{a_+a_-}\sum_{\vec{x}} Tr\left[Re\left(U_-(\vec{x},x_+-a_+)U_-^\dagger(\vec{x},x_+)\right)\right]\right]\!\!\right] \\
&\quad \times \exp\left[\!\!\left[\mathrm{i}\frac{2}{g^2}\sum_{\vec{x},k}\left\{\frac{a_-}{a_+}\eta^2 Tr\left[Re\left(U_k(\vec{x},x_+-a_+)U_k^\dagger(\vec{x},x_+)\right)\right]\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - Tr\left[Im\left(U_k(\vec{x},x_+-a_+)U_k^\dagger(\vec{x},x_+)\right)Im\left(U_{-k}(\vec{x},x_+)\right)\right]\right\}\right]\!\!\right] \\
&\quad \times \exp\left[\!\!\left[-\mathrm{i}\frac{2}{g^2}\sum_{\vec{x}}\left\{\frac{a_+a_-}{a_\perp^2}Tr\left[Re\left(U_{12}(\vec{x},x_+)\right)\right]\right\}\right]\!\!\right] \\
&\quad \times \exp\left[\!\!\left[-\mathrm{i}\frac{2}{g^2}\sum_{\vec{x}} Tr\left[2\eta^2\frac{a_-}{a_+}+\frac{a_\perp^2}{a_+a_-}-\frac{a_+a_-}{a_\perp^2}\right]\right]\!\!\right] \qquad (4)
\end{aligned}
$$

- Hilbert-Space

$$\langle \Psi' | \Psi \rangle = \prod_{\vec{x}} \prod_{j=1,2,-} \left(\int dg \Psi'^{*}_{\vec{x},j}(g) \Psi_{\vec{x},j}(g) \right) \quad |\{U\}\rangle = \prod_{\vec{x},j} |U_j(\vec{x})\rangle \quad j = 1, 2, -$$

$$\mathbb{1} = \int \mathcal{D}\{U\} |\{U\}\rangle \langle \{U\}| \quad \langle \{U'\} | \{U\} \rangle = \prod_{\vec{x},j} \delta \left(U'_j(\vec{x}), U_j(\vec{x}) \right)$$

- Define time translation operator

$$\left\langle \left\{ U^{(x_+ - a_+)} \right\} \middle| \widehat{T} \middle| \left\{ U^{(x_+)} \right\} \right\rangle \equiv T \left(\left\{ U^{(x_+ - a_+)} \right\}, \left\{ U^{(x_+)} \right\} \right)$$

- Introduce Matrix valued operators

$$\begin{aligned} \widehat{U}_j(\vec{x}) |\{U\}\rangle &= U_j(\vec{x}) |\{U\}\rangle \\ \widehat{R}\big(g_j(\vec{x})\big) |\{U\}\rangle &= |\{U'\}\rangle \end{aligned} \qquad |\{U'\}\rangle = \prod_{\vec{x}' \neq \vec{x}, j' \neq j} \left| U_{j'}(\vec{x}') \right\rangle \left| g_j(\vec{x}) U_j(\vec{x}) \right\rangle$$

$$\widehat{R}\big(g_j(\vec{x})\big) = e^{\mathrm{i}\Theta^a_j(\vec{x})\widehat{\Pi}^a_j(\vec{x})} \quad \text{for} \quad g_j(\vec{x}) = e^{\mathrm{i}\Theta^a_j(\vec{x})\lambda^a}$$

$$\begin{aligned} \left[\widehat{\Pi}^a_j(\vec{x}), \widehat{\Pi}^b_{j'}(\vec{x}') \right] &= \mathrm{i} f_{abc} \widehat{\Pi}^c_k(\vec{x}) \delta_{j,j'} \delta_{\vec{x},\vec{x}'} \\ \left[\widehat{\Pi}^a_j(\vec{x}), \widehat{U}_{j'}(\vec{x}') \right] &= -\lambda^a \widehat{U}_j(\vec{x}) \delta_{j,j'} \delta_{\vec{x},\vec{x}'} \\ \left[\widehat{\Pi}^a_j(\vec{x}), \widehat{U}^\dagger_{j'}(\vec{x}') \right] &= \widehat{U}^\dagger_j(\vec{x}) \lambda^a \delta_{j,j'} \delta_{\vec{x},\vec{x}'} \\ \left[\widehat{\Pi}_j(\vec{x})^2, \widehat{\Pi}^b_{j'}(\vec{x}') \right] &= 0 \end{aligned}$$

The lattice Hamiltonian

$$\widehat{H} = \sum_{\vec{x}} \left[\!\left[\frac{1}{2} g^2 \frac{1}{a_-} \sum_{k,a} \frac{1}{\eta^2} \left\{ \widehat{\Pi}^a_k(\vec{x}) - \frac{2}{g^2} Tr\left[\lambda^a Im\left(\widehat{U}_{-k}(\vec{x}) \right) \right] \right\}^2 + \frac{1}{2} g^2 \frac{a_-}{a_\perp^2} \sum_a \widehat{\Pi}^a_-(\vec{x})^2 + \frac{2}{g^2} \frac{a_-}{a_\perp^2} Tr\left[\mathbb{1} - Re\left(\widehat{U}_{12}(\vec{x}) \right) \right] \right]\!\right]$$

- Dominant part is similar to a particle coupled to a vector potential in ordinary QM
- Guided QMC is not applicable: Local energy

$$E_L(\{U\}) = \hat{H}\Phi(\{U\})$$

is complex (branching process) $\Longrightarrow$ large fluctuations once again

- $\Longrightarrow$ variational optimization of the ground state wavefunctional
  - Try to solve the dominant part
  - Variationally optimize the full Hamiltonian with respect to the total energy

Single site Hamiltonian (strong coupling limit)

Polar representation of a SU(2) matrix

$$U_{-k}(\vec{x}) = \cos\left(\frac{1}{2}B_p\right) + i\hat{n}_p^a\tau^a \sin\left(\frac{1}{2}B_p\right) \Rightarrow$$

$$\cos\left(\frac{1}{2}B_p\right) = \frac{1}{2}Tr\left(U_p\right)$$

$$Tr\left[\lambda^a Im\left(U_p\right)\right] = \hat{n}_p^a \sin\left(\frac{1}{2}B_p\right)$$

Dominant part of the Hamiltonian is given by

$$\widehat{H}_0 = \frac{1}{\eta^2}\frac{1}{a_-\beta'}\sum_a\left[\widehat{\Pi}^a - \beta'\hat{n}_p^a \sin\left(\frac{1}{2}B_p\right)\right]^2 \qquad \beta' = \frac{2}{g^2}$$

Momentum operator applied to cos-term yields

$$\widehat{\Pi}^a \cos\left(\frac{1}{2}B_p\right) = \frac{1}{2}Tr\left[U_p\frac{\tau^a}{2}\right] = \frac{i}{2}\hat{n}_p^a \sin\left(\frac{1}{2}B_p\right)$$

Consider the momentum operator as a differential operator acting on Bp and np

$$\widehat{\Pi}^a \cos\left(\frac{1}{2}B_p\right) = -\frac{1}{2}\sin\left(\frac{1}{2}B_p\right)\widehat{\Pi}^a B_p \qquad \widehat{\Pi}^a B_p = -i\hat{n}_p^a$$

One obtains a Schrödinger equation in Bp

$$\widehat{H}_0\Psi(B_p) = \frac{1}{\eta^2}\frac{1}{a_-\beta'}\left[-\frac{\partial^2}{\partial B_p^2} - \cot\left(\frac{1}{2}B_p\right) + \frac{3}{2}\mathrm{i}\beta'\cos\left(\frac{1}{2}B_p\right) + 2\mathrm{i}\beta'\sin\left(\frac{1}{2}B_p\right)\frac{\partial}{\partial B_p} + \beta'^2\sin\left(\frac{1}{2}B_p\right)^2\right]\Psi(B_p) = \epsilon\Psi(B_p)$$

with solution:

$$\Psi(B_p) = e^{-2\mathrm{i}\beta'\cos\left(\frac{1}{2}B_p\right)} \qquad \epsilon_0 = 0$$

Single site Hamiltonian is unitary equivalent to a Hamiltonian of a free particle:

$$\frac{1}{2}\frac{1}{\eta^2}e^{-\mathrm{i}\frac{2}{g^2}Tr\left[Re\left(\widehat{U}_{-k}(\vec{x})\right)\right]}\,\widehat{\Pi}_k^a(\vec{x})^2\,e^{\mathrm{i}\frac{2}{g^2}Tr\left[Re\left(\widehat{U}_{-k}(\vec{x})\right)\right]} = \frac{1}{\eta^2}\widehat{H}_0(\vec{x})$$

Solution of the discrete non-compact Hamiltonian

Discrete non compact Hamiltonian for $A_- = 0$ (violates periodicity) :

$$\widehat{H}_0 = \frac{1}{2}\frac{1}{\eta^2}\sum_{\vec{x}}\sum_{k,a}\left\{\widehat{\Pi}_k^a(\vec{x}) - \frac{1}{2}\frac{\widehat{A}_k^a(\vec{x}+\hat{e}_-) - \widehat{A}_k^a(\vec{x}-\hat{e}_-)}{a_-}\right\}^2 + \mathcal{O}(a$$

Find the ground state with standard many body methods:

- Coupled in „-“ direction $\Longrightarrow$ Fourier space in „-“ direction
- Perform Bogoliubov-Transformation to diagonalize the Hamiltonian
- Express ground state wavefunctional in terms of the original variables

$$\Psi_0(A) = \exp\left\{-\frac{1}{2}\frac{1}{L^2 a_-}\sum_{k,a}\sum_{\vec{x},\vec{x}',\vec{z}}\gamma(x_-,x'_-)\left[A_k^a(\vec{x}) - A_k^a(\vec{z})\right]\left[A_k^a(\vec{x}') - A_k^a(\vec{z})\right]\delta_{\vec{x}_\perp,\vec{x}'_\perp}\delta_{\vec{x}_\perp,\vec{z}_\perp}\right\}$$

$$\gamma(x_-,x'_-) \equiv \sum_{q_-}\left|\sin(q_- a_-)\right|\exp\left[iq_-\cdot\left(x'_- - x_-\right)\right] \quad q_- = \frac{2\pi}{L}n \quad E_0 = \frac{3}{\eta^2}N_1 N_2 \gamma(0,0)$$

Go over to link matrices

$$\Psi_0(U) = \exp\left\{ -\frac{1}{2}\frac{1}{g^2}\frac{1}{N_-^2}\sum_k \sum_{\vec{\mathbf{x}},\vec{\mathbf{x}}',\vec{\mathbf{z}}} \gamma(\mathbf{x}_-,\mathbf{x}'_-)\left[P_k(\vec{\mathbf{x}},\vec{\mathbf{x}}') - W_k(\vec{\mathbf{x}},\vec{\mathbf{x}}',\vec{\mathbf{z}})\right]\delta_{\vec{\mathbf{x}}_\perp,\vec{\mathbf{x}}'_\perp}\delta_{\vec{\mathbf{x}}_\perp,\vec{\mathbf{z}}_\perp}\right\}$$

$$P_k(\vec{x},\vec{x}') \equiv Tr\left[Re\left(U_k(\vec{x})U_k^\dagger(\vec{x}')\right)\right]$$

$$W_k(\vec{x},\vec{x}',\vec{z}) \equiv Tr\left[Re\left(U_k^\dagger(\vec{z})U_k(\vec{x})U_k^\dagger(\vec{z})U_k(\vec{x}')\right)\right]$$

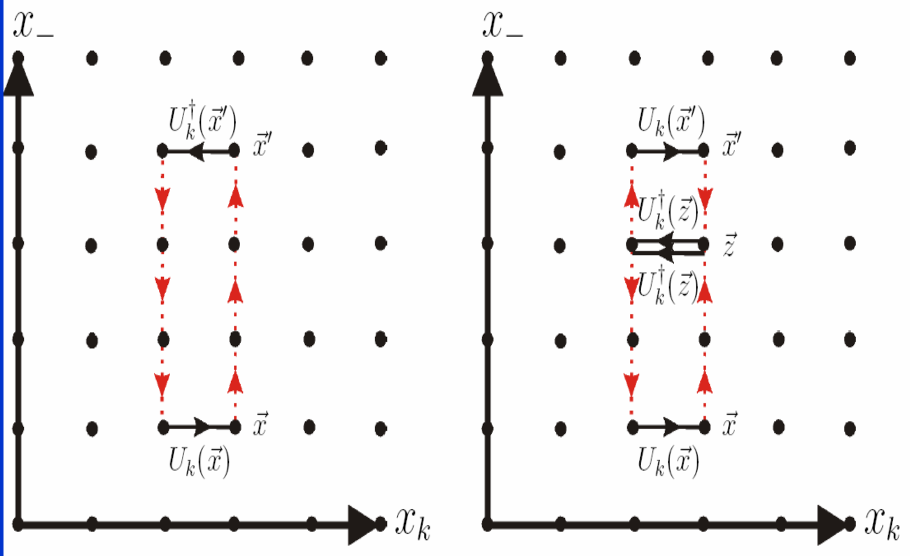


Figure 1: Left panel: $P_k(\vec{x},\vec{x}')$. Right panel: $W_k(\vec{x},\vec{x}',\vec{z})$.
The black arrows represent links in transversal direction. The red arrows represent links in longitudinal direction.

- $U_-(\vec{x}) = 1$
- $\Longrightarrow$ close loops by insertion of links in minus direction
- wavefunctional is gauge invariant

Computation of expectation values

$$\hat{\Pi}|0\rangle = 0 \qquad |\Psi_0\rangle = \hat{\Psi}_0|0\rangle \qquad \hat{\Psi}_0 = e^{F(\hat{U})}$$

Kinetic energy:

$$\langle\Psi_0|\,\widehat{\Pi}^2\,|\Psi_0\rangle = \langle\Psi_0|\left[\widehat{\Pi},\left[\widehat{\Pi},F(U)\right]\right]|\Psi_0\rangle + \langle\Psi_0|\left[\widehat{\Pi},F(U)\right]^2|\Psi_0\rangle$$

$$\langle\Psi_0|\,\widehat{\Pi}^2\,|\Psi_0\rangle = -\langle\Psi_0|\left[\widehat{\Pi},F(U)\right]^2|\Psi_0\rangle$$

$$\langle\Psi_0|\,\widehat{\Pi}^2\,|\Psi_0\rangle = \langle\Psi_0|\frac{1}{2}\left[\widehat{\Pi},\left[\widehat{\Pi},F(U)\right]\right]|\Psi_0\rangle$$

Linear momentum term:

$$\langle\Psi_0|\,\widehat{\Pi}G(\widehat{U}) + G(\widehat{U})\widehat{\Pi}\,|\Psi_0\rangle = 0$$

$$\eta^2\langle H_0\rangle = \frac{2}{\beta}\sum_{\vec{x},k,a}\left\langle\frac{1}{2}\Pi_k{}^a(\vec{x})^2\right\rangle + \beta\sum_{\vec{x},k,a}\left\langle\frac{1}{2}Tr\left[\lambda^a\,\mathrm{Im}\left(U_{-k}(\vec{x})\right)\right]^2\right\rangle$$

Numerical simulation

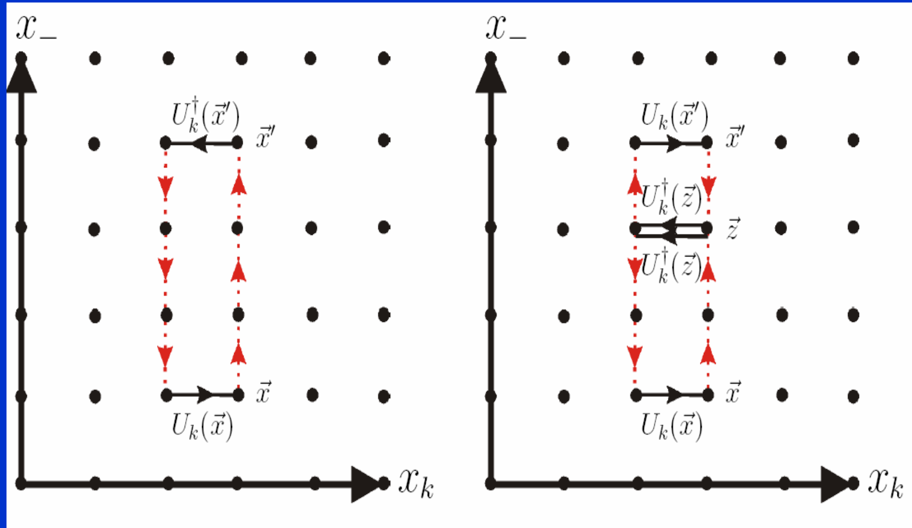
Sample : $\left|\hat{\Psi}_0\right|^2 = e^{2F(\hat{U})}$ Problem: „action" is not linear in the links

- Perform local heatbath update for linearized „action":

$$\exp\{Tr[US_0]+Tr[S_1US_2US_3]\} \to \exp(Tr[US_0]+\frac{1}{2}\{Tr[US_3S_1U_{old}S_2]+Tr[US_2U_{old}S_3S_1])$$

$$=\exp(Tr[US'])$$

- Perform an accept/reject step to correct for the linearization error with the correct „action"
- Perform an overrelaxation update for fixed linear part of the „action" in order to explore the configuration space
- Perform an accept/reject step to correct for the quadratic part



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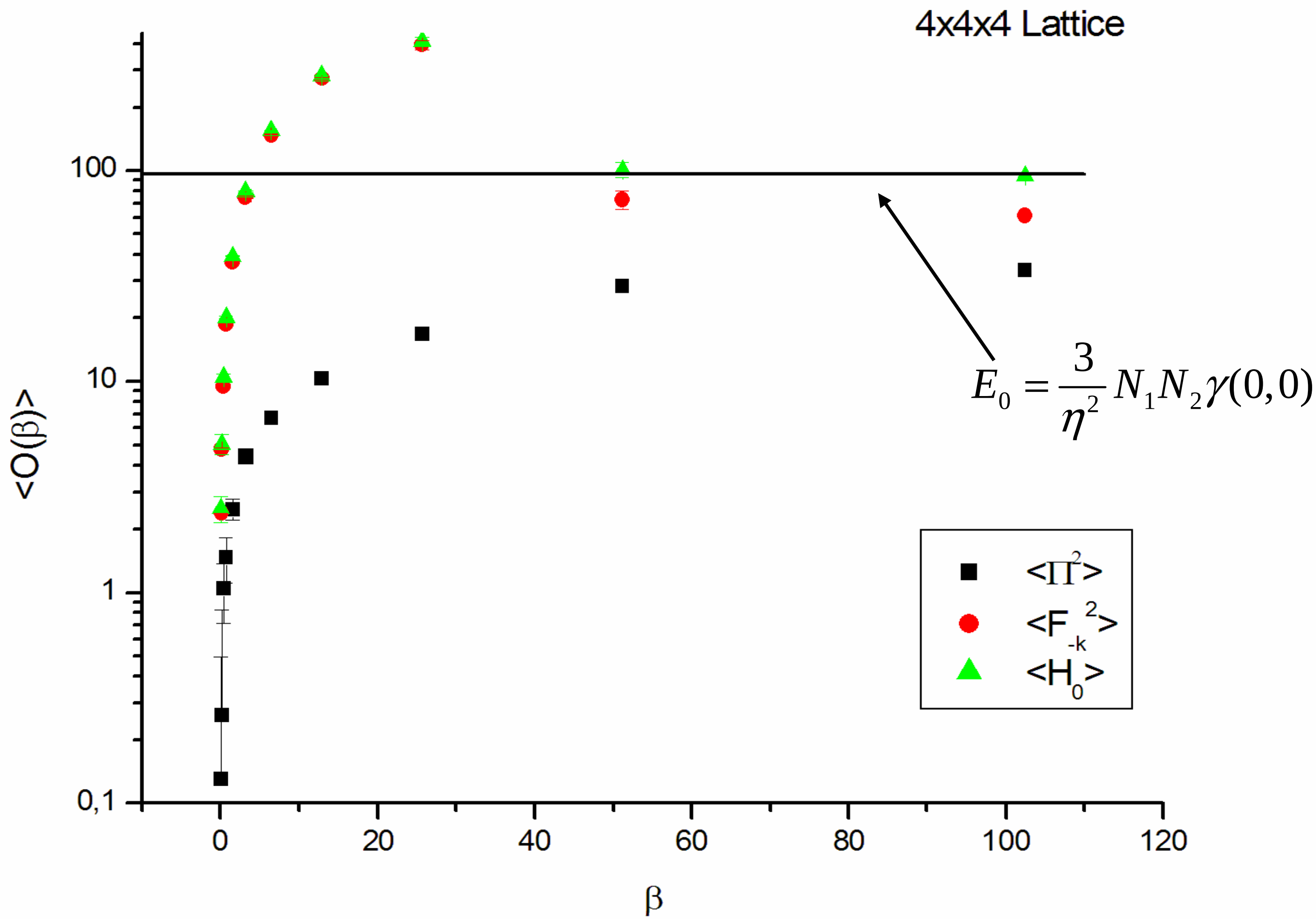
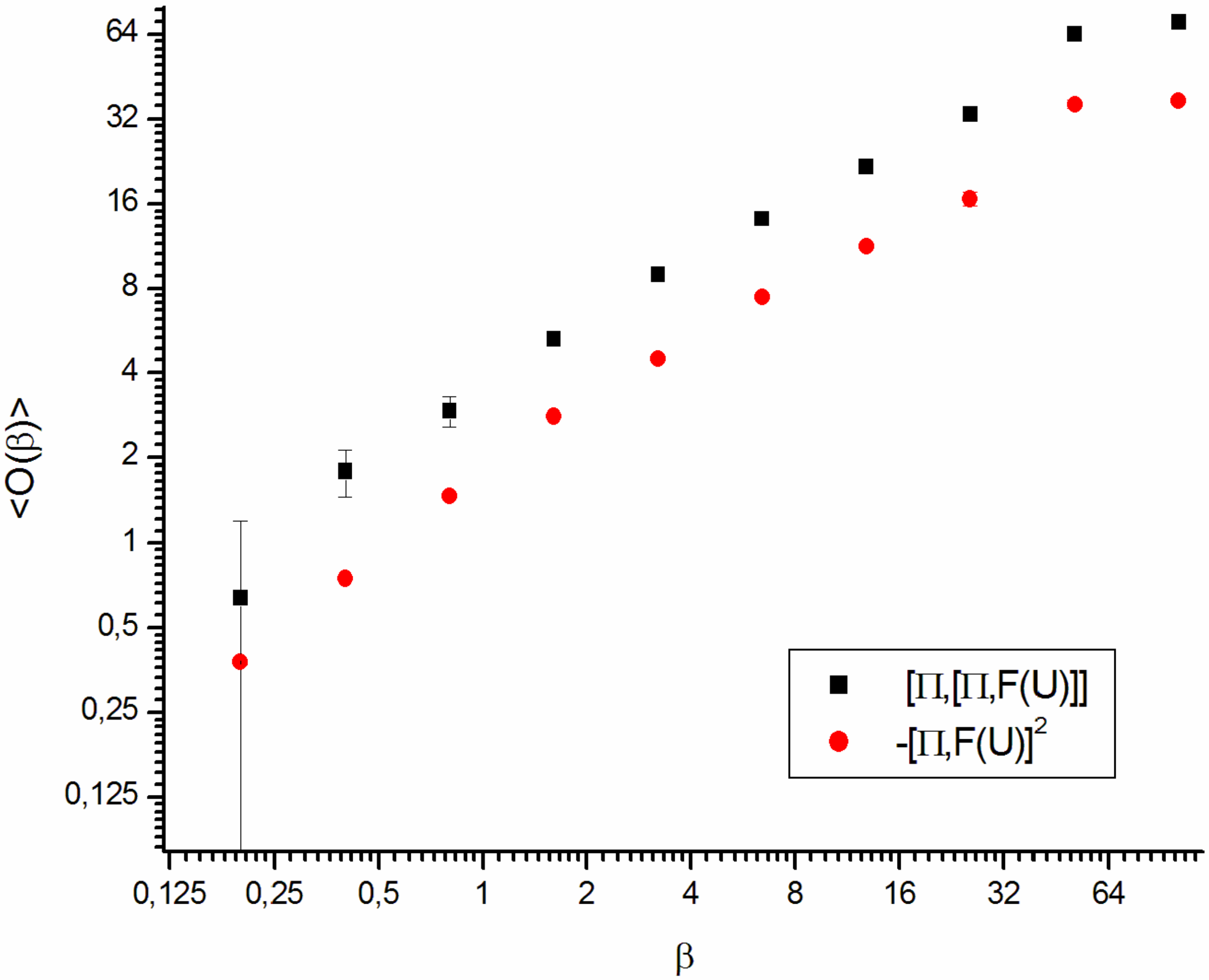

4x4x4 Lattice
<O(β)>
β
100
10
1
0,1
0
20
40
60
80
100
120
$E_0 = \frac{3}{\eta^2} N_1 N_2 \gamma(0,0)$
$<\Pi^2>$
$<F_{-k}^2>$
$<H_0>$

$$\langle\Psi_0|\frac{1}{2}\left[\widehat{\Pi},\left[\widehat{\Pi},F(U)\right]\right]|\Psi_0\rangle = -\langle\Psi_0|\left[\widehat{\Pi},F(U)\right]^2|\Psi_0\rangle$$



Conclusions:

- Near light cone coordinates seem to be a promising tool in order to describe high energy scattering on the lattice
- Eucledian path integral as well as Diffusion Quantum Monte Carlo treatment of the theory are inefficient due to complex phases during the update process
- We are beginning to study a variational approach to the ground state wavefunctional