Quantum Fluctuations during Inflation

1 Recap

In the last talk we introduced Inflation as a phase of accelerated expansion before the classical Hot Big Bang, which solved various important cosmological problems. The easiest way to achieve Inflation was to introduce a scalar field \( \phi \), which has to fulfill the following conditions:

\[
\epsilon = \frac{\dot{\phi}^2}{2H^2} < 1.
\]

\[
\eta = -\frac{\ddot{\phi}}{H\dot{\phi}}, \quad \text{with} \quad |\eta| \ll 1
\]

These parameters are called slow-roll parameters. We understand the meaning of them, if we look at a typical potential like in figure 1. This becomes more intuitive, when we define a different kind of slow-roll parameters in terms of derivatives of the potential:

\[
\epsilon_V = \frac{M_{pl}^2}{2} \left( \frac{V\phi}{V} \right)^2
\]

\[
\eta_V = M_{pl}^2 \frac{V\phi\dot{\phi}}{V}
\]

In the slow-roll limit they are related to the original slow-roll parameters by:

\[
\epsilon \approx \epsilon_V \quad \text{and} \quad \eta \approx \eta_V - \epsilon_V.
\]

2 Motivation/Overview

In this talk we will quantize the Inflaton field. This is motivated mainly by two very important cosmological questions:

1. **Origin of structure**: During Inflation all classical inhomogeneities are wiped out and at the end of Inflation classically we would deal with a perfectly homogeneous Universe. But when we look into the universe, we see a lot of structures on different scales (e.g. earth, sun, galaxies, clusters, . . . ). We also see structure in the CMB, from which todays structures originate. But where did the CMB structures originate? The answer is through quantum fluctuations during inflation. Fluctuations in the Inflaton field \( \phi \) lead to a local time delay, when Inflation ends (see figure 1). So different regions inflate by a different amount, which leads to density perturbations and ultimately to the temperature fluctuations in the CMB. So Inflation on the one side explains, why the Universe is extremely homogeneous, since Inflation wipes away all initial classical inhomogeneities, but on the other side it explains the origin of small inhomogeneities via quantum fluctuations.
2. Testing Inflation models: When choosing a special model for Inflation, we can predict the shape of the CMB structures (i.e. the power-spectrum, which will be introduced later). Comparing with measurements of the CMB (e.g. by the Planck satellite) we can constrain our model for Inflation.

So the goal of todays talk will be to present the calculation of the primordial power-spectrum (=”amount of fluctuations”), which is the seed for the structures in the CMB and in the end to compare briefly the results for different Inflation models with observations.

3 Cosmological Perturbation Theory

Before we can start quantizing the Inflaton field and calculate the power-spectrum we have to introduce a few concepts from cosmological perturbation theory.

The inhomogeneities in the CMB are at the 10^{-5} level so the strategy is to split all quantities $X(t,\mathbf{x})$ (the metric $g_{\mu\nu}$ and the matter quantities $\phi$, $\rho$ and $p$) into an homogeneous background $\bar{X}(t)$ and an inhomogeneous perturbation:

$$\delta X(t,\mathbf{x}) := X(t,\mathbf{x}) - \bar{X}(t).$$ \hspace{1cm} (6)

The evolution of the perturbations is then given by:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}.$$ \hspace{1cm} (7)

Although we won’t discuss it in detail here, we have to introduce the SVT (scalar, vector, tensor) decomposition of the perturbed metric here. This is just a very general
way to write down a perturbed metric:

\[ ds^2 = -(1 + 2\Phi)dt^2 + 2aB_idx^i dt + a^2[(1 - 2\Psi)\delta_{ij} + E_{ij}]dx^idx^j, \]  

(8)

where \( a \) is the scale factor and:

\[ B_i := \partial_i B - S_i, \quad \text{with} \quad \partial^i S_i = 0 \]  

(9)

and

\[ E_{ij} = 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}, \quad \text{with} \quad \partial^i F_i = 0, \quad h_i^i = 0 = \partial^i h_{ij}. \]  

(10)

The quantity \( h_{ij} \) looks similar to gravitational waves, which will turn out to be true. The tensor perturbations \( h_{ij} \) are responsible for the production of primordial gravitational waves.

Let’s count at this point the degrees of freedoms of the whole system:

- 5 scalar modes: \( \Phi, B, \Psi, E \) and the Inflaton perturbations \( \delta \phi \).
- 4 vector modes: We have two vector perturbations \( S_i \) and \( F_i \), but both are constraint by \( \partial^i S_i \) and \( \partial^i F_i \)
- 2 tensor modes: \( h_{ij} \) is symmetric and has four constraints \( h_i^i = 0 = \partial^i h_{ij} \Rightarrow 6 - 4 = 2 \) polarization modes

**Remark on gauge choice:** Consider a completely homogeneous Universe. Now we can make a gauge transformation, e.g.:

\[ t \to t + \tau(t, x). \]  

(11)

Then for example the field \( \phi \) transforms as:

\[ \phi(t) \to \phi(t) + \tau(t, x)\dot{\phi}(t) = \phi(t) + \delta \phi(t, x). \]  

(12)

So it seems like we have an inhomogeneity, although we assumed an homogeneous Universe. This is an coordinate artifact. So our interpretation of fluctuations seems to depend on the choice of gauge. Hence it is necessary to introduce gauge invariant measures for inhomogeneities. One such quantity is the *comoving curvature perturbation*:

\[ R := \Psi - \frac{H}{\dot{\rho} + \dot{p}}\delta q, \]  

(13)

which measures geometrically the spatial curvature of comoving (or constant-\( \phi \)) hypersurfaces. The 3-momentum density \( \delta q \) is defined via \( T_i^0 = \partial_i \delta q \). During Inflation we have \( (T_i^0 = -\dot{\phi}\partial_i \delta \phi) \):

\[ R = \Psi + \frac{H}{\dot{\phi}}\delta \phi. \]  

(14)

The tensor perturbations turn out to be gauge invariant as well.
Now we can reduce our degrees of freedom via gauge choices: A gauge choice eliminates 2 scalar and 2 vector degrees of freedom. Corresponding to these gauge choices there are four constraint equations from the Einstein equations, that eliminate another 2 scalar and 2 vector modes (constraint equations are equations without second derivatives, which don’t describe dynamics). So in the end we are left with 1 scalar and 2 tensor degrees of freedom. The scalar modes are responsible for the density perturbations we observe in the CMB and the tensor modes lead to the production of primordial gravitational waves. These gravitational waves produce B-mode polarizations in the CMB, but have not been detected yet.

We now choose for the rest of the talk the so called comoving gauge \((\delta \phi = 0, \, E = 0)\). In this gauge all degrees of freedom are parametrised by \(\mathcal{R}\) and \(h_{ij}\):

\[
g_{ij} = a^2 [(1 - 2\mathcal{R})\delta_{ij} + h_{ij}] \tag{15}
\]

4 Calculation strategy

Before we start with the calculation of the quantum fluctuations we want to give a short overview of the strategy: During Inflation we have a (quasi-) de Sitter expansion, so the comoving Hubble Horizon shrinks (see figure 2). At some point a comoving fluctuation will exit this horizon. A very important property of \(\mathcal{R}\) is that it is constant outside the horizon. That means \(\mathcal{R}\) becomes superhorizon and will not change anymore until the mode re-enters the horizon after the end of Inflation. So if we compute the fluctuations at horizon crossing it won’t change until re-entry. With a transfer function and a projection onto the sky we can make the last step from re-entry to the observed CMB fluctuations today (we won’t do this last step here in the talk and leave it to the astrophysicists). So our aim will be to compute the fluctuations at horizon crossing, which then can be compared to the CMB.

5 Quantum fluctuations

Now we start with the quantization. The action for the Inflaton field is:

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left( R + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi) \right), \tag{16}
\]

If we expand this action using the metric in our chosen comoving gauge to second order in \(\mathcal{R}\) we get:

\[
S^{(2)} = \frac{1}{2} \int d^4xa^3 \frac{\phi'^2}{H^2} \left[ \mathcal{R}^2 - a^{-2}(\partial_i \mathcal{R})^2 \right]. \tag{17}
\]

When we define:

\[
v := z\mathcal{R}, \quad \text{where} \quad z^2 := a^2 \frac{\phi'^2}{H^2} \tag{18}
\]

we can rewrite the action using the conformal time \(\eta\):

\[
S^{(2)} = \frac{1}{2} \int d\eta d^3x \left[ (v')^2 - (\partial_i v)^2 + \frac{z''}{z} v^2 \right]. \tag{19}
\]
This is exactly the action, that was studied in the talk about de Sitter space. So we wont go through the technical details again. In this talk a special vacuum called Bunch-Davies vacuum was introduced. In this vacuum the mode functions had the form:

$$v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right).$$  \hspace{1cm} (20)

From this we can calculate the vacuum fluctuations of the mode $\hat{v}_k = v_k \hat{a}_k + v_k^* \hat{a}^\dagger_{-k}$:

$$\langle 0 | \hat{v}_k(\eta) \hat{v}_k'(\eta) | 0 \rangle = (2\pi)^3 \delta^{(3)}(k - k') |v_k(\eta)|^2,$$  \hspace{1cm} (21)

where we easily can calculate:

$$|v_k(\eta)|^2 = \frac{1}{2k^3 \eta^2} (1 + k^2 \eta^2).$$  \hspace{1cm} (22)

Now we get the vacuum fluctuations of $\mathcal{R}$ by resubstituting $v = a \dot{r} \mathcal{R}$ and using $a\eta = -\frac{1}{\dot{r}}$:

$$\langle 0 | \mathcal{R}_k(\eta) \mathcal{R}_{k'}(\eta) | 0 \rangle = (2\pi)^3 \delta^{(3)}(k - k') \frac{H^4}{2k^3 \dot{\phi}^2} (1 + k^2 \eta^2).$$  \hspace{1cm} (23)

On superhorizon scales ($|k\eta| \ll 1$) the vacuum fluctuations approach:

$$\langle 0 | \mathcal{R}_k(\eta) \mathcal{R}_{k'}(\eta) | 0 \rangle \rightarrow (2\pi)^3 \delta^{(3)}(k - k') \frac{H^4_*}{2k^3 \dot{\phi}^2_*},$$  \hspace{1cm} (24)

where we evaluated at horizon crossing (a mode crosses the horizon for $H_* = \frac{k}{a'(\eta_c)}$).
Remark on power spectra: The vacuum fluctuations are related to the so-called power spectrum. Power spectra are an important tool in cosmology, since they contain all statistical information of a statistical inhomogeneity (if it is Gaussian). The power spectrum of mode $k$ is defined via:

$$\langle R_k R_{k'} \rangle = (2\pi)^3 P_R(k) \delta^{(3)}(k - k'),$$  \hspace{1cm} (25)$$

So we see that the power spectrum for superhorizon modes becomes:

$$P_R(k) \rightarrow \frac{H_*^4}{2k^3 \dot{\phi}_*^2}$$  \hspace{1cm} (26)$$

Most of the time we will work with the dimensionless power spectrum, which is defined as:

$$\Delta^2_{R}(k) = \frac{k^3}{2\pi^2} P_R(k) = \frac{H_*^4}{(2\pi)^2 \dot{\phi}_*^2} = \Delta^2_s(k),$$  \hspace{1cm} (27)$$

where the subscript $s$ denotes scalar perturbations. In an analogous calculation we get for the tensor fluctuations:

$$\Delta^2_t(k) = 2\Delta^2_{\phi}(k) = \frac{2}{\pi^2} \frac{H_*^2}{M_{pl}^2}.$$  \hspace{1cm} (28)$$

Although these spectra seem to be independent of $k$, since $k$ does not appear explicitly, they still depend on $k$, since the time of horizon-crossing depends on $k$ ($H_* = \frac{k}{a(\eta)}$).

6 Power spectra in terms of slow-roll parameters

At this point we want to remind you of the slow-roll parameters introduced in the beginning:

$$\epsilon = \frac{\dot{\phi}^2}{2H^2}$$  \hspace{1cm} (29)$$

$$\eta = \frac{\ddot{\phi}}{H \dot{\phi}}$$  \hspace{1cm} (30)$$

$$\epsilon_V = \frac{M_{pl}^2}{2} \left( \frac{V_{,\phi}}{V} \right)^2$$  \hspace{1cm} (31)$$

$$\eta_V = \frac{M_{pl}^2 V_{,\phi\phi}}{V}.$$  \hspace{1cm} (32)$$

In the slow-roll limit they are related by:

$$\epsilon \approx \epsilon_V \quad \text{and} \quad \eta \approx \eta_V - \epsilon_V.$$  \hspace{1cm} (33)$$

Now we introduce the spectral indices $n_s$ and $n_t$ as well as the tensor-to-scalar ratio $r$, which measures the amount of primordial gravitational waves, and express them in
terms of the slow-roll parameters:

\[ n_s - 1 = \frac{d \ln \Delta_s^2}{d \ln k} = 2\eta_* - 4\epsilon_* \approx 2\eta_V^* - 6\epsilon_V^* \] (34)

\[ n_t = \frac{d \ln \Delta_t^2}{d \ln k} = -2\epsilon_* \approx -2\epsilon_V^* \] (35)

\[ r = \frac{\Delta_t^2}{\Delta_s^2} = 16\epsilon_* = -8n_t \approx 16\epsilon_V^*. \] (36)

The interpretation of e.g. \( n_s \) can be seen if we assume a power law for the power spectrum \( \Delta_s^2 \):

\[ \Delta_s^2 = A \left( \frac{k}{k_0} \right)^{n_s-1} \Rightarrow \frac{d \ln \Delta_s^2}{d \ln k} = n_s - 1, \] (37)

where \( k_0 \) is an arbitrary reference scale.

The parameters \( n_s, n_t \) and \( r \) can be measured with the CMB. The three parameters are not independent and the equation \( r = -8n_t \) is therefore called consistency relation. Usually \( n_s \) and \( r \) are measured.

### 7 Example \( \phi^2 \)

Now we want to calculate \( n_s \) and \( r \) for the simple \( \phi^2 \) model. In this model we have:

\[ \epsilon_V = \frac{M_{pl}^2}{2} \left( \frac{V_{\phi}}{V} \right)^2 = 2 \left( \frac{M_{pl}}{\phi} \right)^2 = \eta_V. \] (38)

In order to evaluate \( \epsilon_V^* \) and \( \eta_V^* \) we need \( \phi_* \), which we get by the following consideration: The number of e-folds (\( N := \ln a \)) the universe expands during Inflation can be calculated for this model via:

\[ N(\phi_*) = \ln \left( \frac{a_{\text{end}}}{a_*} \right) = \cdots = \int_{\phi_{\text{end}} = \sqrt{2}M_{pl}}^{\phi_*} \frac{d\phi}{\sqrt{2}\epsilon} = \frac{\phi_*^2}{4M_{pl}^2} - \frac{1}{2}. \] (39)

As discussed in the last talk we need \( N \sim 60 \) and hence \( \phi_*^2 \sim 240M_{pl}^2 \) and \( \epsilon_V^* = \eta_V^* \sim 1/120 \). With this we can calculate:

\[ n_s = 1 + 2\eta_V^* - 6\epsilon_V^* \approx 0.966 \] (40)

and

\[ r = 16\epsilon_V^* \approx 0.13 \] (41)

### 8 Planck results

In the end we now want to compare the theoretical predictions for \( n_s \) and \( r \) of different Inflation models with observations of the CMB, in particular the observation from the Planck satellite (see figure 3).

We see that the simple model of \( \phi^2 \) seems to be excluded already, but there are other models that are still possible. We also observe that we only have an upper limit for \( r \) at the moment, since primordial gravitational waves have not been detected yet.
Figure 3: Constraints for $n_s$ and $r$ from Planck and other observations of the CMB and predictions of theoretical models.

9 Additional material: Energy scale of inflation

From the CMB we know that $\Delta^2_\text{S} \sim 10^{-9}$. Using $H^2 \sim V$ we can estimate the energy scale of Inflation:

$$V^{1/4} \sim \left( \frac{r}{0.01} \right)^{1/4} 10^{16} \text{GeV}$$

so $r = 0.01$ corresponds to GUT-scale.

We can also approximate how much the Inflaton-field $\phi$ changes during Inflation by integrating $r = \frac{8}{M_{\text{pl}}^2} \left( \frac{d\phi}{dN} \right)^2$ up to $N = 60$ e-folds:

$$\frac{\Delta \phi}{M_{\text{pl}}} = \mathcal{O}(1) \times \left( \frac{r}{0.01} \right)^{1/2}$$

This relation tells us, that $r > 0.01$ corresponds to $\Delta \phi > M_{\text{pl}}$ and hence to large field Inflation. So if large field Inflation is the right solution we should detect primordial gravitational waves in the near future.