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Control of adiabatic quantum evolution

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Dedica

Vorrei dedicare questa tesi a tutte le persone che mi hanno sostenuto
in questi anni durante il mio percorso universitario.

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Chapter 1

Introduction

Quantum mechanics generally has many applications in physics and quantum control theory is an important branch because it is the theory behind the design of quantum computers, sensors, and cryptography [1, 2, 3]. The goal of quantum control is to guide the evolution of a quantum system without destroying the coherence [4]. Richard Feynman proposed an interesting application as the construction of quantum computers, which permit a huge reduction of computational time due to the entanglement [5].

1.1 Aims of the thesis

Generally, the question of whether a quantum system is reversible or not is one of the most fundamental aspects in the field of quantum mechanics [6]: If our system is reversible during periodic cycles we can reduce significantly the loss of information. The aim of this thesis is to investigate how to control a three-level system, with a focus on accelerated adiabatic evolution by adding a counterdiabatic term to the main Hamiltonian. In the second chapter of this thesis, we will explain the necessary tools to comprehend the topics of this thesis.

1.2 Third chapter

In the third chapter, we will introduce our protocol where time evolution driven by the Schrödinger equation is constructed to be adiabatic. An adiabatic evolution in quantum mechanics occurs when the $|\psi(t)\rangle$ of the Schrödinger equation follows the starting eigenstate with a dynamic and a geometric phase, better known as Berry phase [7]. In order to get an adiabatic evolution we can choose two methods: the first one is using a very long experimental time, while the second one is adding a counterdiabatic term [8] to the main Hamiltonian. The advantage of using it is that we can significantly reduce the experimental time thus we can have a much lower computational load for our simulations [9] and, more importantly decoherence has no time to act.

1.3 Fourth chapter

Subsequently, in the fourth chapter, by adding some time dependent phase to the Hamiltonian I calculate the Berry phase [7], a geometric phase which can be linked to the area enclosed by the path of the Bloch vector on the Bloch sphere in a reduced two-level system evolving under an adiabatic evolution. Essentially, a different path caused by, e.g, some experimental noise implies a different area enclosed by the qubit, and therefore a different Berry phase. In this study the Berry phase serves as a control tool to investigate certain control imperfections.

1.4 Fifth chapter

In the fifth chapter, we will explore population decay by using a γ decay rate both through the Schrödinger equation and a master equation, particularly the Lindblad equation. An important quantum phenomenon that can be observed is the Zeno effect [10]. Specifically, the populations are analyzed as a function of γ and we observe an inversion of the trend: In fact, the populations continue to decrease until they reach a γ value after which they begin to increase. Notably, the

von Neumann entropy as a function of γ reaches the minimum in the same γ transition of the populations.

1.5 Conclusions

The final chapter discusses the conclusions and future prospects and applications of this work.

Chapter 2

Theoretical aspects

In quantum mechanics, the observables are described by linear Hermitian operators. An operator is hermitian if it is equal to its adjoint. In our case we will use a 3x3 matrix representation of the Hamiltonian because we study a three-level system. Since sometimes there will be interactions only between two eigenstates we will reduce a 3x3 Hamiltonian to a 2x2 [11] Hamiltonian to simplify the computations. Since H_{11} is a time-dependent term we obtain a matrix that is very similar to the Landau-Zener model [12, 13].

2.1 Landau-Zener

To understand our protocol, we introduce the Landau-Zener model, which is a two-level system described by a 2x2 Hamiltonian with a $\epsilon(t)$ function on the diagonals and a Δ on the out-diagonal terms [13, 12]. The Δ term is the element that guarantees the avoidance of crossing, where an eigenvalue almost approaches another energy level during the time evolution of the quantum state. The Landau-Zener Hamiltonian can be written as:

$$H(t) = \begin{pmatrix} \epsilon(t) & \Delta \\ \Delta & -\epsilon(t) \end{pmatrix}$$

Since the Hamiltonian is time dependent, the eigenvectors and the eigenvalues are time dependent and the eigenvalues equation is:

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle$$

and the Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle \quad (2.1)$$

The Schrödinger equation can be also rewritten by using time evolution operator U which is unitary in a closed quantum system so the equation becomes:

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \quad (2.2)$$

If H is not time-dependent

$$U(t, t_0) = e^{-\frac{i}{\hbar}H(t-t_0)} \quad (2.3)$$

If the Hamiltonian is time-dependent, like in the Landau-Zener case, if we want to calculate U it is necessary to use the Dyson series, which can be expressed as [14]:

$$U(t, t_0) = \mathcal{T} \exp \left(-\frac{i}{\hbar} \int_{t_0}^t H(t') dt' \right) \quad (2.4)$$

where $\mathcal{T}$ is a time-ordering operator which is necessary, because $H(t)$ does not commute with itself at different times. The resulting series is therefore written as

$$U(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t dt_1 H(t_1) \quad (2.5)$$

$$+ \left(-\frac{i}{\hbar} \right)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H(t_1) H(t_2) \quad (2.6)$$

$$+ \left(-\frac{i}{\hbar} \right)^3 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H(t_1) H(t_2) H(t_3) + \dots \quad (2.7)$$

2.2 QuTip software

Solving analytically the Schrödinger equation is generally impossible because it requires solving a system of differential equations with time-dependent coefficients. In order to have precise computations it is necessary to use advanced software like QuTip which is based on the python library [15]. Now we present a short list of the main commands of this software.

- `sesolve` is necessary to solve Schrödinger equation.
- `mesolve` is necessary to solve master equations.
- `linspace` divides an interval into equal parts.
- `arange` generates a well-defined numerical sequence.
- `Qobj` creates a quantum object.
- `basis` generates a set of basis vector.
- `expect` computes expectation values.

2.3 Fidelity

One of the goal of this work is to obtain a complete transfer of population from the state $|1\rangle$ to the state $|T\rangle$. To measure the efficiency of our protocol of transferring the entire population we must introduce the concept of fidelity [16]. We can define the fidelity as:

$$F(t) = |\langle T|\psi(t)\rangle|^2. \quad (2.8)$$

If we measure the fidelity at a time $t_f/2$, and if the protocol works correctly we should observe a value similar to 1.

2.4 Countediabatic term

To achieve high fidelity we must use a considerable experimental time. However, it is often inconvenient to use extended times for studying quantum protocols. So the only reasonable alternative is to add the counterdiabatic term H_{CD} to the main Hamiltonian. To derive the H_{CD} term we start from $|\psi(t)\rangle$ given by the adiabatic evolution [7]:

$$|\psi(t)\rangle = e^{i\gamma_n(t)} e^{-\frac{i}{\hbar} \int_0^t E_n(t') dt'} |n(t)\rangle \quad (2.9)$$

And now we take the time derivative

$$\partial_t |\psi(t)\rangle = i \dot{\gamma}_n(t) |\psi(t)\rangle - \frac{i}{\hbar} E_n(t) |\psi(t)\rangle + e^{i\gamma_n(t)} e^{-\frac{i}{\hbar} \int_0^t E_n(t') dt'} |\partial_t n(t)\rangle \quad (2.10)$$

And we put it equal to:

$$i\hbar \partial_t |\psi(t)\rangle = (H_0(t) + H_{CD}(t)) |\psi(t)\rangle \quad (2.11)$$

With

$$H_{CD}(t) |n(t)\rangle = -\hbar \dot{\gamma}_n(t) |n(t)\rangle + i\hbar |\partial_t n(t)\rangle \quad (2.12)$$

$$\langle m(t) | H_{CD}(t) | n(t) \rangle = \begin{cases} i\hbar \langle m(t) | \partial_t n(t) \rangle, & m \neq n \\ -\hbar \dot{\gamma}_n(t) + i\hbar \langle n(t) | \partial_t n(t) \rangle, & m = n \end{cases} \quad (2.13)$$

$$\dot{\gamma}_n(t) = i \langle n(t) | \partial_t n(t) \rangle \quad (2.14)$$

Which is the derivative of the Berry's phase

$$H_{CD}(t) = i\hbar \sum_n (|\partial_t n(t)\rangle \langle n(t)| - \langle n(t) | \partial_t n(t) \rangle |n(t)\rangle \langle n(t)|) \quad (2.15)$$

$$H_{CD}(t) = i\hbar \sum_n [\partial_t P_n(t), P_n(t)], \quad P_n(t) = |n(t)\rangle \langle n(t)| \quad (2.16)$$

Otherwise we can write the counterdiabatic term

$$\langle m(t) | \partial_t | n(t) \rangle = \frac{\langle m(t) | \partial_t H_0(t) | n(t) \rangle}{E_n(t) - E_m(t)} \quad (2.17)$$

And by consequence:

$$H_{\text{CD}}(t) = i\hbar \sum_{m \neq n} \frac{\langle m(t) | \partial_t H_0(t) | n(t) \rangle}{E_n(t) - E_m(t)} |m(t)\rangle \langle n(t)| \quad (2.18)$$

Thus in the Schrödinger equation we replace the main Hamiltonian $H_0(t)$ with $H_{\text{total}}(t) = H_{\text{CD}}(t) + H_0(t)$ [17]. The advantage of using the H_{CD} term is that we can obtain an adiabatic evolution with much shorter experimental time.

2.4.1 Berry phase

Thus in an adiabatic regime, if our protocol starts with an eigenstate $|n(t)\rangle$ of the Hamiltonian $H(t)$, its time evolution is given by:

$$|\psi(t)\rangle = e^{i\gamma_n(t)} e^{-\frac{i}{\hbar} \int_0^t E_n(t') dt'} |n(t)\rangle \quad (2.19)$$

where

- $E_n(t)$ is the starting eigenvalue of the Hamiltonian.
- $|n(t)\rangle$ is the associate eigenvector.
- $\gamma_n(t)$ is the geometric phase (Berry phase) defined by

$$\gamma_n(t) = i \int_0^t \langle n(t') | \frac{d}{dt'} | n(t') \rangle dt' \quad (2.20)$$

2.5 Bloch sphere

Since we often reduce our 3x3 Hamiltonian to a 2x2 Hamiltonian is possible to describe its evolution by using a Bloch vector on the Bloch sphere. The Bloch vector can be described by [18]:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle \quad (2.21)$$

with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.22)$$

where θ is the polar angle while ϕ is the azimuthal angle. When our evolution is adiabatic and the Bloch vector makes a closed loop the Berry phase has an important relationship with the solid angle [19]:

$$\gamma_n(t) = i \int_0^t \langle n(t') | \frac{d}{dt'} | n(t') \rangle dt' = -\frac{\Omega}{2} \quad (2.23)$$

In the following image we can see an example of Bloch sphere:

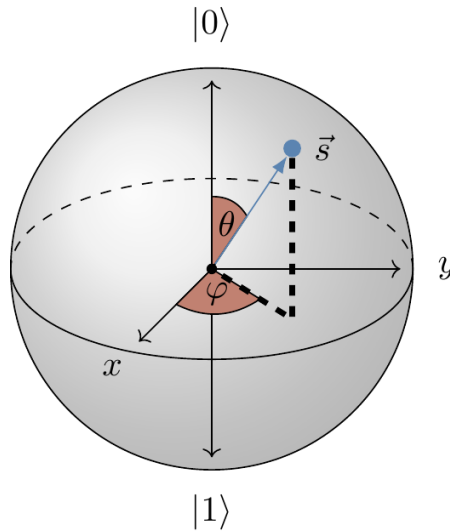


Figure 2.1: An example of Bloch sphere [18]

2.6 Density matrix

Another way to describe a quantum system is by using the density matrix ρ . If we have the maximum information of the quantum state it is a pure quantum state and we can define ρ as [20]:

$$\rho = |\psi\rangle\langle\psi|, \quad (2.24)$$

where $|\psi\rangle$ is the ket of the Schrödinger equation.

Instead if we do not have the maximum information we write ρ as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad (2.25)$$

where p_i is the probability of each state. Is it possible to determine the mean value of an operator A by using the density matrix:

$$\langle A \rangle = \text{Tr}(\rho A) \quad (2.26)$$

while the time evolution of the density matrix is described by this following equation :

$$i\hbar \frac{d\rho}{dt} = [H, \rho] \quad (2.27)$$

To demonstrate the previous relationship we start from (2.24) and we take the time derivative of the density matrix. So we obtain:

$$\frac{d\rho}{dt} = \frac{d}{dt} (|\psi(t)\rangle\langle\psi(t)|) = \left(\frac{d}{dt} |\psi(t)\rangle \right) \langle\psi(t)| + |\psi(t)\rangle \left(\frac{d}{dt} \langle\psi(t)| \right) \quad (2.28)$$

And developing the calculations we have:

$$\frac{d\rho}{dt} = \frac{1}{i\hbar} H |\psi(t)\rangle\langle\psi(t)| - \frac{1}{i\hbar} |\psi(t)\rangle\langle\psi(t)| H \quad (2.29)$$

$$\frac{d\rho}{dt} = \frac{1}{i\hbar} (H\rho - \rho H) = \frac{1}{i\hbar} [H, \rho] \quad (2.30)$$

Multiplying both terms for $i\hbar$ we obtain the original equation. For pure and mixed states, the density matrix has some important properties [20]:

- $\rho = \rho^\dagger$: Density matrix is Hermitian.
- $\text{Tr}(\rho) = 1$: Trace is always equal to 1.
- The eigenvalues of ρ are real and positive or equal to 0.
- $\rho^2 = \rho$ if and only if it's a pure state.
- $\text{Tr}(\rho^2) = 1$: if we have a pure state.
- $\text{Tr}(\rho^2) < 1$: if we have a mixed state.

2.7 Master equations

Until now we have described a closed quantum system but this type of protocols are rare in nature. We can often observe interactions between quantum systems and the external environment. To describe this phenomena we must use master equations particularly written as [21]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_k \left(\hat{L}_k \rho \hat{L}_k^\dagger - \frac{1}{2} \hat{L}_k^\dagger \hat{L}_k \rho - \frac{1}{2} \rho \hat{L}_k^\dagger \hat{L}_k \right) \quad (2.31)$$

- ρ is the density matrix.
- H is the Hamiltonian.
- $\hat{L}_k$ are the Lindblad operators which tune the interaction between the quantum system and the environment
- $[H, \rho]$ is the commutator H e ρ , i.e. $H\rho - \rho H$.

For example we can describe a decay of a level state with a decay rate γ by writing $\hat{L} = \sqrt{\gamma} |4\rangle \langle T|$.

Chapter 3

Protocol description

One of the main goals of this work is to control the time evolution of a few level system, particularly, a three-level system. Specifically, we want that if our evolution begins in a specific eigenstate, it must follow that state during time evolution possibly multiplied by a phase. This type of evolution is known as adiabatic evolution. To achieve this, we can choose two options: using a long experimental time, which guarantees a good control on our evolution or better we append a counterdiabatic term to the main Hamiltonian which permits an adiabatic evolution in a much shorter experimental time [22, 23, 24].

3.1 Molecular system

In some cases quantum control theory can be applied to a molecule. We start from an original structure of a lithium dimer, which is shown, e.g., in Fig. 2 in [25]. We have the starting state, the ground state $|1\rangle$ that corresponds to the rotational vibrational state $X^1\Sigma_g^+(\nu = 2, J = 22)$ in Li_2 , and the spin orbit coupled states $|T\rangle$ and $|S\rangle$ which correspond to $1^3\Sigma_g^-(\nu = 1, J = 21, f)$ and $G^1\Pi_g(\nu = 12, J = 21, f)$, respectively. The $|S\rangle$ and $|T\rangle$ Lithium molecular states of [26] have mixing coefficients $\alpha^2 = 0.87$ and $\beta^2 = 0.13$, with spin-orbit splitting $\Delta_{so} = 2\pi \cdot 0.75$ GHz equivalent to 4.71 ns^{-1} .

Molecules have energy levels that can be divided into triplet state

$|T\rangle$, which corresponds to a spin-symmetric wave function, and the singlet state $|S\rangle$, which has a spin-antisymmetric wave function [14]. Our protocol is based on a four-level system : the ground state $|1\rangle$, which has the lowest energy, the triplet state $|T\rangle$ which has an intermediate energy, and finally the singlet state $|S\rangle$ and the state $|2\rangle$ with the highest energy [26, 27, 28]. The intention of [29] is to obtain a complete population transfer from the state $|1\rangle$ to the triplet state $|T\rangle$. Our advantage is that the $(|S\rangle, |T\rangle)$ manifold of states has a mixed singlet-triplet symmetry owing to spin-orbit coupling. They originate from the $(|S_0\rangle, |T_0\rangle)$ states corresponding to unperturbed singlet and triplet states with original energy separation Δ_0 . It's possible to notice the presence of a spin-orbit interaction V that can be written as $H_{so} = V|S_0\rangle\langle T_0| + H.c$ with $\hbar = 1$. The mixed states $|S\rangle$ and $|T\rangle$ are then equal to

$$|S\rangle = \alpha|S_0\rangle - \beta|T_0\rangle, \quad |T\rangle = \beta|S_0\rangle + \alpha|T_0\rangle, \quad (3.1)$$

where *i.e.*, $|\alpha|^2 + |\beta|^2 = 1$. There is a relationship between the term Δ_0 which is the unperturbed energy separation, the perturbation V and effective energy splitting Δ_{so} which is [30]

$$\Delta_0 = (\alpha^2 - \beta^2)\Delta_{so} \quad V = \alpha\beta\Delta_{so}. \quad (3.2)$$

We can reach the singlet-triplet manifold due to a pump (p) laser which connects the $|1\rangle$ singlet to the $|S\rangle$ state with detuning δ_p . The singlet component of both $|S\rangle, |T\rangle$ eigenstates of Eq. (3.1) determines their excitation by the p laser characterized by Rabi frequencies $\alpha\Omega_p$ and $\beta\Omega_p$, respectively. The excitation is driven by the pump detuning from those states.

The quantum control of the singlet-triplet transfer introduced in [26, 27] is obtained by modulating the energy separation of the spin-orbit $|S\rangle, |T\rangle$ manifold. Such a modification is produced by a control (δ_c) off-resonant laser linking the $|T\rangle$ state to an additional $|2\rangle$ triplet one with detuning δ_c . The Rabi frequencies for the mixed manifold components are $-\beta\Omega_c$ and $\alpha\Omega_c$, respectively.

Overall, the time evolution of the optically driven four levels ($|1\rangle, |S\rangle, |T\rangle, |2\rangle$) is described by the Hamiltonian

$$H^{(4)} = \begin{pmatrix} \delta_p & \alpha\Omega_p/2 & -\beta\Omega_p/2 & 0 \\ \alpha\Omega_p/2 & 0 & 0 & \beta\Omega_c/2 \\ -\beta\Omega_p/2 & 0 & -\Delta_{so} & \alpha\Omega_c/2 \\ 0 & \beta\Omega_c/2 & \alpha\Omega_c/2 & -\delta_c - \Delta_{so} \end{pmatrix}. \quad (3.3)$$

By our choice the $|S\rangle$ state has an energy equal to zero as we can notice in the H_{22} term of the Hamiltonian.

3.1.1 The adiabatic reduction

As we can see in [29], for $|\delta_c| \gg \Omega_c$, corresponding to a virtual excitation of the $|2\rangle$ state, $|2\rangle$ can be adiabatically eliminated. To do this we begin with a system of four differential equations derived by the Schrödinger equation using the coefficients: $C_1(t), C_S(t), C_T(t), C_2(t)$

$$i\dot{C}_1 = \delta_p C_1 + \alpha \frac{\Omega_p}{2} C_S - \beta \frac{\Omega_p}{2} C_T \quad (3.4)$$

$$i\dot{C}_S = \alpha \frac{\Omega_p}{2} C_1 + \beta \frac{\Omega_c}{2} C_2 \quad (3.5)$$

$$i\dot{C}_T = -\beta \frac{\Omega_p}{2} C_1 - \Delta_{so} C_T + \alpha \frac{\Omega_c}{2} C_2 \quad (3.6)$$

$$i\dot{C}_2 = \beta \frac{\Omega_c}{2} C_S + \alpha \frac{\Omega_c}{2} C_T - (\Delta_{so} + \delta_c) C_2 \quad (3.7)$$

Since $|\delta_c| \gg \Omega_c$, we can adiabatically eliminate the state $|2\rangle$ putting $\dot{C}_2 \approx 0$, then we have:

$$C_2 \approx \frac{1}{2(\delta_c + \Delta_{so})} (\beta\Omega_c C_S + \alpha\Omega_c C_T) \quad (3.8)$$

Substituting this expression into the preceding equations yields the

reduced three-level system.

$$i\dot{C}_1 = \delta_p C_1 + \frac{\Omega_p}{2} C_S - \beta \frac{\Omega_p}{2} C_T \quad (3.9)$$

$$i\dot{C}_S = \alpha \frac{\Omega_p}{2} C_1 + \frac{\beta^2 \Omega_c^2}{4(\delta_c + \Delta_{so})} C_S + \frac{\alpha \beta \Omega_c^2}{4(\delta_c + \Delta_{so})} C_T \quad (3.10)$$

$$i\dot{C}_T = -\beta \frac{\Omega_p}{2} C_1 + \frac{\alpha \beta \Omega_c^2}{4(\delta_c + \Delta_{so})} C_S + (-\Delta_{so} + \frac{\alpha^2 \Omega_c^2}{4(\delta_c + \Delta_{so})}) C_T \quad (3.11)$$

We can notice that the terms in front of the coefficients are the same elements of the 3x3 Hamiltonian that we have introduced in the previous subsection

3.2 The state reduction from four to three levels

So now we have a new three-level system [16] .

$$H^{(3)} = \begin{pmatrix} \delta_p & \frac{\alpha \Omega_p}{2} & -\frac{\beta \Omega_p}{2} \\ \frac{\alpha \Omega_p}{2} & \frac{\beta^2 \Omega_c^2}{4(\delta_c + \Delta_{so})} & \frac{\alpha \beta \Omega_c^2}{4(\delta_c + \Delta_{so})} \\ -\frac{\beta \Omega_p}{2} & \frac{\alpha \beta \Omega_c^2}{4(\delta_c + \Delta_{so})} & -\Delta_{so} + \frac{\alpha^2 \Omega_c^2}{4(\delta_c + \Delta_{so})} \end{pmatrix}, \quad (3.12)$$

in the basis $\{|1\rangle, |S\rangle, |T\rangle\}$. Notice that within the effective energy of the state $|T\rangle$ the light-shift or ac shift δ_{ls} given by

$$\delta_{ls} = \frac{\alpha^2 \Omega_c^2}{4(\delta_c + \Delta_{so})}, \quad (3.13)$$

associated to the Autler-Townes process.

3.3 3x3 Hamiltonian

In our three-level system, we can distinguish the ground state $|1\rangle$ which has the lowest energy, the triplet state $|T\rangle$ and the singlet state $|S\rangle$. Our goal is to start the adiabatic evolution from the state $|1\rangle$ in order to observe the avoiding crossing with the triplet state $|T\rangle$ and after this returning symmetrically to state $|1\rangle$. By doing this we expect a complete transfer of population from the ground state to the triplet state. The following image shows precisely our protocol scheme:

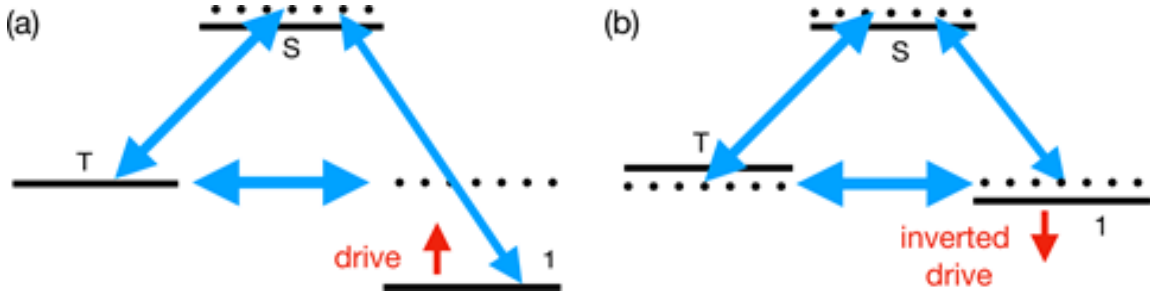


Figure 3.1: Image which explains our protocol scheme.

For simplicity we can rewrite the Hamiltonian as it follows:

$$H_0(t) = \begin{pmatrix} \epsilon_1(t) & \Omega_{1S} & \Omega_{1T} \\ \Omega_{1S}^* & \epsilon_S & \Omega_{ST} \\ \Omega_{1T}^* & \Omega_{ST}^* & \epsilon_T \end{pmatrix}. \quad (3.14)$$

where the exact values are listed in the appendix C. In our Hamiltonian, we have the $\epsilon(t)$ term which drives the time evolution of the eigenstate with the lowest energy. Since our goal is the avoiding crossing of the ground state with the triplet state symmetrically, we choose two symmetric functions: one is based on the atan function and the second one is an eight-degree polynomial fit of the atan function. In the next subsection we report the two sweeps.

3.4 The two sweeps

The first sweep that we take into consideration is the atan function. This because is the best function that we can have to reach a complete transfer of population from $|1\rangle$ to the triplet state $|T\rangle$ [31]. This can be demonstrated by using the principle of minimal action on the Bloch sphere. In our case we write the atan function:

$$\epsilon_1(\tau) = f(\tau) \equiv \begin{cases} a \arctan(b\tau) - c & 0 \leq \tau < 0.5, \\ a \arctan(b(1-\tau)) - c & 0.5 \leq \tau \leq 1, \end{cases} \quad (3.15)$$

for real parameters $a = 10ns^{-1}$, $b = 20$, and $c = 18.6ns^{-1}$. The transfer to the state $|T\rangle$ occurs in the interval $0 \leq \tau < 0.5$, and back to the initial state in $0.5 < \tau \leq 1$. A slightly softer evolution is obtained by an eight-degree polynomial fit of the function $f(\tau)$, that reads:

$$p(\tau) = a_8\tau^8 + a_7\tau^7 + a_6\tau^6 + a_5\tau^5 + a_4\tau^4 + a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0, \quad (3.16)$$

where the coefficients are listed in appendix A. These two functions are similar but there is a key difference: the atan function has a cusp at a $\tau = 0.5$ caused by the discontinuity of the derivative in that temporal point while the polynomial fit does not have this problem. Since the derivative discontinuity of the atan can cause imperfections during the experiments and for the computation of the counterdiabatic term, we will adopt the polynomial fit in all the QuTip simulations.

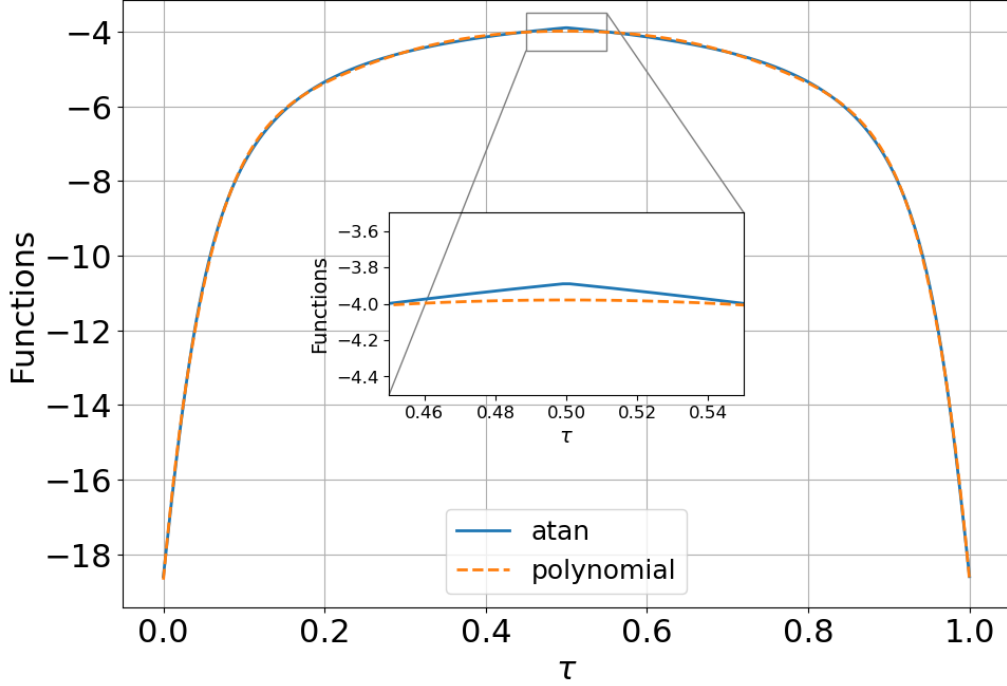


Figure 3.2: Confront between the two sweeps: it's possible to see that the two functions are almost equal if we exclude the cusp at a $\tau = 0.5$ of the atan function.

These functions control the time evolution of the ground state, and since it is the only time-dependent Hamiltonian term, it also influences the other two states $|T\rangle$ and $|S\rangle$. It's possible to visualize the eigenvalues of the three states guided by the atan function and the polynomial fit. Because these two sweeps drive the evolution of the ground state, we can notice that the eigenvalue of $|1\rangle$ has an evolution that reflects the graph behavior of the two functions. In the following image it's possible to see the evolution of the time-dependent eigenvalues.

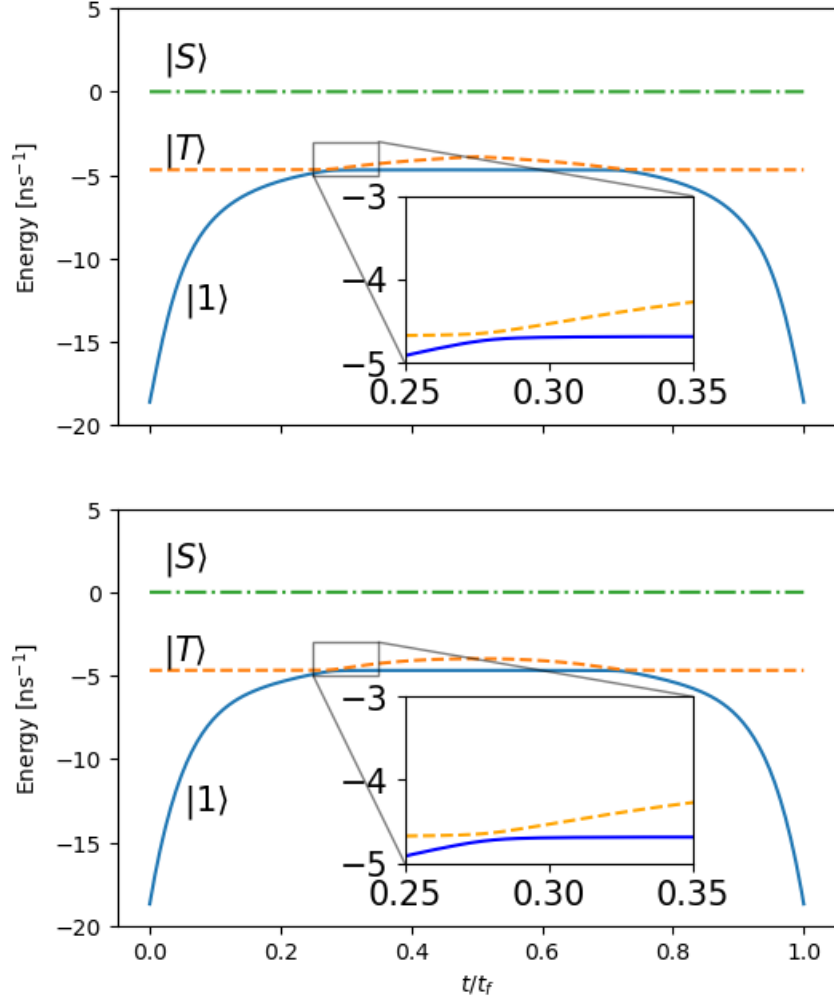


Figure 3.3: Instantaneous eigenvalues of $|1\rangle$, $|T\rangle$, and $|S\rangle$. We can notice the avoiding crossing between the ground state and the triplet state. In the first graphic the ground state is driven by the atan sweep, while in the second graphic the ground state is driven by the polynomial.

It's possible to notice the avoiding crossing between the ground state $|1\rangle$ and the triplet state $|T\rangle$ during time evolution. Now we compare time evolution of the populations of the ground state, the

triplet state, and the singlet state driven by the polynomial with the case driven by the atan function. We use a short experimental time $t_f = 500ns$ to demonstrate that it is impossible to achieve adiabatic evolutions without using the H_{CD} term in a short time. The following image shows the comparison between the populations driven by the two sweeps:

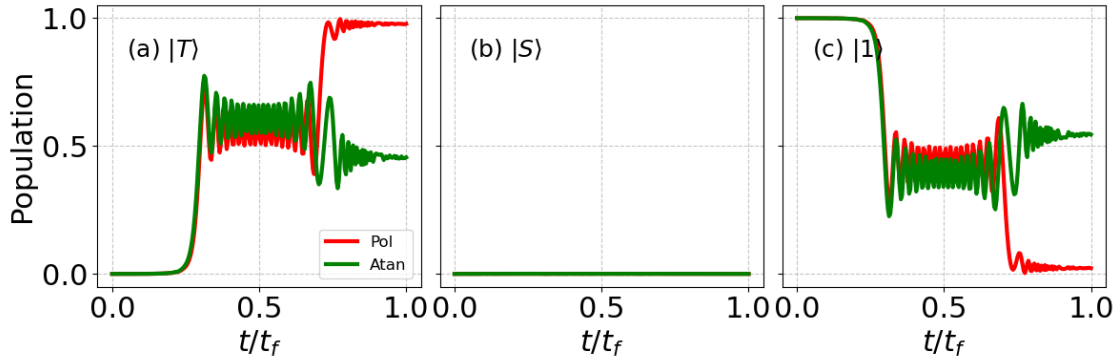


Figure 3.4: Populations trend at an experimental time $t_f = 500ns$ using the polynomial (red) and atan (green). It's possible to notice that neither populations $|T\rangle$ are able to reach 1 and go back to 0 symmetrically.

As we see, we can observe a high number of oscillations of the ground state and the triplet state during evolution. Since we want to avoid it, we subsequently use a much longer t_f . When we use a significantly longer experimental time, we can see a substantial reduction of oscillations and we can notice that, during time evolution, the population of the triplet state $|T\rangle$ reaches 1 in the middle of the graph, while the ground state population tends to 0. This effect is due to the particular evolution of the ground state, which is time dependent. In fact, when it reaches the energy of triplet state it becomes similar to the triplet state so the eigenvector resembles the $|T\rangle$ state, and we have a fidelity close to 1. This is because, during an adiabatic evolution, the state $\psi(t)$ governed by the Schrödinger equation can be written as the initial time-dependent eigenstate multiplied by a dynamic and a geometric phase. Therefore, the probability depends only on the time-

dependent eigenstate, since the squared modulus of the phases is equal to 1. The following image shows the effects of the two sweeps:

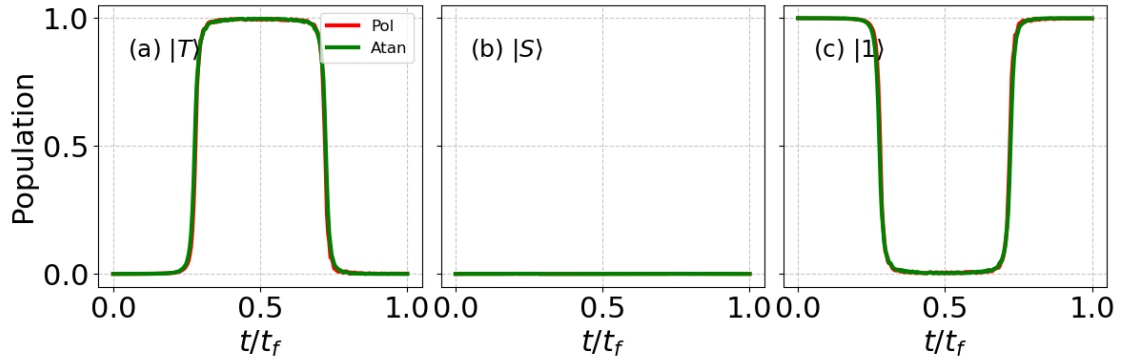


Figure 3.5: Populations of the three states $|1\rangle$, $|T\rangle$, $|S\rangle$ at the experimental time $t_f = 5000ns$. In this case, since the experimental time is long, there is an important reduction of the oscillations and the population of the triplet state is able to reach the maximum. This demonstrates that we can obtain adiabaticity using a very long t_f without H_{CD} .

Now we study the fidelity by using a time range t_f between 150ns and 5000ns inserting in the `sesolve` $t_f/2$ instead of t_f to observe the physical behavior at $\tau = 0.5$. Our goal is to verify that the two fidelities at a $t/t_f = 0.5$ reach 1. Two fidelities are listed in the following image:

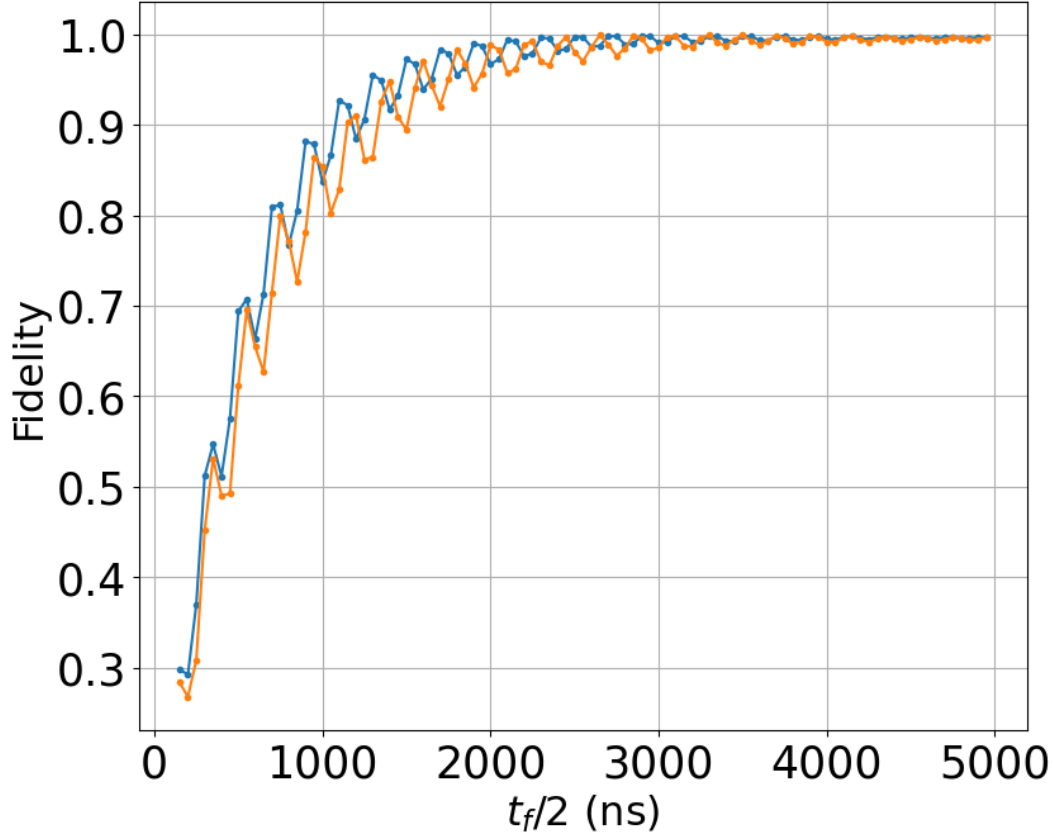


Figure 3.6: The two fidelities of the polynomial and the atan with a $150ns \leq t_f < 5000ns$. In this case in the `sesolve` we insert $t_f/2$ instead of t_f without using H_{CD} in order to check if the two fidelities reach 1 in the middle of the experiment. It's possible to notice that the oscillations undergo to a significant reduction as t_f increase. We use 50 points for the graph in order to avoid rapid oscillations. Anyway the two fidelities are very similar.

As mentioned earlier the eight-degree polynomial fit has not a derivative discontinuity at a $t/t_f = 0.5$ so in our simulations we choose it. In the next figure we can see the evolution of the three populations at a $t_f = 5000ns$.

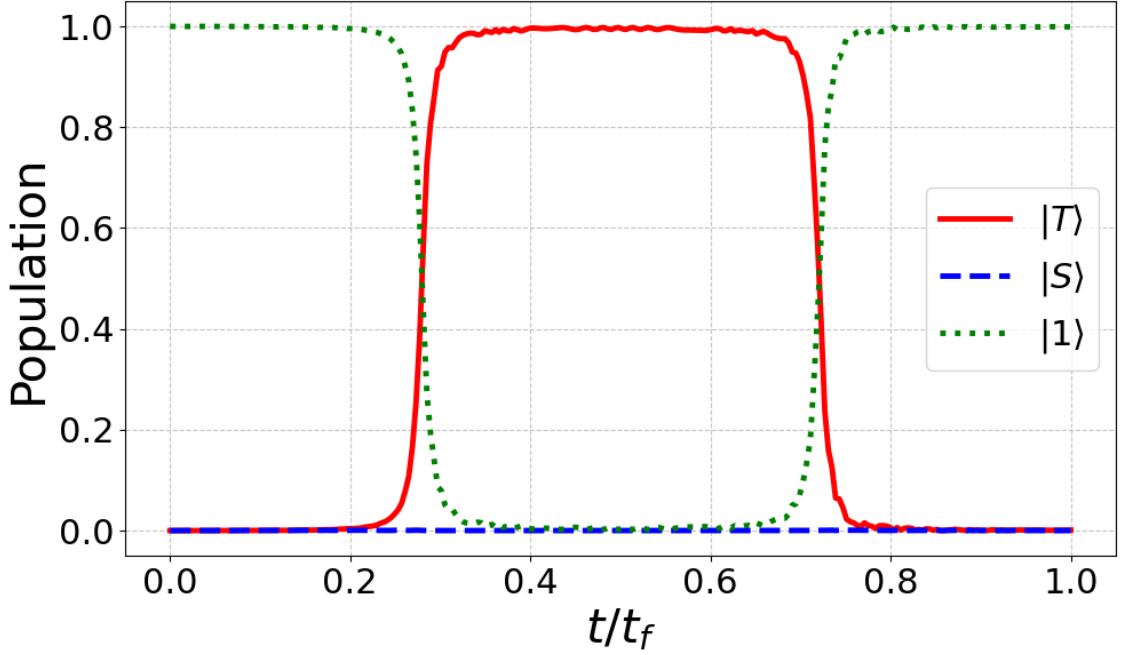


Figure 3.7: Time dependence of the populations at a experimental time $t_f = 5000ns$ using the polynomial. The oscillations are negligible due to the long t_f .

3.5 Populations with H_{CD}

Since we need a much shorter experimental time for experimental and computational reasons, we add the counterdiabatic term to the main Hamiltonian. The next figure shows the time evolution of $|1\rangle, |T\rangle$, and $|S\rangle$ driven by the polynomial sweep. Subsequently, we use H_{CD} to have a perfect adiabatic evolution using a shorter time t_f . It's worth noting the almost complete disappearance of the oscillations and some other non-adiabatic effects. The following image shows this:

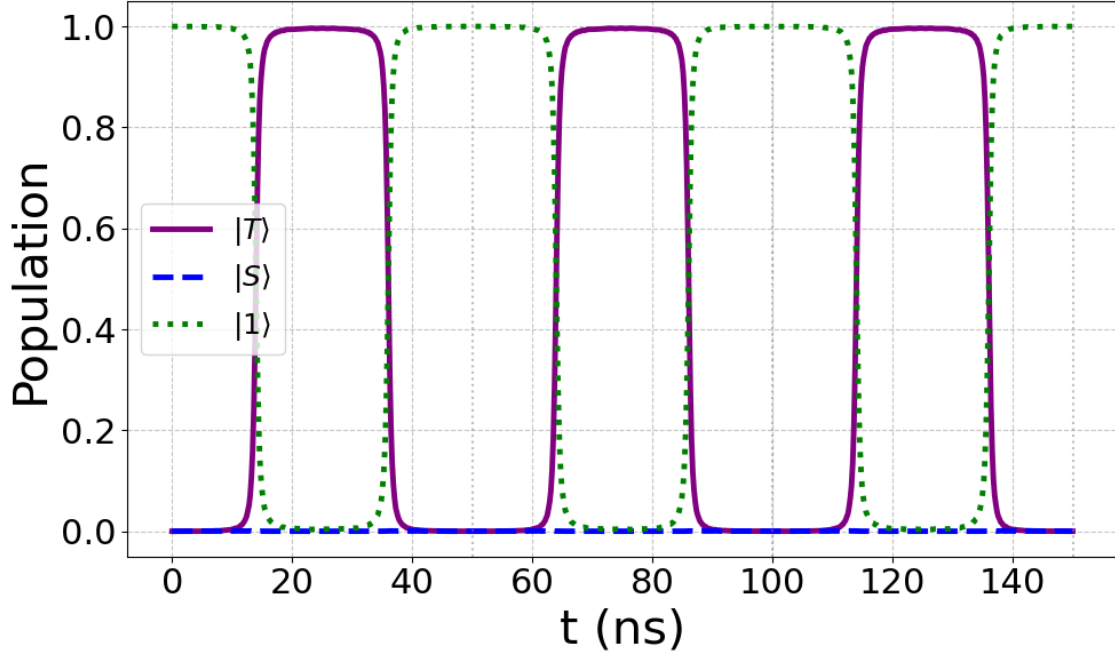


Figure 3.8: In this figure is possible to see the populations of $|1\rangle$, $|T\rangle$, $|S\rangle$, during a perfect adiabatic evolution.

3.6 Conclusions

We can observe an almost perfect reversible population transfer from $|1\rangle$ to $|T\rangle$. This demonstrates the efficiency of the counterdiabatic term since we use a smooth function. In fact H_{CD} , reaches the maximum efficiency when we use polynomials [32] since they have simple derivatives. This permits us to have an adiabatic evolution with multiple cycles which can be applied to some quantum devices like and quantum computers. As will be shown in chapter 5 fast evolutions counteract dissipation effects, which influence the dynamics only on long scales.

Chapter 4

The Berry phase

The Berry phase is a geometric phase which makes sense when the evolution of a quantum system is adiabatic. If the Bloch vector on the Bloch sphere follows a closed path, the Berry phase can be linked to the area subtended by the curve.

4.1 Application to our system

In our protocol, the reversibility driven by the symmetric polynomial fit ensures the path closure. Since the singlet does not play a significant role, we can reduce our 3x3 system to a 2x2 Hamiltonian, and this gives us the opportunity of visualizing the area enclosed on the Bloch sphere. Following [33], we insert in the matrix a time dependent complex term in order to get a non-zero Berry phase. The evolution of the Bloch vector specifically depends on the value and time dependence of such a complex term, see for instance refs. [34, 35, 36]. In the case of the experiments with molecules described in [26, 27], the amplitudes α and β quantify the mixing of the singlet and triplet states, due to the Autler-Townes effect [26, 25], and can be made complex. This would realize complex coupling elements $\Omega_{1T} = |\Omega_{1T}| e^{i\phi}$.

As we have explained previously, the singlet does not play a significant role in the time evolution of the Bloch vector so we can use a 2x2 matrix instead of a 3x3 matrix. Thus we can write the effective

Hamiltonian as:

$$H_0^{(\text{eff})}(\tau) = \begin{pmatrix} p(\tau) & |\Omega_{1T}| \text{sign}(\tau - \frac{1}{2}) e^{i\phi_k(\tau)} \\ \text{H. c.} & \epsilon_T \end{pmatrix}. \quad (4.1)$$

The function $\text{sign}(\tau - \frac{1}{2})$ changes the sign of the coupling along σ_x at half the protocol time, while the phase $\phi_k(\tau)$ induces a mixing of the operators σ_x and σ_y . We choose

$$\text{sign}\left(\tau - \frac{1}{2}\right) \approx \frac{2}{1 + \exp[-d(\tau - 0.5)]} - 1, \quad (4.2)$$

with $d = 20$ for computations without a time-dependent phase $\phi_k(\tau)$.

It's possible to create a closed loop in the evolution on the Bloch sphere with the sign function and ϕ_k constant or zero. Alternatively, this can be done without sign function and a time-dependent phase $\phi_k(\tau)$. In the latter case, we choose $\phi_k(\tau) = 2\pi k\tau$ with integer k . An evolution along a closed contour on the Bloch sphere can give rise to a geometric phase. Since our evolution is perfectly adiabatic, it is interesting to deal with the Berry phase [7], arising from the evolution of the diagonal part in the instantaneous eigenbasis $|n(\tau)\rangle$. Berry phase is expressed in terms of the so-called Berry connection $A(\tau)$, integrated over a closed path drawn by the time evolution. More in detail, $A(\tau)$ is generally defined by the diagonal part of the time evolution in the instantaneous eigenstates $|n(\lambda(t))\rangle$ of $H_0(t)$ [37], when changing some parameter $\lambda(t)$ over time t

$$A(\lambda) = -i \langle n(\lambda) | \frac{\partial n(\lambda)}{\partial \lambda} \rangle. \quad (4.3)$$

Integrating the Berry connection over a closed loop C in the parameter space, we obtain the Berry phase [7, 38, 39, 40]:

$$\begin{aligned} \gamma_B(T) &= \oint_C A(\lambda) d\lambda = -i \int_{t=0}^T \langle n(t) | \frac{\partial n(t)}{\partial t} \rangle dt = \\ &= -i \int_{\tau=0}^1 \langle n(\tau) | \frac{\partial n(\tau)}{\partial \tau} \rangle d\tau, \end{aligned} \quad (4.4)$$

corresponding to the definition in sec in sec 2.4.1. In the next section we can also find an alternative formula for the Berry phase.

4.2 Berry connection

We consider the eigenstate $n(\tau)$ written with a dependence on the azimuthal and polar angles:

$$|n(\tau)\rangle = \cos\left(\frac{\theta(\tau)}{2}\right) |0\rangle + e^{i\phi(\tau)} \sin\left(\frac{\theta(\tau)}{2}\right) |1\rangle. \quad (4.5)$$

Following the [19], we calculate the time derivative::

$$\begin{aligned} \frac{d}{d\tau}|n(\tau)\rangle = & -\frac{1}{2}\sin\left(\frac{\theta(\tau)}{2}\right)\frac{d\theta}{d\tau}|0\rangle \\ & + e^{i\phi(\tau)}\left(\frac{1}{2}\cos\left(\frac{\theta(\tau)}{2}\right)\frac{d\theta}{d\tau} + i\sin\left(\frac{\theta(\tau)}{2}\right)\frac{d\phi}{d\tau}\right)|1\rangle \end{aligned} \quad (4.6)$$

The inner product :

$$\begin{aligned} \left\langle n(\tau) \left| \frac{d}{d\tau} \right| n(\tau) \right\rangle &= \cos\left(\frac{\theta(\tau)}{2}\right) \left(-\frac{1}{2}\sin\left(\frac{\theta(\tau)}{2}\right)\frac{d\theta}{d\tau} \right. \\ &\quad \left. + \sin\left(\frac{\theta(\tau)}{2}\right) \left(\frac{1}{2}\cos\left(\frac{\theta(\tau)}{2}\right)\frac{d\theta}{d\tau} + i\sin\left(\frac{\theta(\tau)}{2}\right)\frac{d\phi}{d\tau} \right) \right) \\ &= -\frac{1}{2}\cos\left(\frac{\theta(\tau)}{2}\right)\sin\left(\frac{\theta(\tau)}{2}\right)\frac{d\theta}{d\tau} \\ &\quad + \frac{1}{2}\sin\left(\frac{\theta(\tau)}{2}\right)\cos\left(\frac{\theta(\tau)}{2}\right)\frac{d\theta}{d\tau} + i\sin^2\left(\frac{\theta(\tau)}{2}\right)\frac{d\phi}{d\tau} \\ &= i \cdot \frac{1 - \cos\theta(\tau)}{2} \frac{d\phi}{d\tau}. \end{aligned} \quad (4.7)$$

So the Berry phase is:

$$\gamma = i \int i \cdot \frac{1 - \cos\theta(\tau)}{2} \frac{d\phi}{d\tau} d\tau = -\frac{1}{2} \int (1 - \cos\theta(\tau)) \frac{d\phi}{d\tau} d\tau. \quad (4.8)$$

If the parameter $\phi(\tau)$ changes as $\Delta\phi$ we obtain:

$$\gamma = -\frac{1}{2}\Delta\phi \cdot \int (1 - \cos\theta(\tau)) d\tau. \quad (4.9)$$

Since the z component of $\mathbf{n}(\tau)$ is:

$$n_z(\tau) = \cos \theta(\tau). \quad (4.10)$$

Numerically, $\theta(\tau)$ it's calculated from $\theta(\tau) = \arccos(\langle \sigma_z \rangle)$, where $\langle \sigma_z \rangle$ is the expected value of σ_z using the starting eigenstate. The integral $\int (1 - \cos \theta(\tau)) d\tau$ is evaluated by using a cubic spline interpolation of $\theta(\tau)$ and numerical integration (for example the trapezoidal rule). Since to the β term is appended a time-dependent phase $\phi_k(\tau) = 2\pi k\tau$ the Berry phase can be written as:

$$\gamma = -k\pi \cdot \int (1 - \cos \theta(\tau)) d\tau. \quad (4.11)$$

Thus the Berry phase depends on the geometry of the path di $\mathbf{n}(\tau)$ on the Bloch sphere. The derivative of $\mathbf{n}(\tau)$ defines the evolution of $\theta(\tau)$ e $\phi(\tau)$, and the Berry's phase is proportional to the integral of $(1 - \cos \theta(\tau))$, and $\phi(\tau)$. Now we demonstrate that $\Delta\phi = -2\pi k$

4.3 Eigenvectors and mean value of σ_x e σ_y

No we want demonstrate that in our case $\Delta\phi = -2\pi k$ where k is present on the phase in the Hamiltonian.

$$H(\tau) = \begin{pmatrix} a(\tau) & -\beta \text{sign}(\tau) e^{i2\pi k\tau} \\ -\beta \text{sign}(\tau) e^{-i2\pi k\tau} & C_2 \end{pmatrix}, \quad (4.12)$$

where β , C_2 are constants and the sign function is equal to $\text{sign}(\tau) = \frac{2}{1+e^{20(\tau-0.5)}} - 1$, $a(\tau)$ is the eight-degree polynomial and k is an integer.

4.3.1 Eigenvector with the lower energy

In order to find the mean values of σ_x and σ_y , we start to calculate the eigenvector with the lower energy. To do this we first find the eigenvalue using the determinant as:

$$\det((H(\tau) - \tilde{\lambda}I) = (a(\tau) - \tilde{\lambda})(C_2 - \tilde{\lambda}) - \beta^2 \text{sign}(\tau)^2 = 0.$$

$$\tilde{\lambda}_- = \frac{a(\tau) + C_2 - \sqrt{(a(\tau) - C_2)^2 + 4\beta^2 \text{sign}(\tau)^2}}{2}.$$

Per $\tilde{\lambda}_-$, the eigenvector $|n(\tau)\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfies:

$$(a(\tau) - \tilde{\lambda}_-)x - \beta \text{sign}(\tau)e^{i2\pi k\tau}y = 0.$$

$$x = \frac{\beta \text{sign}(\tau)e^{i2\pi k\tau}y}{a(\tau) - \tilde{\lambda}_-}.$$

We normalized $|x|^2 + |y|^2 = 1$:

$$y = \frac{a(\tau) - \tilde{\lambda}_-}{\sqrt{(a(\tau) - \tilde{\lambda}_-)^2 + \beta^2 \text{sign}(\tau)^2}}, \quad x = \frac{\beta \text{sign}(\tau)e^{i2\pi k\tau}}{\sqrt{(a(\tau) - \tilde{\lambda}_-)^2 + \beta^2 \text{sign}(\tau)^2}}.$$

$$|n(\tau)\rangle = \frac{\beta \text{sign}(\tau)e^{i2\pi k\tau}}{\sqrt{(a(\tau) - \tilde{\lambda}_-)^2 + \beta^2 \text{sign}(\tau)^2}}|0\rangle + \frac{a(\tau) - \tilde{\lambda}_-}{\sqrt{(a(\tau) - \tilde{\lambda}_-)^2 + \beta^2 \text{sign}(\tau)^2}}|1\rangle.$$

Now we have the eigenvector written as a function of the matrix parameters and of the minimum eigenvalue. We can define also:

$$D(\tau) = \sqrt{(a(\tau) - \tilde{\lambda}_-)^2 + \beta^2 \text{sign}(\tau)^2}.$$

We simplify the notations in the subsequent subsection:

4.3.2 Mean values σ_x e σ_y

So we have:

$$D(\tau) = \sqrt{(a(\tau) - \tilde{\lambda}_-)^2 + \beta^2 \text{sign}(\tau)^2}.$$

$$|n(\tau)\rangle = \begin{pmatrix} \frac{\beta \text{sign}(\tau) e^{i2\pi k\tau}}{D(\tau)} \\ \frac{a(\tau) - \tilde{\lambda}_-}{D(\tau)} \end{pmatrix}, \quad \langle n(\tau)| = \begin{pmatrix} \frac{\beta \text{sign}(\tau) e^{-i2\pi k\tau}}{D(\tau)} & \frac{a(\tau) - \tilde{\lambda}_-}{D(\tau)} \end{pmatrix}.$$

We can calculate the mean value of σ_x as

$$\begin{aligned} \langle \sigma_x \rangle &= \begin{pmatrix} \frac{\beta \text{sign}(\tau) e^{-i2\pi k\tau}}{D(\tau)} & \frac{a(\tau) - \tilde{\lambda}_-}{D(\tau)} \end{pmatrix} \begin{pmatrix} \frac{a(\tau) - \tilde{\lambda}_-}{D(\tau)} \\ \frac{\beta \text{sign}(\tau) e^{i2\pi k\tau}}{D(\tau)} \end{pmatrix} \\ &= \frac{2\beta \text{sign}(\tau)(a(\tau) - \tilde{\lambda}_-) \cos(2\pi k\tau)}{(a(\tau) - \tilde{\lambda}_-)^2 + \beta^2 \text{sign}(\tau)^2}. \end{aligned}$$

And we can calculate the mean value of σ_y

$$\begin{aligned} \langle \sigma_y \rangle &= \begin{pmatrix} \frac{\beta \text{sign}(\tau) e^{-i2\pi k\tau}}{D(\tau)} & \frac{a(\tau) - \tilde{\lambda}_-}{D(\tau)} \end{pmatrix} \begin{pmatrix} -i \cdot \frac{a(\tau) - \tilde{\lambda}_-}{D(\tau)} \\ i \cdot \frac{\beta \text{sign}(\tau) e^{i2\pi k\tau}}{D(\tau)} \end{pmatrix} \\ &= -\frac{2\beta \text{sign}(\tau)(a(\tau) - \tilde{\lambda}_-) \sin(2\pi k\tau)}{(a(\tau) - \tilde{\lambda}_-)^2 + \beta^2 \text{sign}(\tau)^2}. \end{aligned}$$

Now if we take the ratio we obtain

$$\frac{\langle \sigma_y \rangle}{\langle \sigma_x \rangle} = \tan(\phi) = \tan(-2\pi k\tau) \quad (4.13)$$

And we demonstrate that $\phi = -2\pi k\tau$. In the following image we can see the path of the Bloch's vector for $k=0,1,2$

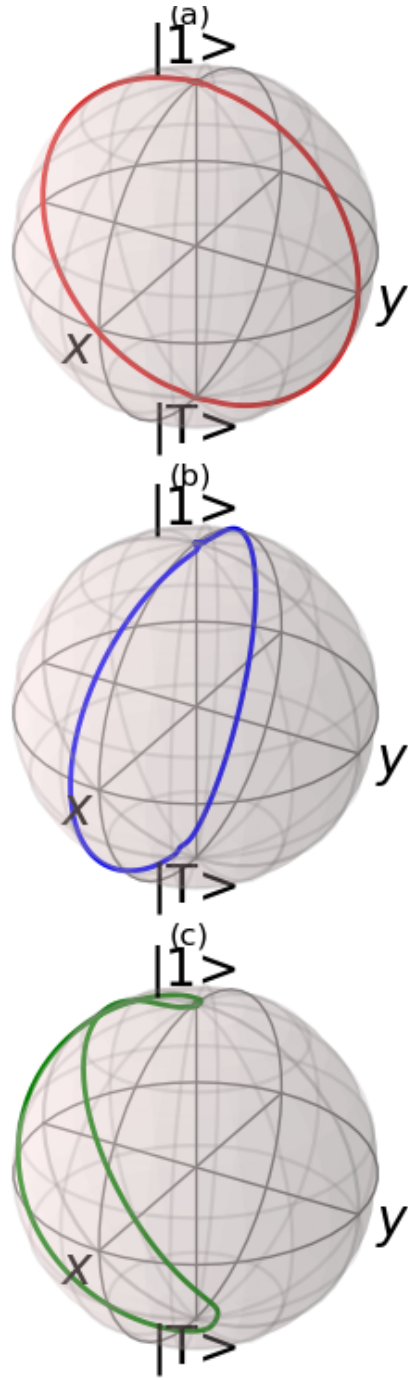


Figure 4.1: Image that shows the path for $k=0$ in red, $k=1$ in blue and $k=2$ in green. For $k = 1$ $\gamma_B = 2.8$ rad while for $k = 2$ $\gamma_B=5.5$ rad.

4.4 Berry phase as a sensor

Berry's phase can be also used to detect some anomalies during the population transfer between the state $|1\rangle$ and the state $|T\rangle$. In fact, it changes significantly when a constant parameter v is added to the polynomial fit in the equation: $p(\tau) \rightarrow p(\tau) + v$. The values of v are taken from the interval $[-10, -0.25]$. The constant shift of the instantaneous energy of state $|1\rangle$ affects the coupling between the instantaneous eigenstates of $H_0^{(\text{eff})}(\tau)$ (and of $H_0(\tau)$ during the evolution. This change, particularly relevant at the avoided level crossings, is visible in the time-dependent fidelity (2.8), as well as in the values of Berry phase gathered along a cycle. Through this adjustment, it is possible to establish a relationship between the maximal values of the $|T\rangle$ population and the Berry phase. For $v \lesssim -3.5$, the transfer to the state $|T\rangle$ no longer works perfectly. Here we obtain populations of $|T\rangle$ smaller than one, even in the presence of CD control, and hence at perfect adiabatic evolution. The reason is the insufficient coupling between the levels $|1\rangle$ and $|T\rangle$. The correlation is best fitted by a polynomial similar to the applied sweep in Eq. (3.16)

$$P(\gamma_B) = b_8\gamma_B^8 + b_7\gamma_B^7 + b_6\gamma_B^6 + b_5\gamma_B^5 + b_4\gamma_B^4 + b_3\gamma_B^3 + b_2\gamma_B^2 + b_1\gamma_B + b_0, \quad (4.14)$$

where the coefficients are listed in appendix B. The following image shows the fit

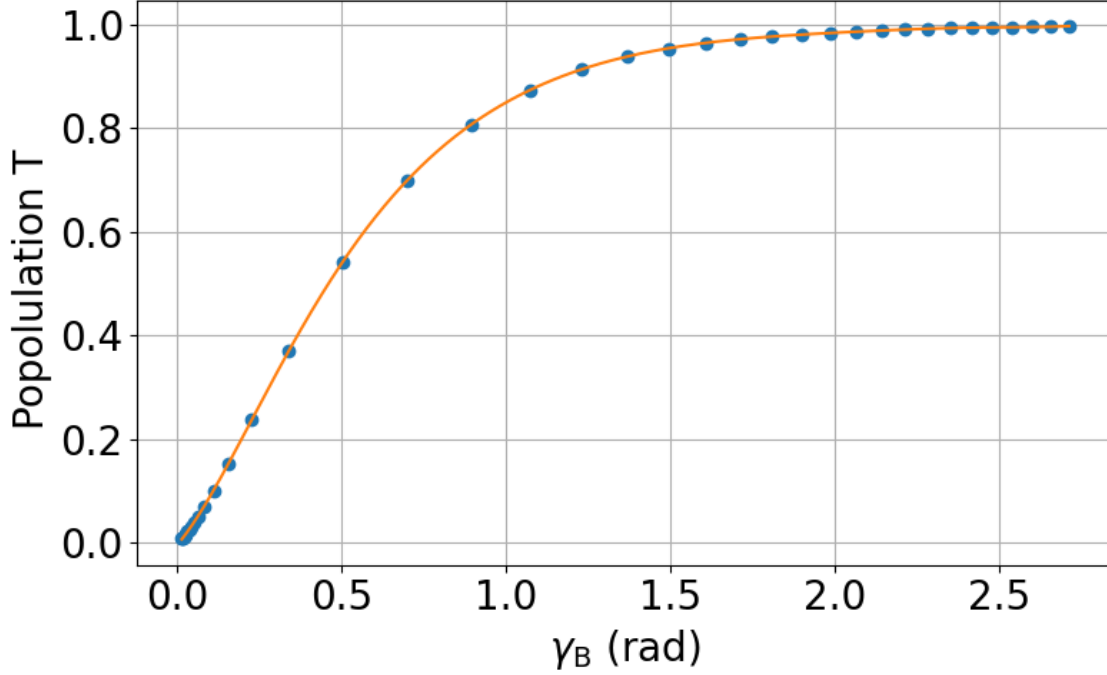


Figure 4.2: Polynomial fit that shows the dependence of the maximal population $|T\rangle$ vs the Berry phase.

This relates the maximal population of the triplet state, $P_T(t = t_f/2)$ with the Berry phase $\gamma_B(T)$ after a full round trip from $t = 0$ to $t = T = t_f$. We have chosen this polynomial fit since it has the same form of the eight-degree polynomial fit of the atan function. The observed correlation indeed seems to reflect the functional form of our sweep $p(\tau)$. Hence we see that the value of the Berry phase can be tuned with the rate of change of the additional phase $\phi_k(\tau)$ (it systematically increases with the integer k) and also by the sweep function, for instance, by the constant parameter v added to Eq. (3.16). We can see that if the protocol is not transferring perfectly population to the triplet state $|T\rangle$, the geometric phase approaches zero. In the limit of no evolution of state $|1\rangle$, this means that only a vanishing solid angle is spanned in the Bloch spheres.

4.5 Conclusions

From this chapter we can deduce an interesting applications of the Berry phase to our protocol. As we have seen, we want to obtain a complete transfer of population from the ground state $|1\rangle$ to the triplet state $|T\rangle$ by using an eight-degree polynomial sweep. But at the same time if our polynomial undergoes some changes, like being translated downward, there is no more a complete population transfer. Then the path on the Bloch sphere becomes smaller and smaller as v decrease until the area enclosed becomes equal to zero. So generally, if we have a Berry phase equal to zero we can deduce that there is no population transfer.

Chapter 5

Decays, Zeno Effect and von Neumann entropy

Now we modify our Hamiltonian adding a complex term γ on the H_{33} element in order to simulate a decay of the triplet state $|T\rangle$ [41]. Specifically, we want to simulate a population loss caused by an interaction between the original closed quantum system and an external environment. Here we study that loss.

5.1 Decay with Schrödinger equation

We start analyzing the populations of the ground state, the triplet, and the singlet with a check on the norm of $\psi(t)$ state of the Schrödinger equation. In a closed quantum system, the norm should always be equal to 1 but since γ simulates a dissipation caused by the external environment we expect a decrease of the norm, as the $\tau = t/t_f$ increases. First of all, we begin with the case without H_{CD} so we must use a long experimental time t_f . The following image shows the populations and the norm vs τ with $\gamma = 0.0001$ and using $t_f = 5000ns$. As we can see, the norms $\psi(t)$ show a decreasing behavior with increasing of τ .

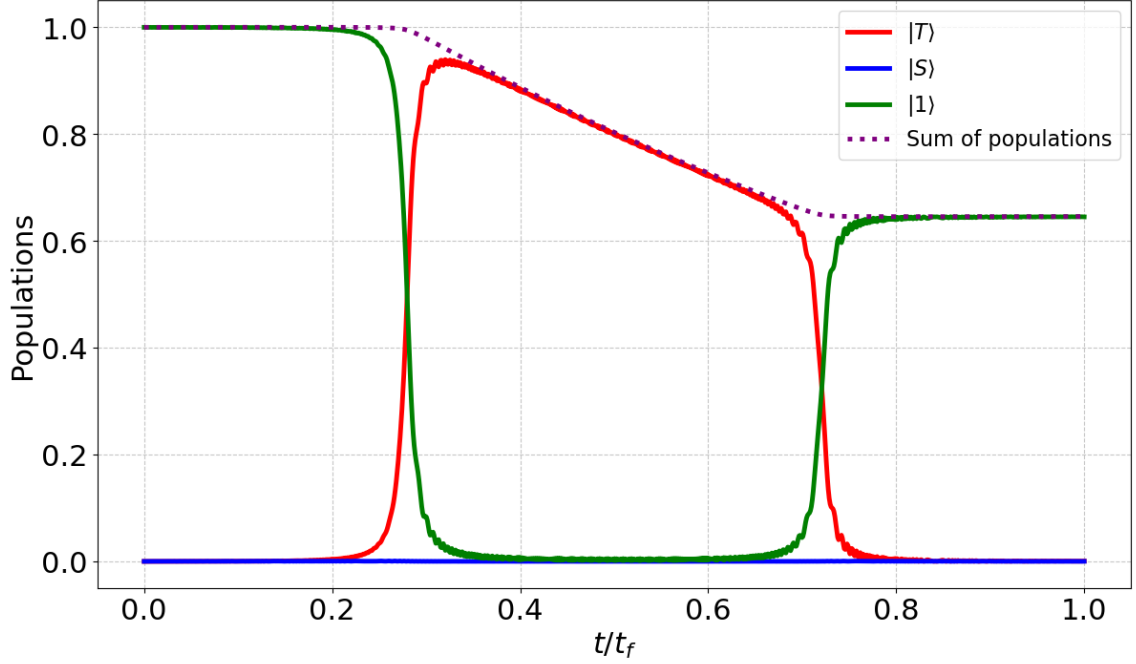


Figure 5.1: It's possible to note the norm decline at the increasing of τ . We can notice that the population $|T\rangle$ decline in the interval $0.25 < \tau < 0.75$ compared to the case without γ while the norm of the $\psi(t)$ goes from 1 to about 0.7 during time evolution.

Instead, in the following image we can see the decrease of the population sum taken at $\tau = 1$ at a $\gamma = 0.00005$ with the increase of the experimental time t_f . We can notice that the curve fit is an exponential as expected. We can write the curve fit as:

$$y = a \cdot \exp(-bt_f) + c \quad (5.1)$$

where the coefficients are equal to:

$$\begin{aligned} a &= 0.998 \pm 0.008 \\ b &= (4.379 \pm 0.07994) \times 10^{-5} \\ c &= 0.002 \pm 0.010 \end{aligned}$$

Notice that c has an error that is bigger than itself, probably because c term is incompatible with the fit.

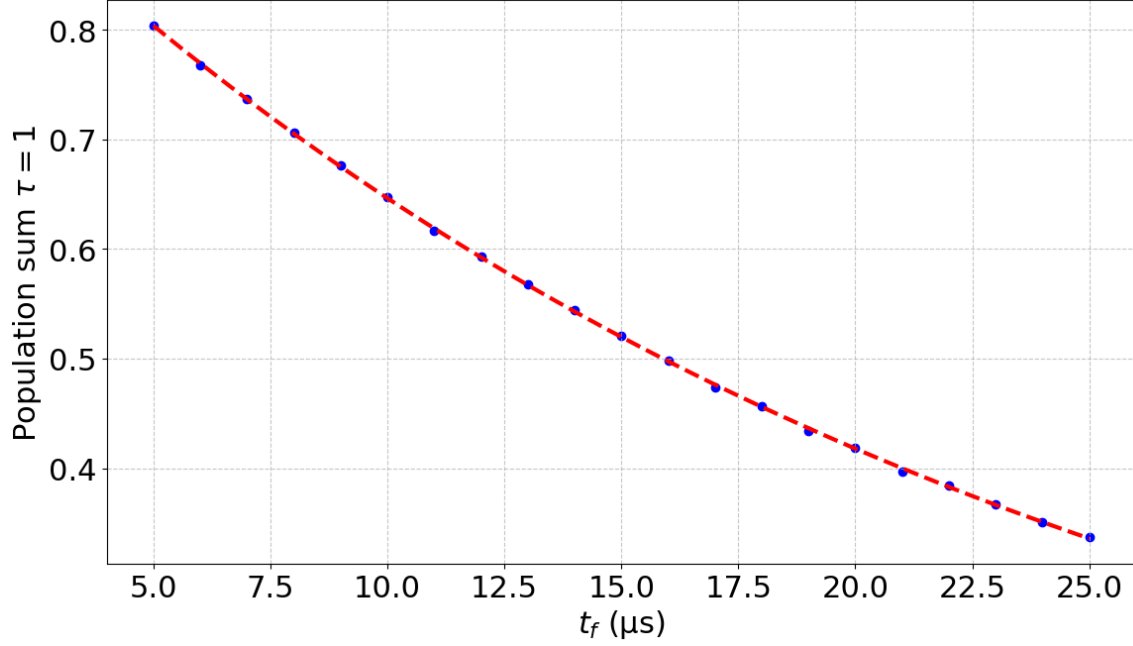


Figure 5.2: In this image we can see the the trend of population sum measured at a $\tau = 1$ vs the experimental time t_f . It's worth noting that even if we have a small value of the decay rate the population decline is remarkable since it falls from 0.8 to 0.15-0.2. This because we use a very long experimental time t_f that begins from 5000ns and finishes to 25000ns.

We now consider the case involving the counterdiabatic term H_{CD} . The inclusion of this term enables the system to reach adiabaticity without requiring long experimental durations as we have seen in chapter 3. As a result, the final time t_f is significantly reduced, which in turn leads to a diminished effect of decay processes. It should be noted that the presence of the dissipation factor $\gamma = 0.0001$ does not alter the dynamics of the population. This observation is supported by the following graph:

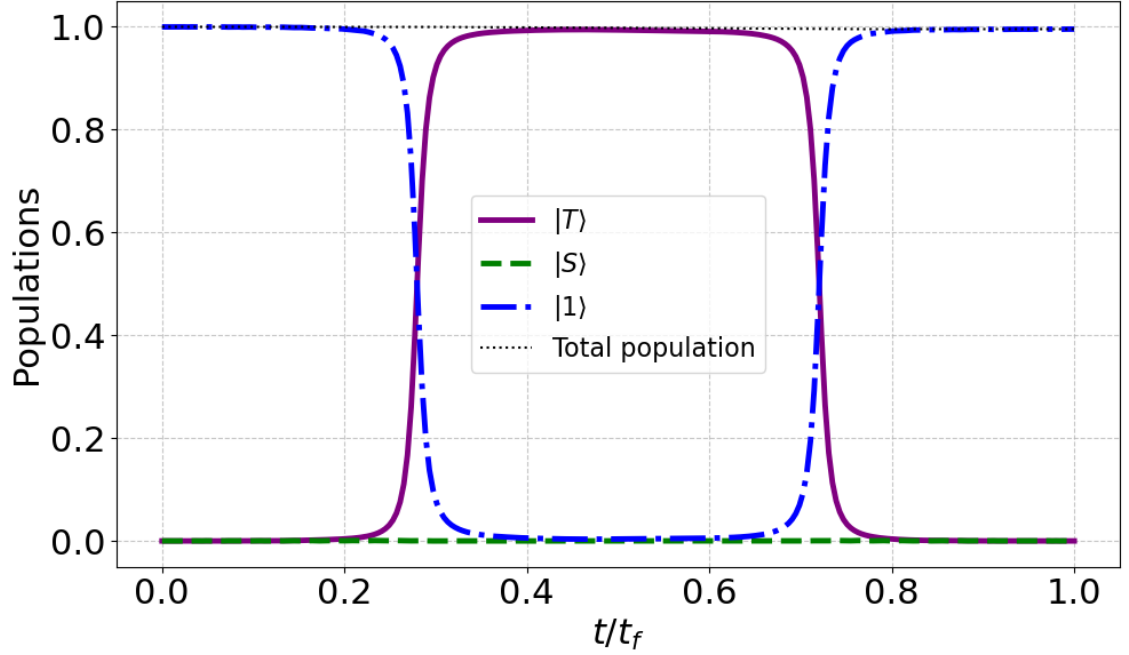


Figure 5.3: In this image we can see the the trend of the populations vs t/t_f . In this case since the experimental time is much shorter than in the previous without the counterdiabatic term we do not notice specific changes in the evolution of the populations of $|1\rangle$, $|T\rangle$, $|S\rangle$ neither in the norm of $\psi(\tau)$.

Here we do not observe any considerable dissipation. This is because it's not necessary to use a long t_f due to the presence of the counterdiabatic term. If t_f is much shorter at the same γ the population loss will be significantly reduced. Hence, we find another important application of H_{CD} : if we append this term to the main Hamiltonian we are also able to reduce dissipation effects and make the protocol more robust against external dissipation. In the following image, we can see the population sum decay using a much shorter experimental time and with $\gamma = 0.0001$.

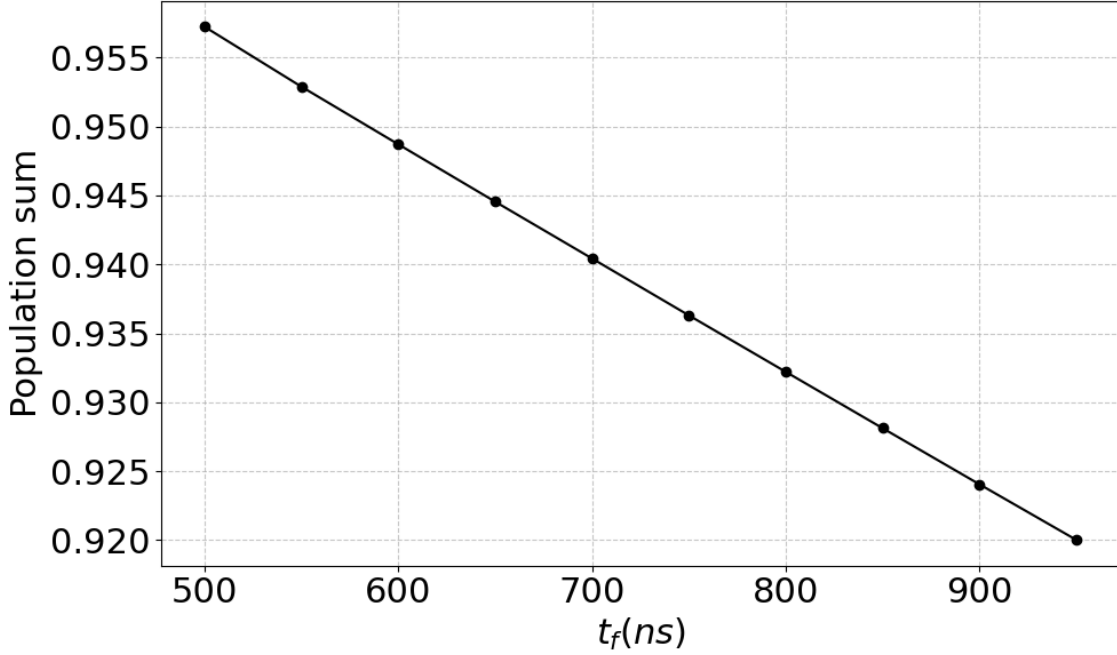


Figure 5.4: Population sum vs t_f with H_{CD} . The graph is similar to a line because of the Taylor expansion.

It is worth noting that, since the population loss is negligible, the graph appears linear. Of course, this does not mean it is truly linear, but rather a consequence of the Taylor expansion. Another approach is to use the Lindblad equations to study the time evolution of the populations under a decay rate. For scaling reasons our Hamiltonian is multiplied by t_f and since we have only a γ decay we write the operator $\hat{L} = \sqrt{\gamma t_f} |4\rangle \langle T|$ and then we calculate the population variation depending on τ . In order to do this we append a fourth level state where the triplet state $|T\rangle$ can decay. So we use a 4x4 Hamiltonian with an energy term in H_{44} and

$$H(t) = \begin{pmatrix} \epsilon_1(t) & \Omega_{1S} & \Omega_{1T} & 0 \\ \Omega_{1S}^* & \epsilon_S & \Omega_{ST} & 0 \\ \Omega_{1T}^* & \Omega_{ST}^* & \epsilon_T & 0 \\ 0 & 0 & 0 & E_4 \end{pmatrix} \quad (5.2)$$

The following graph shows the population trend of the four energy

levels using in this case the master equations without H_{CD} at a $t_f = 5000ns$ and with $\gamma = 0.0001$.

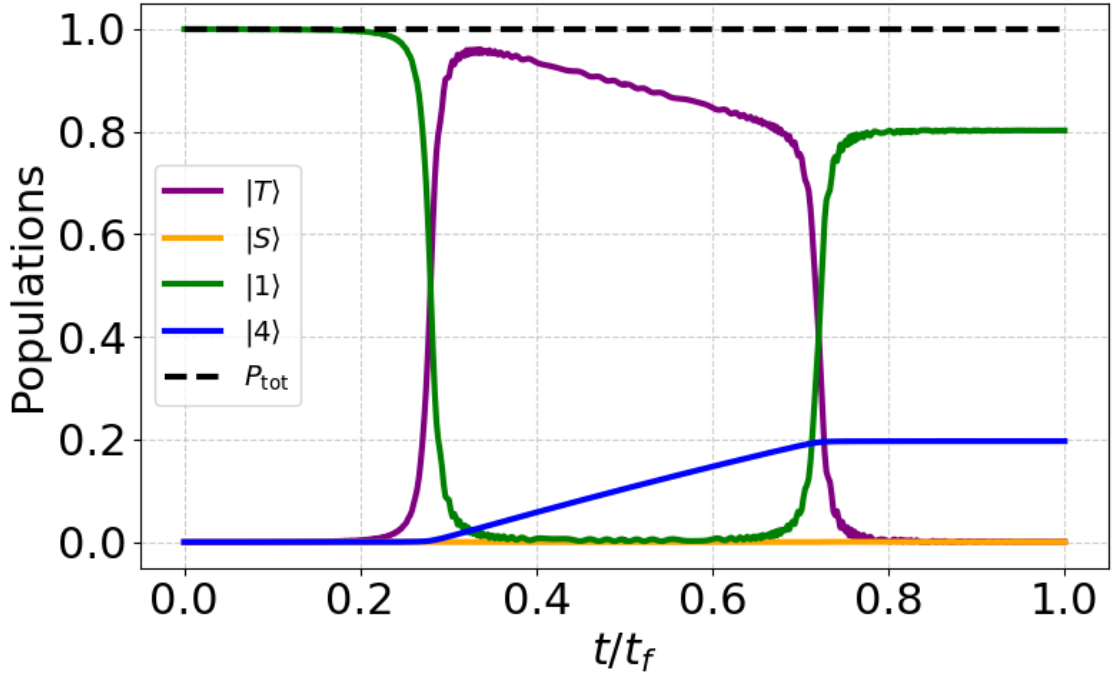


Figure 5.5: Like in the Schrödinger case we can observe the reduction of the sum of the populations of $|1\rangle$, $|T\rangle$, $|S\rangle$, while P_4 of the fourth state increases. The total sum anyway $P_1+P_T+P_S+P_4$ is equal to 1.

It is worth noting the increase in the population of state $|4\rangle$ compared to the loss observed in the sum of other states. As in the case of using the Schrödinger equation, here too we measure the decay coefficients of the population sum as a function of t_f . Specifically, we start from $t_f = 5000ns$ and go up to $t_f = 25000ns$ with $\gamma = 0.0001$.

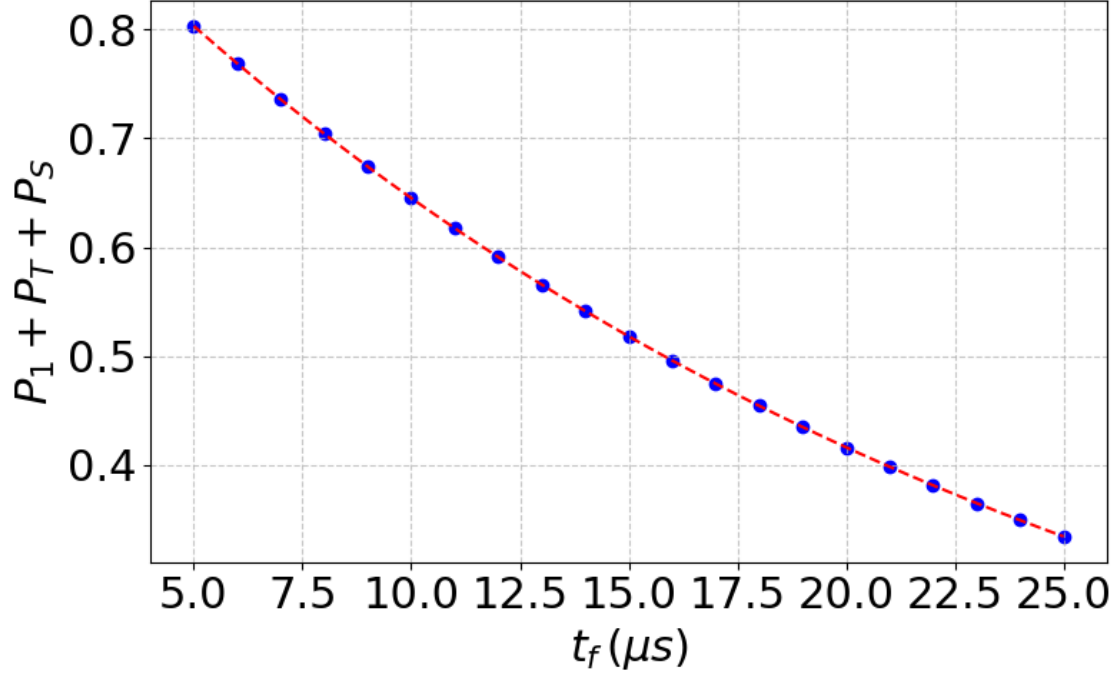


Figure 5.6: Image which shows the population decline using a population sum of $|1\rangle + |T\rangle + |S\rangle$ using master equations.

In the case of the master equation the coefficients of the exponential fit are:

$$\begin{aligned}
 a &= (1.00 \pm 4.75 \times 10^{-5}) \\
 b &= (4.38 \times 10^{-5} \pm 5.02 \times 10^{-9}) \\
 c &= (-6.04 \times 10^{-5} \pm 6.11 \times 10^{-5})
 \end{aligned}$$

If we exclude c the coefficients are very similar to the case where we used the Schrödinger equation but the errors are much smaller.

In the last step, we add the counterdiabatic term H_{CD} in order to obtain an adiabatic evolution. Like in the Schrödinger case we use a much shorter experimental time $t_f = 50ns$ and a $\gamma = 0.0001$. Then we expect a considerable conservation of the sum of the populations of the ground state, the triplet and the singlet.

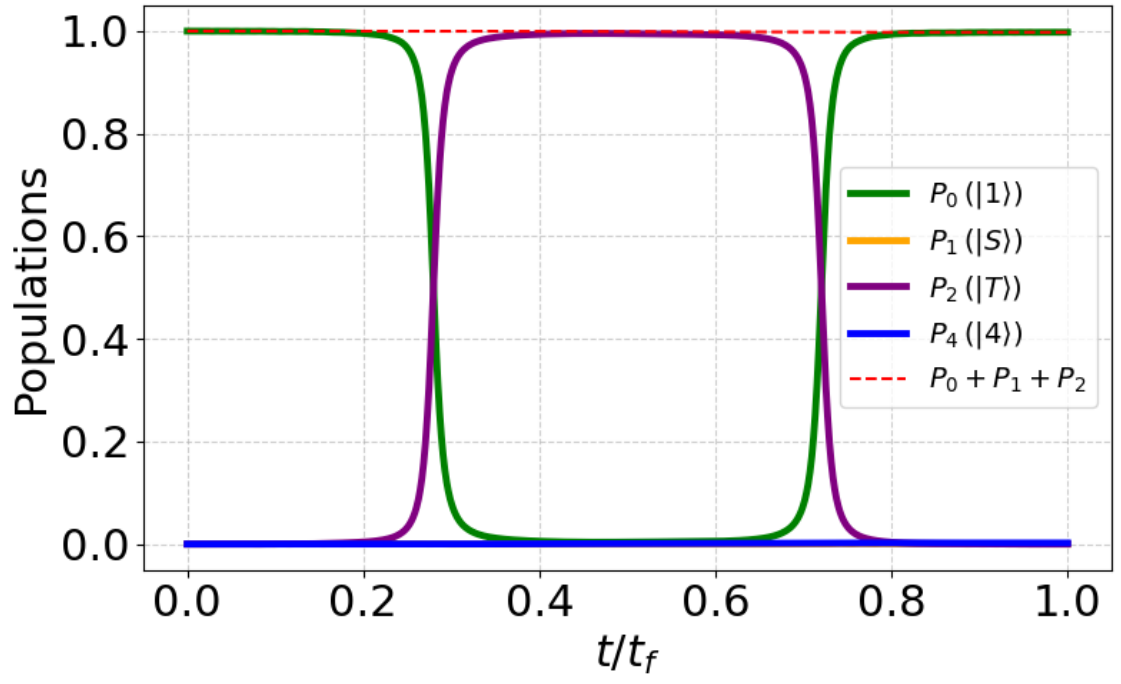


Figure 5.7: Various populations evolution using a short $t_f = 50ns$.

5.2 Zeno effect

We now introduce the Quantum Zeno effect [10, 42]. This phenomenon occurs when a quantum state ceases to evolve due to an excessively high decay rate. Essentially, a strong decay rate corresponds to frequent measurements, which tend to freeze. Instead if we use master equations it's possible to notice an increasing trend of the population sum after a specific γ , which can be named $\gamma_{inversion}$.

5.2.1 Zeno effect without H_{CD}

To verify the presence of the Zeno effect, we study the evolution of the sum of the ground-state population plus the triplet and the singlet population as a function of the decay rate γ . For this purpose, we consider the evolution without the counterdiabatic term H_{CD} at a final time $t_f = 5000$ ns, and the evolution with H_{CD} at a final time $t_f = 50$ ns. We begin with the case without H_{CD} . The following image shows the population sum of the ground state, the triplet state and the singlet state:

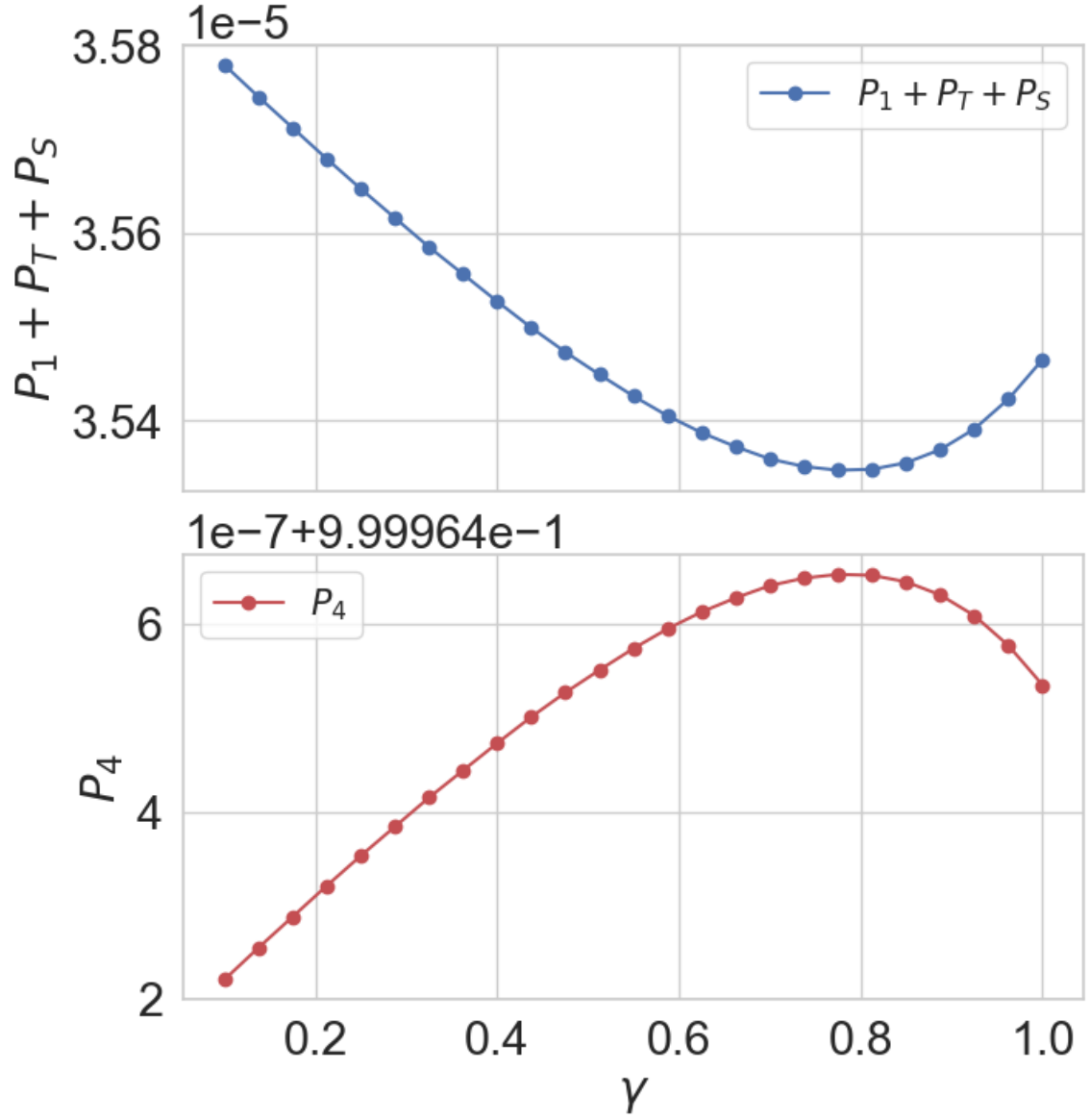


Figure 5.8: These two graphs show the confront between the $P_1+P_T+P_S$ and P_4 varying γ at a $t_f = 5000ns$. We can notice that the total population $P_1+P_T+ P_S+ P_4$ is almost to 1, and that there is a γ where the population sum of the three states stops decreasing and begin to increase. At the same time, complementarily, P_4 stops increasing and begins to decrease.

5.2.2 Zeno effect with H_{CD}

It is possible to notice that the sum of $|T\rangle + |S\rangle + |1\rangle$ it is very small, because as shown in the preceding chapter we use a considerable experimental time t_f so the sum of the populations vanishes. Anyway, if we use the counterdiabatic term H_{CD} we can always reduce the necessary experimental time with different consequences: in fact, if we observe the populations in the following images we have:

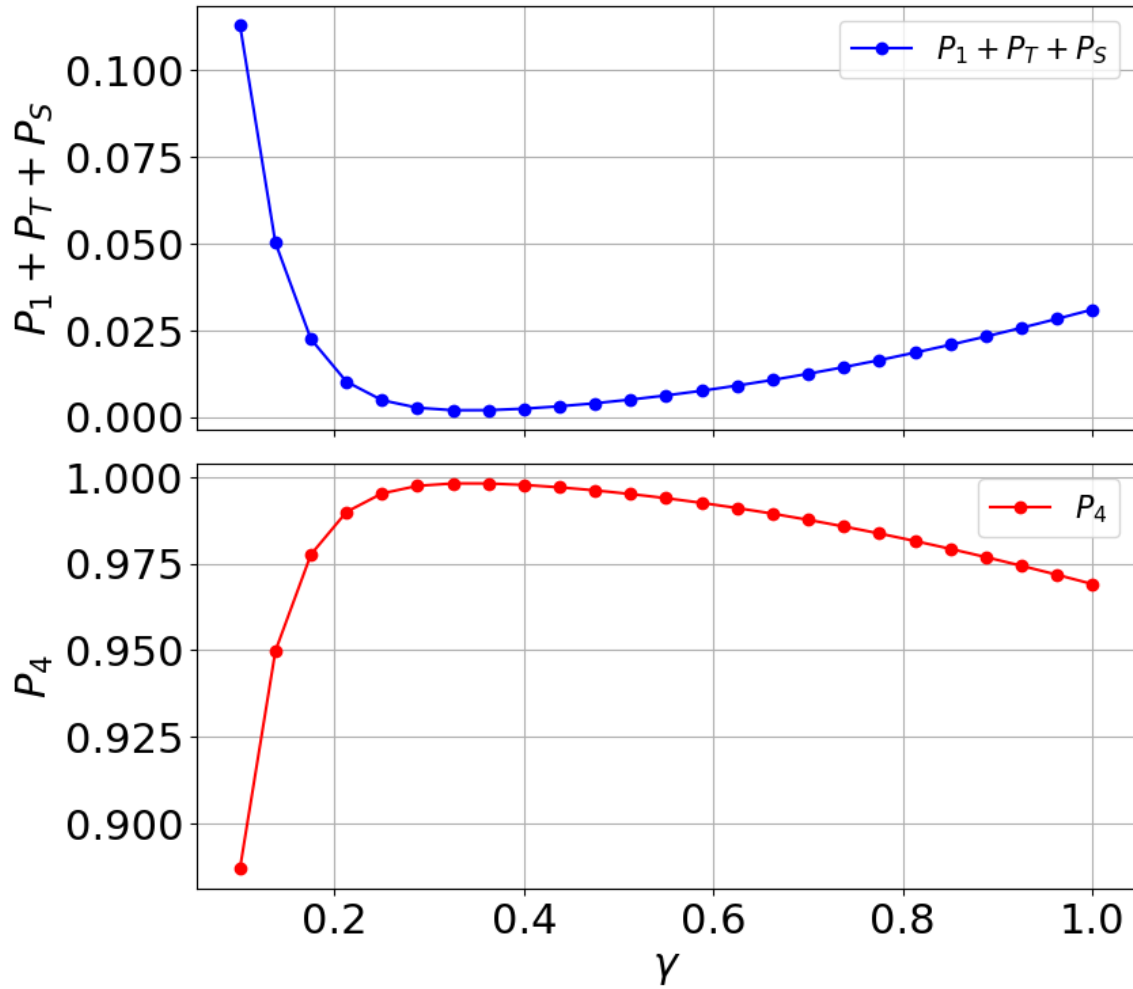


Figure 5.9: Confront between $P_1+P_T+P_S$ and P_4 by using H_{CD} at experimental time $t_f = 50ns$.

5.2.3 γ inversion

Now using H_{CD} we derive the $\gamma_{inversion}$, the point where there is the inversion of the trend of the sum of the populations of the three level states, $|1\rangle$, $|T\rangle$, and $|S\rangle$. To do this, we take a range of t_f between 50ns and 200ns in 10 steps using the linspace function in QuTip. This is necessary to measure the decay of these populations using anyway a second-degree polynomial for each gamma taken in a range from 0.15 to 0.35 in a linspace of 20 steps.

$$P(t_f) = at_f^2 + bt_f + c \quad (5.3)$$

We use this polynomial instead of a function based on the exponentials because of the instability of these fits due to the very low values of sum of the three populations. After these computations we make a fourth-degree polynomial fit b vs γ in order to find the γ where b is equal to zero. The point where b is zero corresponds to the point where the sum of the populations begins to increase. The new polynomial is

$$b(\gamma) = c_4\gamma^4 + c_3\gamma^3 + c_2\gamma^2 + c_1\gamma + c_0 \quad (5.4)$$

Where the coefficients are listed in the appendix D. We can see in the following image the key point is where γ is 0.2895.

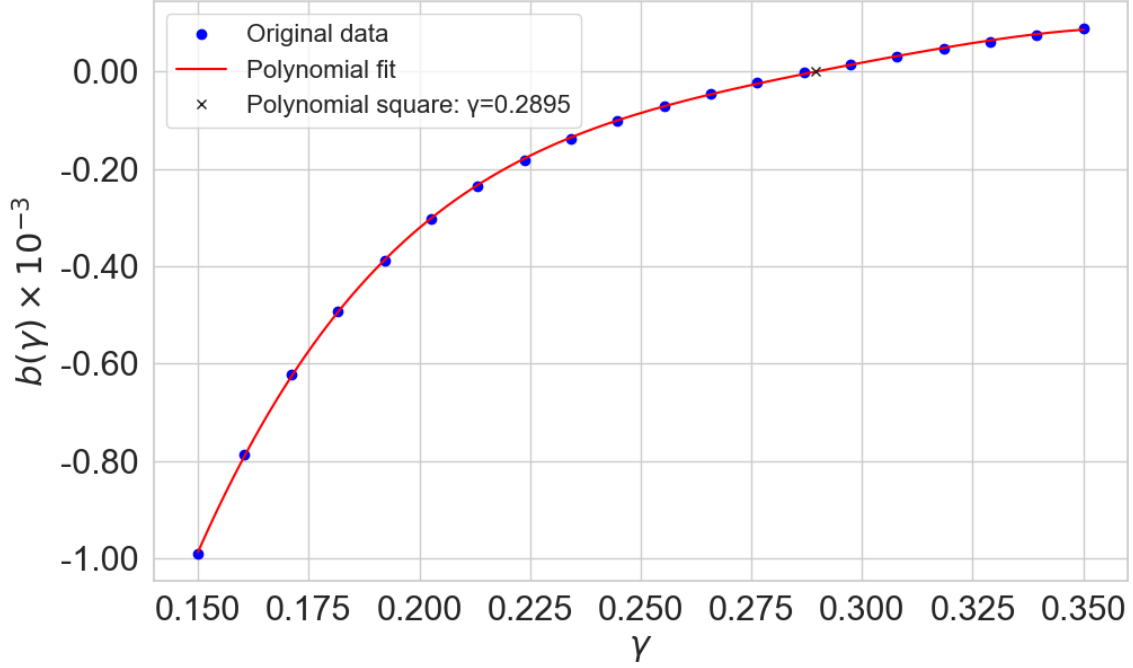


Figure 5.10: The polynomial fit vs γ . The population sum begins to increase where b is equal to zero.

5.3 von Neumann entropy and populations

Now we investigate the relationship between the sum of the populations $P_1 + P_T + P_S$ using the counterdiabatic term H_{CD} and the von Neumann entropy. We can define the von Neumann entropy as [43]:

$$S(\rho) = -\text{Tr}(\rho \log \rho) \quad (5.5)$$

Here, ρ represents the density matrix as defined in the Lindblad equation. The entropy is an additive quantity which is correlated to the degrees of freedom of a system [44]: if we have more degrees of freedom we have a greater entropy.

5.3.1 von Neumann entropy with H_{CD}

Interestingly, we observe a correlation between the minimum of the sum of the three populations and the minimum of the von Neumann entropy. The following image illustrates how the von Neumann entropy and the population sum vary with respect to the parameter γ at a $t_f = 50ns$.

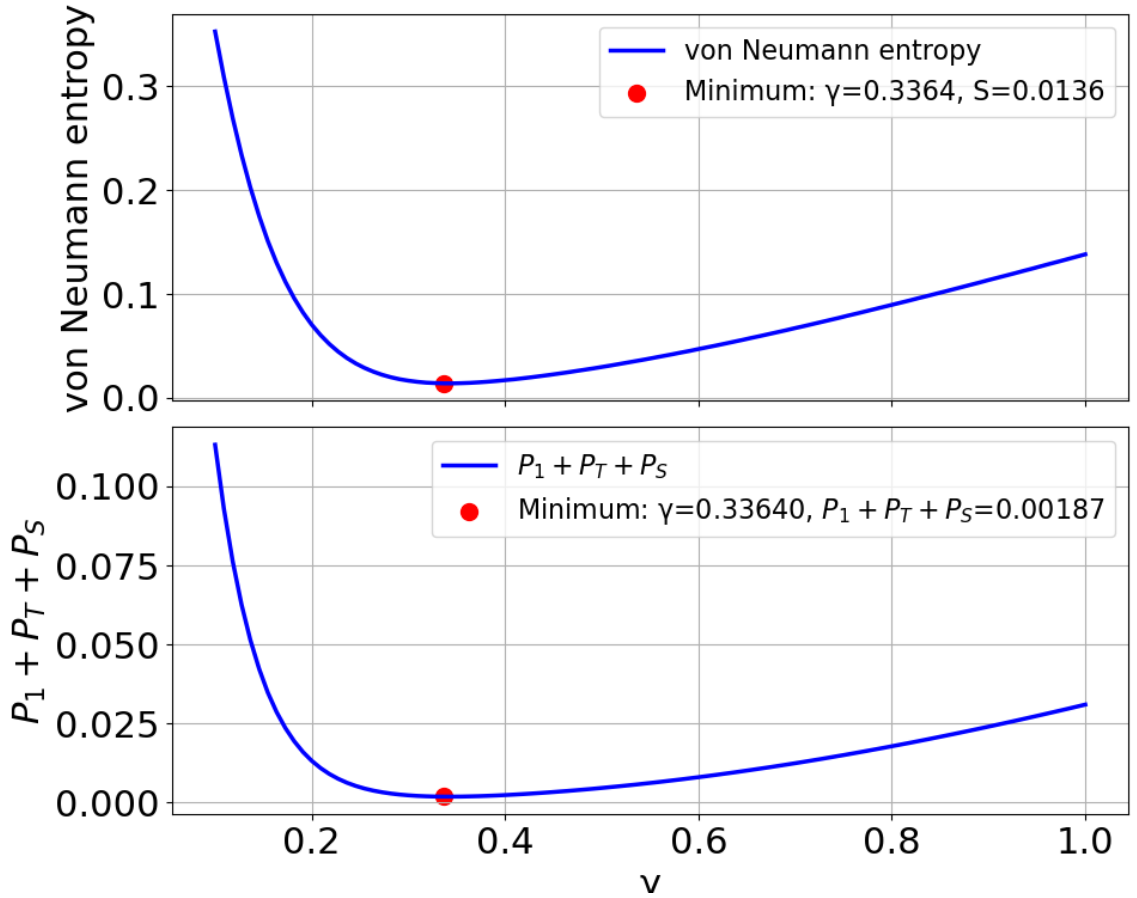


Figure 5.11: The figure illustrates the behavior of the population sum and the entropy. It's possible to notice that the γ corresponding to the minimum of the population sum is the same as the one for which the entropy reaches its minimum.

We can notice that the two γ transitions are equal. This happens because, at the point of the minimum of the population sum, we have

a transition caused by the Zeno effect as we have seen in the previous section. The transition point corresponds to the entropic freezing point since the quantum system remains frozen because the possible distribution of the populations are reduced. The following image shows the confront between the population sum and the von Neumann entropy both normalized.

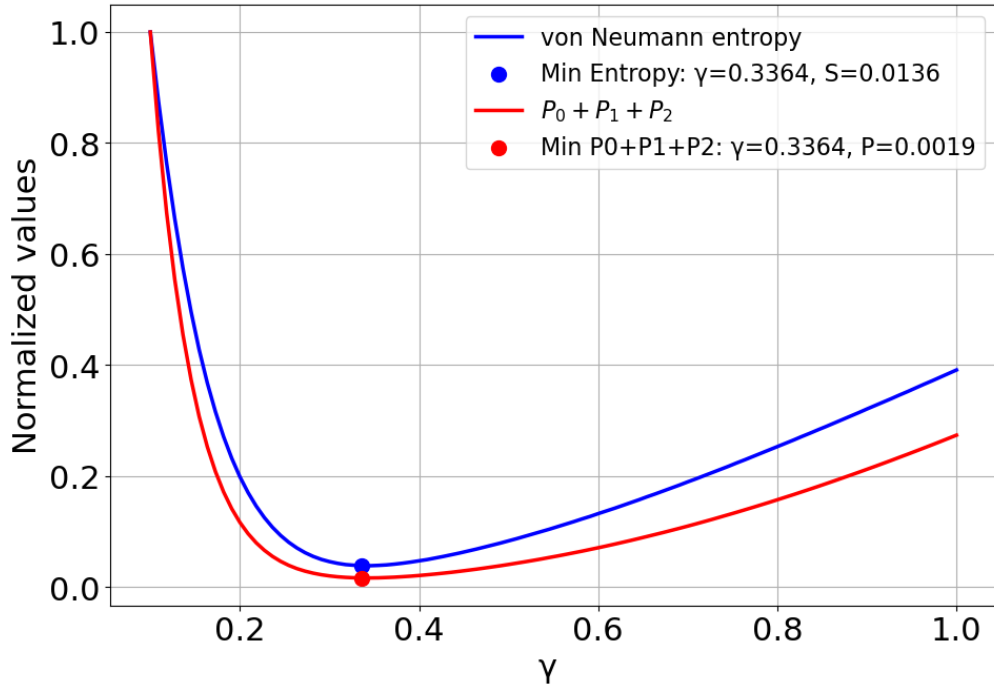


Figure 5.12: In this image there is the confront between the population sum normalized and the von Neumann entropy normalized. The two minima coincide perfectly at the same $\gamma = 0.3364$.

5.4 Conclusions

In this chapter, we have simulated the decay of the triplet state by either adding a complex term γ to the H_{33} Hamiltonian term or by using a master equation. Specifically, we have noticed that the presence of

the H_{CD} term lets us use a much shorter t_f , which guarantees a significant reduction of the dissipation. Since QuTip does not work very well with non-Hermitian operators, we have used Lindblad equation in the successive sections. We also verified the Zeno effect by observing an inversion of the behavior of the population sum as γ increases. This phenomenon is confirmed by the minimum of the von Neumann entropy, which corresponds perfectly to the minimum of the population sum.

Chapter 6

Conclusions

6.1 The three-level system

In this present work, we have studied time evolution of a three-level quantum system. The main goal was to obtain a complete population transfer between the ground state and the triplet state reversibly. To obtain reversibility it was necessary that the ground state evolution was driven by a symmetric function without cusps. Remarkable is the efficacy of the counterdiabatic term H_{CD} , which reduces by a factor of 100 the necessary time for the simulations without a loss of fidelity. This speed up is necessary to best fight dissipation process as studied in chapter 5.

6.2 The Berry phase

We have also studied Berry phase and utilized it as a detector for the presence of some imperfections in our quantum system. Particularly we have used it to check the goodness of the population transfer from $|1\rangle$ to $|T\rangle$ in presence of noise and we have found an interesting correlation between the maximal values of $|T\rangle$ and the Berry phase by using a eight-degree polynomial fit. Due to this polynomial it's possible to control whether there is a correct population transfer during time evolution.

6.3 Zeno effect

At the same time it was possible to observe a dynamic Zeno effect by varying the populations vs the decay rate γ using master equations instead of Schrödinger equation since in QuTip the Schrödinger equation does not work very well by adding complex terms to the Hamiltonian. Obviously, for the decays it's possible to modify the Hamiltonian's parameter in order to get a faster or a slower population decay using specific algorithms.

6.4 Future perspectives

The studies of this thesis could have some applications to quantum technologies. In fact, during the formation of cold molecules there can be a transfer of populations between the singlet state and the triplet state [45]. There are some possible future developments and unexplored fields of these theoretical concepts. First of all the Berry phase and the counterdiabatic term H_{CD} are concepts related to closed quantum systems but there is no counterdiabatic term to an open quantum systems. There are some methods like repumping techniques that can restore dissipation lost [46], and bath engineering [47], which consists in shaping the external environment to suppress dissipation effects but there is a lack of generalisation.

6.5 Applications

In this thesis we investigated a particular situation realized with Li_2 dimers [28]. Our results can be extended to study more complex such as Rb_2 ground state molecules [48]. Also excitation and detection mechanism of forming in atom-atom collisions in traps could be investigated. Here a few-state model would represent a much simplified of loosely bound molecules. Their excitation and subsequent detection by dissociation occurs in the high excited state. An open quantum system

description, such as used in chapter 5 can be applied and the coupling and decay parameters could be optimized to effectively detect the formation of cold molecules [49, 50]. Hence, the methods, which we developed in this thesis have potentially a plethora of applications.

Appendix

Appendix A

The coefficients of the polynomial fit of the atan function are, respectively:

$$\begin{aligned}a_8 &= -6.95332 \times 10^3, & a_7 &= 2.78132 \times 10^4, \\a_6 &= -4.71818 \times 10^4, & a_5 &= 4.41991 \times 10^4, \\a_4 &= -2.49587 \times 10^4, & a_3 &= 8.70096 \times 10^3, \\a_2 &= -1.85047 \times 10^3, & a_1 &= 2.30968 \times 10^2, \\a_0 &= -1.86840 \times 10^1.\end{aligned}$$

Appendix B

The coefficients of the eight-degree polynomial which describes the maximal population value of the Triplet state $|T\rangle$ vs the Berry's phase are:

$$\begin{aligned}b_0 &= -0.00435 & b_1 &= 0.74855 \\b_2 &= 2.41404 & b_3 &= -5.60260 \\b_4 &= 5.62913 & b_5 &= -3.26138 \\b_6 &= 1.12098 & b_7 &= -0.21196 \\b_8 &= 0.01696.\end{aligned}$$

Appendix C

The Hamiltonian's elements are related to the H of the (3.12) by the relations :

- $\alpha = 0.933$
- $\beta = 0.36$
- $\Delta_{so} = 4.71 \text{ ns}^{-1}$
- $\Omega_p = 0.24 \text{ ns}^{-1}$
- $\Omega_c = 3.8 \text{ ns}^{-1}$
- $\delta_c = 100 \text{ ns}^{-1}$

$$\begin{aligned}\epsilon_1(t) &= \delta_p(t), \quad \epsilon_S = \frac{\beta^2 \Omega_c^2}{4(\delta_c + \Delta_{so})}, \\ \epsilon_T &= -\Delta_{so} + \frac{\alpha^2 \Omega_c^2}{4(\delta_c + \Delta_{so})} = -4.74001, \\ \Omega_{1S} &= \frac{\alpha \Omega_p}{2} = 0.11196, \quad \Omega_{1T} = -\frac{\beta \Omega_p}{2} = -0.0432, \\ \Omega_{ST} &= \frac{\alpha \beta \Omega_c^2}{4(\delta_c + \Delta_{so})} = 0.01158.\end{aligned}$$

Appendix D

The coefficients of the fourth degree polynomial that finds the γ_c transition in section 5.2.3 is

$$\begin{aligned}c_0 &= -1.156305 \times 10^{-2} \\ c_1 &= 1.474306 \times 10^{-1} \\ c_2 &= -7.233213 \times 10^{-1} \\ c_3 &= 1.604387 \times 10^0 \\ c_4 &= -1.341608 \times 10^0\end{aligned}$$

Bibliography

- [1] Richard R Allen, Francisco Machado, Isaac L Chuang, Hsin-Yuan Huang, and Soonwon Choi. Quantum computing enhanced sensing. *arXiv preprint arXiv:2501.07625*, 2025.
- [2] Emils Bagirovs, Grigory Provodin, Tuomo Sipola, and Jari Hautamäki. Applications of post-quantum cryptography. *arXiv preprint arXiv:2406.13258*, 2024.
- [3] Farbod Khoshnoud, Ibrahim Esat, Shayan Javaherian, and Behnam Bahr. Quantum entanglement and cryptography for automation and control. *Instrumentation*, 6(4):1–12, 2019.
- [4] Michael V Berry. Transitionless quantum driving. *Journal of Physics A: Mathematical and Theoretical*, 42(36):365303, 2009.
- [5] Richard P. Feynman. Simulating physics with computers. *International Journal of Theoretical Physics*, 21(6-7):467–488, 1982.
- [6] Joel L. Lebowitz and Herbert Spohn. A Gallavotti–Cohen-Type Symmetry in the Large Deviation Functional for Stochastic Dynamics. *Journal of Statistical Physics*, 95(1):333–365, 1999.
- [7] M. V. Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 392:45–57, 1984.
- [8] D. L. Sounas and A. Alù. Non-reciprocal photonics based on time modulation. *Nature Photonics*, 11(12):774–783, December 2017.

- [9] F. Petiziol, B. Dive, S. Carretta, R. Mannella, F. Mintert, and S. Wimberger. Accelerating adiabatic protocols for entangling two qubits in circuit QED. *Physical Review A*, 99:042315, 2019.
- [10] B. Misra and E. C. G. Sudarshan. The Zeno’s paradox in quantum theory. *Journal of Mathematical Physics*, 18(4):756–763, 1977.
- [11] Jonas Larson and Stig Stenholm. Effective hamiltonians for three-level systems: A pedagogical introduction. *Physics Reports*, 417(1-4):1–28, 2006.
- [12] L. D. Landau. Zur Theorie der Energieübertragung. ii. *Physics of the Soviet Union*, 2(4):46–51, 1932.
- [13] C. Zener. Non-adiabatic crossing of energy levels. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 137(5):696–702, 1932.
- [14] J.J. Sakurai and Jim Napolitano. *Modern Quantum Mechanics*. Addison-Wesley, revised edition edition, 1994.
- [15] J. R. Johansson, P. D. Nation, and F. Nori. Qutip: An open-source python framework for the dynamics of open quantum systems. *Computer Physics Communications*, 183(3):1760–1772, 2012.
- [16] Leonardo Romanato, Najmeh Eshaqi-Sani, Luca Lepori, Teodora Kirova, Ennio Arimondo, and Sandro Wimberger. Reversing adiabatic state preparation. *arXiv preprint arXiv:2507.08439*, July 2025. <https://arxiv.org/abs/2507.08439>.
- [17] Zhiqiang Zhao, Yu-Chen Zhang, Yu Liu, and Ren-Bao Liu. Shortcuts to adiabaticity with inherent robustness. *arXiv preprint arXiv:2211.02543*, 2022.
- [18] Qubit Guide. 2.10 the bloch sphere. <https://qubit.guide/2.10-the-bloch-sphere>, 2023. Accessed: Sep 24, 2025.

- [19] David J. Griffiths. *Introduction to Quantum Mechanics*. Cambridge University Press, 3rd edition, 2018.
- [20] Claude Cohen-Tannoudji, Bernard Diu, and Franck Laloë. *Quantum Mechanics, Volume 1*. Wiley-Interscience, 1977.
- [21] M. M. Wolf. A short introduction to the Lindblad master equation. *arXiv preprint arXiv:1906.04478*, 2019.
- [22] Zhen Li, Hao Wang, and Xi Chen. Counterdiabatic driving for quantum shortcuts. *arXiv preprint arXiv:2403.01854*, 2024.
- [23] Jae Kim, Rohan Patel, and Anjali Singh. Towards adiabatic quantum computing using compressed quantum circuits. *PRX Quantum*, 5:020362, 2024.
- [24] Elena Martinez and Wei Zhou. Advancements in counterdiabatic driving for quantum computing. *SciSimple*, 2025.
- [25] E. H. Ahmed and A. M. Lyyra. Frequency domain control of quantum state singlet/triplet character and prospects for an all-optical spin switch. *Journal of Modern Optics*, 61(7):595–601, 2014.
- [26] E. H. Ahmed, S. Ingram, T. Kirova, O. Salihoglu, J. Huennekens, J. Qi, Y. Guan, and A. M. Lyyra. Quantum control of the spin-orbit interaction using the Autler-Townes effect. *Physical Review Letters*, 107:163601, 2011.
- [27] E. H. Ahmed and A. M. Lyyra. Frequency domain control of quantum state singlet/triplet character and prospects for an all-optical spin switch. *Journal of Modern Optics*, 61(7):595–602, 2014.
- [28] E. H. Ahmed, X. Pan, J. Huennekens, and A. M. Lyyra. Optical control of collisional population flow between molecular electronic states of different spin multiplicity. *Physical Review A*, 89:061401, 2014.

- [29] M. Delvecchio, T. Kirova, E. Arimondo, D. Ciampini, and S. Wimberger. High-fidelity quantum control via Autler-Townes splitting. *Physical Review A*, 106:052802, 2022.
- [30] T. Kirova and F. C. Spano. Designing molecular eigenstates in a four-level λ system. *Physical Review A*, 71:063816, 2005.
- [31] Simon Dengis, Sandro Wimberger, and Peter Schlagheck. Multi-mode noon states generation with ultracold atoms via geodesic counterdiabatic driving. *Physical Review A*, 2025. Accepted for publication.
- [32] Kevin Morawetz and Anatoli Polkovnikov. Universal counterdiabatic driving. *arXiv preprint arXiv:2503.01952*, 2025.
- [33] J. Kölbl, A. Barfuss, M. S. Kasperczyk, L. Thiel, A. A. Clerk, H. Ribeiro, and P. Maletinsky. Initialization of single spin dressed states using shortcuts to adiabaticity. *Physical Review Letters*, 122:090502, 2019.
- [34] D. Hoang Minh and H. B. Nguyen. Controllable optical switching in a closed-loop three-level lambda system. *Physica Scripta*, 94(11):115510, 2019.
- [35] S. Li, T. Chen, J. Liu, and Z.-Y. Xue. Interaction induced non-reciprocal three-level quantum transport. *Chinese Physics B*, 30(6):060314, 2021.
- [36] D. V. Kosachiov, B. G. Matisov, and Y. V. Rozhdestvensky. Coherent phenomena in multilevel systems with closed interaction contour. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 25:2473–2487, 1992.
- [37] F. Petiziol, F. Mintert, and S. Wimberger. Quantum control by effective counterdiabatic driving. *Europhysics Letters*, 145:15001, 2024.

- [38] M. V. Berry. Quantum phase corrections from adiabatic iteration. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 414:31–46, 1987.
- [39] J. Zak. Berry’s phase for energy bands in solids. *Physical Review Letters*, 62:2747–2750, 1989.
- [40] F. Wilczek and A. D. Shapere. *Geometric Phases in Physics*. Advanced Series in Mathematical Physics. World Scientific, Singapore, 1989.
- [41] Carl M. Bender. Making sense of non-hermitian hamiltonians. *Reports on Progress in Physics*, 70(6):947–1018, 2007.
- [42] Mikhail Lemesko and Bretislav Friedrich. Quantum zeno effect. *arXiv preprint arXiv:0903.4560*, 2009. Preprint.
- [43] H. Baumgärtel. Entropy, von Neumann and the von Neumann entropy. *arXiv preprint math-ph/0102013*, 2001.
- [44] Dharm Veer Singh and Shobhit Sachan. How are the degrees of freedom responsible for entropy in btz spacetime? *arXiv preprint arXiv:1612.09387*, 2016.
- [45] M. Viteau, M. Marangoni, D. J. Papoular, M. Allegrini, E. Arimondo, D. Comparat, and P. Pillet. Coherent control of ultracold molecule dynamics in a magneto-optical trap by use of chirped femtosecond laser pulses. *Physical Review A*, 78:023413, Aug 2008.
- [46] T. Lindvall, T. Fordell, I. Tuttonen, and M. Merimaa. Unpolarized, incoherent repumping light for prevention of dark states. *Physical Review A*, 87(1):013439, 2013.
- [47] Dirk Witthaut, Florian Trimborn, and Sandro Wimberger. Dissipation Induced Coherence of a Two-Mode Bose-Einstein Condensate. *Physical Review Letters*, 101(20):200402, 2008.

- [48] E. H. Ahmed, X. Pan, J. Huennekens, and A. M. Lyyra. Experimental study of the $3^3\pi_g$ and $4^3\sigma_g^+$ states in rubidium dimer. *Physical Review A*, 105(3):032823, 2022.
- [49] Jing-Lun Li, Johannes Hecker Denschlag, and José P. D’Incao. Controlling few-body reaction pathways using a Feshbach resonance. *Nature Physics*, 21:228–233, 2025.
- [50] Jing-Lun Li, Paul S. Julienne, Johannes Hecker Denschlag, and José P. D’Incao. Spin hierarchy in van der Waals molecule formation via ultracold three-body recombination. *Physical Review A*, 111(1):013308, 2025.

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