Introduction to Loop Quantum Gravity

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Abstract

We give a short introduction to Loop Quantum Gravity focusing on one of the key concepts underlying its covariant formulation, namely the quantisation of constrained dynamics. First, we discuss a general covariant formulation of classical mechanics that treats space and time equally and then define the quantum theory of such a covariant system. In the end, we review an explicit example of a system that has no classical counter-part and serves as a role model for the understanding of Loop Quantum Gravity.

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1 Introduction

One of the major unresolved problems in theoretical physics is the quantisation of General Relativity (GR), the two most prominent approaches to achieve this goal being String Theory and Loop Quantum Gravity (LQG). The latter is less ambitious in the sense that it 'only' provides a theory of Quantum Gravity and no unification of all known forces. There exist different formulation of LQG, the oldest and best developed one being Canonical LQG (see e.g [3]). However, in the last few years another description has been explored that makes explicit the general covariant nature of GR and LQG.

In this write-up, we give a brief introduction to this so-called covariant formulation of LQG. We focus on the basic idea underlying the construction of Covariant LQG, i.e. general covariance. The structure is as follows. In the second section, we show that Classical Mechanics can be reformulated in a way that treats position and time variables equally. This covariant formulation generalises Newtonian Mechanics to a larger class of systems including GR. However, the new description is largely redundant due to gauge transformations, which in turn leads to the existence of certain constraints on the physical motions. Consequently, the third section deals with the quantisation of general covariant systems whose dynamics are constrained. We arrive at the famous Wheeler-deWitt equation and motivate this hypothesis by showing that we recover standard Quantum Mechanics. In section 4, we take the next step and carry out the developed quantisation procedure for a general covariant model system that cannot be assigned an intuitive meaning of time. We conclude with a last section giving a brief outlook to the remaining steps that lead to the construction of Covariant LQG.

The main reference for this talk is the book [2], especially chapter 2 therein. The example of a timeless system is also studied in great detail in [1].

2 General Covariant Formulation of Classical Mechanics

2.1 Parametric Lagrangian

In the following we will present a formulation of Classical Mechanics that treats space and time on the same footing. Consider a simple system described by the action

\[ S[q] = \int_{t_a}^{t_b} dt \mathcal{L}(q(t), \dot{q}(t)) \]  

The physical motion is determined by Hamilton’s principle, i.e. it is given by the trajectory \( q(t) \) that minimises the action. To be specific, consider the Lagrangian

\[ \mathcal{L}(q, \dot{q}) = \frac{m}{2} \dot{q}^2 - V(q) \]  

The Euler-Lagrange equations yield the Newtonian equations of motion

\[ \frac{d}{dt} m \dot{q} = -\nabla_q V \]  

In this formulation the trajectory is found as the solution of (3) as a function parametrised by \( t \). Instead we can also choose an arbitrary parameter \( \tau \) and treat \( q(\tau) \) and \( t(\tau) \) as depend variables. More precisely, we consider the action as a functional of \( q \) and \( t \)

\[
S[q,t] = \int_{\tau_a}^{\tau_b} dt(\tau) \frac{d}{d\tau} \left( q(\tau), \frac{dq(\tau)/d\tau}{dt(\tau)/d\tau} \right) \tag{4}
\]

Motions \((q(\tau), t(\tau))\) that minimise this action, determine trajectories \( q(t) \) that minimise the original action. Indicating \( \tau \)-derivatives by a dot from now on, we find the so-called parametric form of the Lagrangian (2) to be

\[
\mathcal{L}(q, t, \dot{q}, \dot{t}) = \frac{m}{2} \dot{q}^2 - \frac{m}{2} \dot{t}^2 V(q) \tag{5}
\]

The equations of motion read

\[
\frac{d}{d\tau} \left( \frac{m}{2} \dot{q}^2 \right) + i \nabla_q V = 0 \tag{6}
\]

\[
\frac{d}{d\tau} \left( -\frac{m}{2} \left( \frac{\dot{q}}{\dot{t}} \right)^2 - V(q) \right) = 0 \tag{7}
\]

Obviously, (6) is equivalent to (3) and (7) is just energy conservation, which in turn is a consequence of the Newtonian equations of motion. Therefore the relation between \( q \) and \( t \) is the same and it is precisely this relation that describes the physical content of the theory.

### 2.2 Hamiltonian Constraint

Let us analyse the Hamiltonian formulation of the parametric Lagrangian introduced above. The canonical momenta are calculated as

\[
p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -\frac{m}{2} \left( \frac{\dot{q}}{\dot{t}} \right)^2 - V(q) \tag{8}
\]

\[
p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{m \dot{q}}{\dot{t}} \tag{9}
\]

In order to write the canonical Hamiltonian as a function of the momenta we need to invert the map \( \phi : (i, \dot{q}) \mapsto (p_t, p_q) \) given by equations (8) and (9). However, the image of \( \phi \) in \((p_t, p_q)\)-space is constrained by

\[
C(t, q, p_t, p_q) \equiv p_t + H_0(p_q, q) \equiv p_t + \frac{p_q^2}{2m} + V(q) = 0 \tag{10}
\]

Since \( H_0 \) is the Hamiltonian of the classical system we started from, we recognise that this equation constrains the momentum canonical to \( t \) as (minus) the energy of
our system. The canonical Hamiltonian as the Legendre transform of the parametric Lagrangian vanishes identically due to the existence of the constraint (10), namely

$$H = i\dot{p}_t + \dot{q}_p - L = i\dot{p}_t + \frac{p_t}{m} p_q - \frac{m}{2\hbar} \left( \frac{p_t}{m} \right)^2 + iV(q) = iC(t, q, p_t, p_q) = 0$$

(11)

For this reason, (10) is sometimes called the Hamiltonian constraint. The existence of a constraint is in fact not very surprising: Recall that we introduced a completely arbitrary parameter $\tau$. A general covariant system is characterised by the fact that it is not affected by a gauge transformation of the type

$$q(\tau) \mapsto q(\tau'(\tau)) \quad \quad t(\tau) \mapsto t(\tau'(\tau))$$

(12)

with a diffeomorphism $\tau'(\tau)$. Invariance under such a transformation means that $\tau$ is pure gauge. Consequently, there is no evolution in this parameter and its generator, i.e. the canonical Hamiltonian $H$, vanishes.

### 2.3 The Hamilton Function

The Hamilton Function of a classical system is defined on a physical trajectory (i.e. one minimising the action), e.g. $(Q(\tilde{\tau}), T(\tilde{\tau})) = (q_{\tau t}, q_{\tau t'}(\tilde{\tau}), t_{\tau t}, t_{\tau t'}(\tilde{\tau}))$ connecting two points $(q, t)$ at $\tau$ and $(q', t')$ at $\tau'$ as the corresponding value of the action

$$S(q, t, \tau, q', t', \tau') = \int_{\tau}^{\tau'} d\tilde{\tau} \mathcal{L}(Q, T, \dot{Q}, \dot{T})$$

(13)

This function is not to be confused with the action functional $S[q, t]$. Since the action is invariant under gauge transformation, $S(q, t, \tau, q', t', \tau') = S(q, t, q', t')$ is actually independent of the evolution parameter $\tau$. It only depends on the physical boundaries of the process and agrees with the Hamilton function of the system in the standard Newtonian description. The Hamilton-Jacobi equation for the Hamilton function of a general covariant system reads

$$\frac{\partial S}{\partial \tau} + H \left( \frac{\partial S}{\partial t}, \frac{\partial S}{\partial q}, q, t \right) S(q, t, q', t') = 0$$

(14)

which is equivalent to

$$C \left( \frac{\partial S}{\partial t}, \frac{\partial S}{\partial q}, q, t \right) S(q, t, q', t') = 0$$

(15)

because of (11) and $\frac{\partial S}{\partial \tau} = 0$. This is the general covariant form of the Hamilton-Jacobi equation. It contains the full dynamics of the system encoded in the Hamiltonian constraint. For the simple case considered in (10), this equation reduces to the well-known expression from classical mechanics. However, it can take more general forms where the relations between $q$ and $t$ cannot be entangled (i.e. writing $q$ as $q(t)$) and thus allow no canonical interpretation of $t$ as a time variable.
2.4 Classical Physics without Time

In order not to confuse the quantities $q, t$ with the degrees of freedom of a classical system, we refer to them as partial observables. Obviously, a general covariant system has always more partial observables than quantities that can be predicted from the knowledge of an initial state. However, solving the Hamilton-Jacobi equation given by the Hamiltonian constraint allows us to determine the relations among the partial observables. We denote by $\mathcal{C}_{\text{ext}}$ the space of all partial observables, called the extended configuration space. In our case $(q, t) = x \in \mathcal{C}_{\text{ext}} = \mathcal{C} \times \mathbb{R}$, where $\mathcal{C}$ is the classical configuration space of a single particle.

Summarising the ideas of this section, we have adopted the following formulation of Classical Mechanics. Let $\mathcal{C}_{\text{ext}}$ be an extended configuration space and $x \in \mathcal{C}_{\text{ext}}$. The action of a general covariant system is a functional of the trajectories $x(\tau)$ which is invariant under diffeomorphisms of the parameter $\tau$

$$S[x] = \int d\tau \mathcal{L}(x, \dot{x}) \quad \text{invariant under} \quad x(\tau) \mapsto x(\tau'(\tau)) \quad (16)$$

This implies the vanishing of the canonical Hamiltonian and the existence of a constraint

$$C(p, x) = 0 \quad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \quad (17)$$

which determines the general covariant Hamilton-Jacobi equation

$$C \left( \frac{\partial S}{\partial x}, x \right) = 0 \quad p = -\frac{\partial S(x, x')}{\partial x} \quad (18)$$

Solving for the Hamilton function and fixing the initial momenta $p$ and partial observables $x$, this gives the relation among the partial observables $x'$, which is the predictive content of the theory.

3 Quantisation of Constrained Dynamics

We define the quantum theory of a covariant system by

1. A Hilbert space $\mathcal{K}$ where self-adjoint operators $x$ and $p_x$ are defined such that they correspond to classical coordinates $x \in \mathcal{C}_{\text{ext}}$ and their canonical momenta.

2. A constraint operator $C$ with classical limit $C(x, p)$.

$\mathcal{K}$ is called the kinematical Hilbert space. Notice that our definition in contrast to canonical quantisation does not make use of a time variable or a Hamiltonian. We are interested in the transition amplitudes $W(x, x')$ defined by the constraint operator. Let us do this more concretely and consider the discrete case first.
3.1 Discrete Spectrum

Assuming that zero is in the discrete spectrum of \( C \), we see that

\[
\mathcal{H} = \{ \psi \in \mathcal{K} \mid C\psi = 0 \} \subset \mathcal{K}
\]

forms a proper subspace, hence it also is a Hilbert space. It is called the physical state space. The equation

\[
C\psi = 0
\]

is called the Wheeler-deWitt equation. There exists an operator \( P : \mathcal{K} \to \mathcal{H} \) given by the orthogonal projection, such that we can define the transition amplitude as

\[
W(x, x') = \langle x' \mid P \mid x \rangle
\]

where \( |x\rangle \) denotes the eigenstates of the operator \( x \) as usual. Formally, this can be rewritten as a path integral over all paths connecting \( x \) with \( x' \)

\[
W(x, x') = \langle x' \mid \delta(C) \mid x \rangle \sim \int_{-\infty}^{\infty} d\tau \langle x' \mid e^{i\tau C} \mid x \rangle \sim \int_{x}^{x'} Dx(\tau) e^{iS[x]}
\]

since the constraint \( C \) generates evolution in \( \tau \). However, due to gauge invariance, this integral includes a large redundancy to be factored out. Fixing e.g. \( \tau = t \), we recover the path integral known from standard quantum mechanics.

3.2 Continuum Spectrum

Assuming zero to be in the continuous spectrum of \( C \), a similar construction allows us to treat the continuous case. We choose a dense subspace \( S \) of \( \mathcal{K} \) whose dual \( S^* \) defines the space of generalised states. The triple \( S \subset \mathcal{K} \subset S^* \) is usually called a Gelfand triple. We then define the physical state space by implementing the constraint as

\[
\mathcal{H} = \{ \psi \in S^* \mid \psi(C\phi) = 0 \forall \phi \in S \}
\]

The map \( P : S \to \mathcal{H} \) is given by

\[
(P\phi)(\phi') = \int_{-\infty}^{\infty} d\tau \langle \phi \mid e^{i\tau C} \mid \phi' \rangle
\]

and the physical state space \( \mathcal{H} \) can shown to be a Hilbert space with scalar product \( \langle P\phi | P\phi' \rangle = (P\phi)(\phi') \). Note that although \( P \) no longer is a projection operator in the mathematical sense, we will still call it the projector.

Considering our example from the previous section, the kinematical Hilbert space can taken to be \( \mathcal{K} = L^2(\mathbb{R}^2, dq \, dt) \) with partial observables represented by the diagonal
operators $q$ and $t$ and their canonical conjugates $p_q = -i\hbar \frac{\partial}{\partial q}$ and $p_t = -i\hbar \frac{\partial}{\partial t}$. From (10), we deduce the constraint operator to be

$$C = -i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2}$$

(25)

such that the Wheeler-deWitt equation becomes the Schrödinger equation. It can be shown that $S$ and $S^*$ can taken to be the Schwartz space and the space of tempered distributions, respectively. The transition amplitude is calculated as follows

$$W(q, t; q', t') = \int d\tau \langle q', t' | e^{\frac{i}{\hbar} \tau C} | q, t \rangle$$

(26)

$$= \int d\tau dp dp_q \langle q', t' | e^{\frac{i}{\hbar} \tau p + \frac{i}{2\hbar} \frac{p_q^2}{2m}} | p_q, p_t \rangle \langle p_q, p_t | q, t \rangle$$

(27)

$$= \frac{1}{2\pi \hbar} \int d\tau dp dp_q \exp \left( \frac{i}{\hbar} \tau \left( p_t + \frac{p_q^2}{2m} \right) + (q' - q)p_q + (t' - t)p_t \right)$$

(28)

$$= \frac{1}{2\pi \hbar} \int dp dp_q \delta \left( p_t + \frac{p_q^2}{2m} \right) e^{\frac{i}{\hbar}(q' - q)p_q + (t' - t)p_t}$$

(29)

$$= \frac{1}{2\pi \hbar} \int dp_q e^{\frac{i}{\hbar}(q' - q)p_q} \frac{p_q^2}{2m} e^{\frac{i}{\hbar}m(q' - q)^2} = \sqrt{\frac{m}{2\pi \hbar(t' - t)}} e^{\frac{i}{\hbar}m(q' - q)^2}$$

(30)

where we have inserted a resolution of the identity, used the relation $\langle q', t' | p_q, p_t \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{\frac{i}{\hbar}(q'p_q + t'p_t)}$ and performed a Gaussian integral in the last step. The final result agrees with the well-known expression from standard quantum mechanics.

### 3.3 Interpretation

The transition amplitude $W(q, t; q', t')$ yields the physical overlap of the kinematical states representing the events 'particle at $(q, t)$' and 'particle at $(q', t')$'. This is accomplished by the operator $P$ projecting onto the physical state space. We are dealing with states that represent events, while a quantum process is given by the events that form its boundaries. This interpretation turns out to be very efficient in Loop Quantum Gravity, where the conventional way of thinking of objects propagating in time fails.

The Wheeler-deWitt equation (20) is the central physical hypothesis we make in order to define the quantum theory of a general covariant system. It is a wave equation whose classical limit is the Hamilton-Jacobi equation (18) and reduces to the Schrödinger equation as seen above. Therefore the corresponding quantum theory has the correct form for Newtonian systems and yields the correct classical limit. This suffices to take the hypothesis seriously and apply the above procedure to general covariant systems that have no Newtonian counter-part. The covariant method treats space and time variables on equal footing as promised in the introduction. While the
canonical quantisation procedure relies on the definition of a Hamiltonian and the associated time evolution and thus can not be used in the context of Quantum Gravity, the general covariant procedure can be used to construct the transition amplitudes of Covariant LQG.

4 Example of a Timeless System

We are now in a position to apply the new concepts to the quantisation of a general covariant system that cannot be described in terms of Newtonian mechanics and thus lies beyond the realm of standard quantum mechanics. Our example system is given by the extended configuration space $C_{\text{ext}} = \mathbb{R}^2$ parametrised by $a, b$ and the Lagrangian

$$\mathcal{L}(a, b, \dot{a}, \dot{b}) = \sqrt{(2E - a^2 - b^2) \left(\dot{a}^2 + \dot{b}^2\right)}$$

(31)

with $E = \text{const.}$

4.1 Classical Solution

Under a reparametrisation $\tau \rightarrow \tau'$ the action remains invariant

$$S[a, b] = \int d\tau \mathcal{L}(a, b, \dot{a}, \dot{b})$$

(32)

$$= \int d\tau' \frac{d\tau(\tau')}{d\tau'} \sqrt{(2E - a^2 - b^2) \left(\frac{da(\tau(\tau'))}{d\tau'} \frac{d\tau'(\tau)}{d\tau}\right)^2 + \left(\frac{db(\tau(\tau'))}{d\tau'} \frac{d\tau'(\tau)}{d\tau}\right)^2}$$

(33)

$$= \int d\tau' \mathcal{L} \left(a, b, \frac{da}{d\tau'}, \frac{db}{d\tau'}\right)$$

(34)

Consequently, the canonical Hamiltonian vanishes identically

$$H(a, b, p_a, p_b) \equiv 0,$$

(35)

which implies the existence of a constraint $C(a, b, p_a, p_b)$. The canonical momenta are

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = \frac{(2E - a^2 - b^2)\dot{a}}{\mathcal{L}}$$

(36)

$$p_b = \frac{\partial \mathcal{L}}{\partial \dot{b}} = \frac{(2E - a^2 - b^2)\dot{b}}{\mathcal{L}}$$

(37)

and we can directly read off

$$p_a^2 + p_b^2 = 2E - a^2 - b^2.$$  

(38)
Thus the constraint can chosen to be

\[ C(a, b, p_a, p_b) = \frac{1}{2} \left( p_a^2 + a^2 + p_b^2 + b^2 \right) - E \]  \hspace{1cm} (39)

We can easily solve the equations of motion by noting that the constraint is just the sum of two harmonic oscillator Hamiltonians. Thus

\[ a(\tau) = A \cos(\tau + \theta_a) \] \hspace{1cm} (40) \\
\[ b(\tau) = B \cos(\tau + \theta_b) \] \hspace{1cm} (41)

with the constraint

\[ A^2 + B^2 = 2E \] \hspace{1cm} (42)

Due to gauge invariance, we can set \( \theta_b = 0 \) by \( \tau \mapsto \tau - \theta_b \). Therefore the space of solution is two-dimensional and we can choose to parametisze it by \( A \) and \( \theta = \theta_a \). Eliminating the gauge parameter \( \tau \), we find the relation among \( a \) and \( b \)

\[ \arccos \left( \frac{a}{A} \right) - \arccos \left( \frac{b}{\sqrt{2E - A^2}} \right) = \theta \] \hspace{1cm} (43)

where \( A \) and \( \theta \) are determined by the initial conditions. This shows that the motions in extended configuration space are ellipses and thus can not be disentangled to define a global meaning of time. Equivalently, the motions are always bounded, whereas in Newtonian Mechanics, there always is a variable \( t \in (-\infty, \infty) \). Thus our system has no Newtonian counter-part and we need the covariant quantisation procedure to define a corresponding quantum theory. Note that we avoided solving the Hamilton-Jacobi equation explicitly because our system resembles classical harmonic oscillators. However, one can show that a possible family of solutions reads

\[ S(a, b) = \frac{a}{2} \sqrt{A^2 - a^2} + \frac{A^2}{2} \arctan \left( \frac{a}{\sqrt{A^2 - a^2}} \right) - \frac{b}{2} \sqrt{B^2 - b^2} + \frac{B^2}{2} \arctan \left( \frac{b}{\sqrt{B^2 - b^2}} \right) \] \hspace{1cm} (44)

With some algebra, one can check that this yields the same relation between \( a \) and \( b \) as in (43).

4.2 Quantum Theory

Let us now turn to the quantum theory with the kinematical Hilbert space given by \( \mathcal{K} = L^2(\mathbb{R}^2, da \, db) \). Since the constraint is just the sum of two harmonic oscillators, we choose the energy eigenbasis where the Wheeler-deWitt equation becomes diagonal

\[ C|n_a, n_b\rangle = \left( (n_a + \frac{1}{2}) + (n_b + \frac{1}{2}) - E \right) |n_a, n_b\rangle = 0 \] \hspace{1cm} (45)
Solutions only exist for integer \( E = N + 1 \) with \( N = n_a + n_b \) and are in general built by linear combinations of \(|n_a, n_b\rangle\) for fixed \( N \). Thus the physical state space \( \mathcal{H} \) is formed by states of the form

\[
|\psi\rangle = \sum_{n=1}^{N} c_n |n, N - n\rangle \tag{46}
\]

and the projector \( P : \mathcal{K} \rightarrow \mathcal{H} \) is given by

\[
P = \sum_{n=1}^{N} |n, N - n\rangle \langle n, N - n| \tag{47}
\]

From this, we can explicitly calculate the transition amplitude

\[
W(a, b; a', b') = \langle a', b'| P |a, b\rangle = \sum_{n=1}^{N} \langle a', b'| n, N - n\rangle \langle n, N - n|a, b\rangle \tag{48}
\]

\[
= \sum_{n=1}^{N} \langle a'| n\rangle \langle b'| N - n\rangle \langle n| a\rangle \langle N - n|b\rangle = \sum_{n=1}^{N} \psi_n(a') \tilde{\psi}_n(a) \psi_{N-n}(b') \tilde{\psi}_{N-n}(b) \tag{49}
\]

with \( \psi_n(x) \) being the eigenstates of the quantum harmonic oscillator in position space representation. Since we lack a global notion of time, we can not speak of one degree of freedom that is propagating. Rather the quantum theory of the model system discussed above describes the correlations among the two partial observables. This very general observation also holds true in LQG.

### 5 Application to General Relativity

So far we have presented a method to quantise systems with constraints, but have only applied it to a simple model example. In this last section, we want to give a brief outline of the steps that are needed to apply the formalism to GR. Thus we will now briefly sketch the construction of Covariant Loop Quantum Gravity.

GR is a general covariant theory with an action invariant under diffeomorphisms of the underlying manifold and hence contains a Hamiltonian constraint. However, it is highly non-trivial to deal with the corresponding Wheeler-deWitt equation. One reason is the fact that we are dealing with a field theory with an infinite number of degrees of freedom. To get around this problem, one can define a discretisation of GR that yields the full theory in the continuum limit. This is also done in standard Quantum Field Theory to rigorously define the path integral. It is most convenient (and also necessary when considering fermions coupling to the gravitational field) to work in the tetrad formulation of GR. In the case of four-dimensional GR, it is possible to take a triangulation of a compact region of space-time by chopping it into 4-simplices. It turns
out to be useful to consider its dual object, i.e. a 2-complex in the mathematical sense. On the boundary of the space-time region this 2-complex reduces to a graph consisting of some nodes connected by certain links. Next, one assigns $SL(2, \mathbb{C})$ group elements to the links and $sl(2, \mathbb{C})$ algebra elements to the faces bounded by these links by means of a holonomy. The latter are responsible for the term loop in LQG. One can show that these variables conspire to give an action to approximates standard GR.

In the next step, one has to define the Hilbert space of LQG. The idea is to consider $L^2[SL(2, \mathbb{C})^l]$ for a graph consisting of $l$ links. Now, the constraints of GR have to be implemented, which is the technical core of LQG. It is possible to rephrase the Hamiltonian constraint as a constraint between magnetic and electric parts of the classical gravitational field. This latter constraint can be used to define a map from $L^2[SU(2)]$ to functions on $SL[2, \mathbb{C}]$ whose image yields the quantum states of GR, which are for this reason often called spin networks. At this point, an interesting result drops out. Due to the definition of the quantum states of gravity, i.e. the quantum states of space-time, there exist a lowest length scale and hence space-time itself becomes discrete. It is important to note that this is a consequence of the quantisation procedure, which is based on general covariance and the principles of Quantum Mechanics. It is not a postulate in order to secure renormalisability of the quantum theory.

Finally, it remains to define the transition amplitudes of LQG by means of the projector onto the physical Hilbert space. They turn out to obey a superposition principle, locality and Lorentz invariance as it is expected for a consistent quantum theory of GR. One can further show that this construction of Covariant LQG is connected to GR in the classical limit, it remains UV finite and (including the cosmological constant) IR finite. It reproduces the $n$-point correlation functions of perturbative GR, the Friedmann equations in cosmology and the Bekenstein-Hawking entropy of black holes. In conclusion, this makes Covariant LQG a promising candidate for the quantum theory of GR. In current research LQG is extended to the matter sector in order to include interactions beyond gravity.

References

