1 Introduction

Fast charged particles may create electromagnetic waves while flying through a medium, which is called Cherenkov radiation. This only happens if the particle is faster than the speed of light in the particular medium. One can compare this to the supersonic flight and in analogy one can observe a Mach cone.

Nowadays there are many areas of application for Cherenkov radiation including the indirect detection of cosmic rays and the measuring of radiation in spent fuel pools.

2 Historic events

Cherenkov radiation was first experimentally discovered and described in 1934 by sovjet physicist Pavel Cherenkov and his supervisor Sergei Wawilow. To honour both discoverers the effect is still called Wawilow-Cherenkov radiation in russia. Only three years later sovjet physicists Ilja Frank and Igor Tamm performed the theoretical description and developed the so called Frank-Tamm formula which describes the whole spectrum completely. In 1958 Cherenkov, Tamm and Frank were eventually awarded the Nobel Prize in Physics "for the discovery and the interpretation of the Cherenkov effect".
3 Creation

If charged particles move through an isolator of (initially constant) index of refraction \( n \), the latter is polarized by the electromagnetic field of the flying particle. Usually these polarisation superpose destructively and the perturbation can relax while the particle moves on. But if the speed of the particle \( \beta \) is higher than the phase velocity of light in the medium \( c_n = n^{-1} \) the perturbation can no longer decay since the reaction time of the medium is not high enough and the perturbation remains behind the particle. Due to the Huygens principle there occurs a wave front at \( \beta > c_n \) but only single spherical waves at \( \beta < c_n \).

\[ \cos(\theta_c) = \frac{1}{\beta n} \quad (1) \]

Due to radial symmetry there is a Mach cone around the moving particle.
4 Spectrum

The amount of photons $dN$ produced in a distance interval $dx$ is generally given by the Frank-Tamm formula:

$$\frac{d^2N}{d\omega dx} = q^2 \frac{\mu}{4\pi} \sin^2(\theta_c)$$  \hspace{1cm} (2)

with particle charge $q$ and permeability $\mu$.

Frank and Tamm derived it in their original work „Coherent Visible Radiation of fast Electrons Passing Through Matter“ as follows:

1. Go to Fourier space, i.e. define quantities $\vec{A}_w$, $\vec{\phi}_w$, $\vec{E}_w$, $\vec{H}_w$ by Fourier transformation.

2. Derive the polarisation $\vec{P}_w = (n^2 - 1) \vec{E}_w$.

3. Rewrite Maxwell equations by using $\partial_t \rightarrow i\omega$:
   - Lorenz gauge: $\vec{\nabla} \vec{A}_w + i\omega n^2 \vec{\phi}_w = 0$  \hspace{1cm} (3)
   - Field definitions: $\vec{H}_w = \vec{\nabla} \times \vec{A}_w$  \hspace{1cm} (4)
   - $\vec{E}_w = -\frac{i}{\omega n^2} \vec{\nabla} (\vec{\nabla} \vec{A}_w) - i\omega \vec{A}_w$  \hspace{1cm} (5)
   - Wave equation: $(\Delta + \omega^2 n^2) \vec{A}_w = -4\pi \vec{J}_w$  \hspace{1cm} (6)

4. Current produced by the particle:
   - $j_z(x, y, z, t) = q \beta \delta(x) \delta(y) \delta(z - \beta t)$  \hspace{1cm} (7)
   - $j_z(\rho, \phi, z, \omega) = \frac{q}{4\pi^2 \rho} e^{-\frac{i\omega z}{\rho}} \delta(\rho)$  \hspace{1cm} (8)

5. Solve $A_z(\omega) = u(\rho) e^{-\frac{i\omega z}{\rho}}$ using Bessel functions:
   $$\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + s^2 \right) u = -\frac{q}{\pi \rho} \delta(\rho)$$  \hspace{1cm} (9)

6. Calculate $\frac{dE}{dx}$ by using the Poynting vector:
   $$\frac{dE}{dx} = 2\pi \rho \int_{-\infty}^{\infty} \left[ \vec{E} \times \vec{H} \right]_\rho \, dt$$
   $$= q^2 \int \omega \left( 1 - \frac{1}{\beta^2 n^2} \right) d\omega$$  \hspace{1cm} (10)

7. Include $\frac{\mu}{4\pi}$ for non-trivial permeability.
This eventually gives the following continuous spectrum:

\[ dE = \frac{q^2}{4\pi} \mu \omega \left( 1 - \frac{1}{\beta^2 n^2} \right) dx d\omega \]  

and a total Cherenkov energy of:

\[ \left( \frac{dE}{dx} \right)_{\text{Cherenkov}} = \frac{q^2}{4\pi} \int_0^\infty \mu(\omega) \omega \left( 1 - \frac{1}{\beta^2 n^2(\omega)} \right) d\omega \Theta \left( n(\omega) - \beta^{-1} \right) \]  

where the Heaviside function \( \Theta \left( n(\omega) - \beta^{-1} \right) \) appears, because there is no Cherenkov radiation for \( \beta n < 1 \).

To further analyse this formula we first have to deal with the properties of the index of refraction. It is an important material constant describing the phase velocity of light in matter which can be calculated via \( n = \sqrt{\varepsilon \mu} \) using permittivity and permeability. In general these values are dependent on the particular frequency which is called dispersion. This means also the Cherenkov angle is frequency dependent and there is a different Mach cone for every wavelength.

This becomes very important looking at the Cherenkov energy. If we assume the index of refraction is universal \( (n(\omega) \equiv n) \) one gets:

\[ \left( \frac{dE}{dx} \right)_{n=\text{const.}} = \frac{q^2 \mu}{4\pi} \left( 1 - \frac{1}{\beta^2 n^2} \right) \int_0^\infty \omega d\omega \]  

which obviously does not converge.

To solve this problem dispersion alone is not enough, since 13 does not converge, if \( (n(\omega) - \beta^{-1} > 0) \) for almost all frequencies. We therefor have to demand the existance of a frequency \( n(\omega > \omega_{max}^{(\beta)}) < \beta^{-1} \) for every given \( \beta \).

\[ n(\omega > \omega_{max}^{(\beta)}) < \beta^{-1} \]  

Since this has to hold for every \( \beta \) we can also write:

\[ n(\omega > \omega_{max}^{(1)}) \leq 1 \]  

At first sight this seems unlikely, because it means that the phase velocity of light would exceed the vacuum speed of light. However we can find such maximum frequencies with \( n < 1 \) in the x-ray realm. For even higher frequencies the index of refrection is usually unity.

Comment: This does not violate SRT, since it only restricts the group velocity which is responsible for the propagation of information. This group velocity may be (and indeed is) still smaller than unity.
5 Application

There are quite a lot of areas of application for the Cherenkov effect. The probably most important one is the detection of cosmic radiation using so called Cherenkov flashes. If high energy photons e.g. from a supernova enter earth’s atmosphere they decay into a cascade of (partially charged) secondary particles. Due to the high energy they move with nearly speed of light and Cherenkov radiation occurs for a very short time until the particle energy (and velocity) has decreased sufficiently. Since these flashes have the same direction as the path of the particle one can draw conclusions from spectrum and intensity distribution about the charged particles and finally the $\gamma$-rays. The first project of this kind was designed in 1989 which allowed the observation of gamma sources for the first time. Nowadays the most important project is the H.E.S.S. in Namibia, which is amongst others supported by the “MPI für Kernphysik” in Heidelberg.

This method is also used in the neutrino research. For example in the IceCube at the south pole a block of ice of $1 \text{ km}^3$ is used. Neutrinos react with the ice and create muons which then radiate Cherenkov rays due to the high index of refractation of ice.

The possibly best-known application is the measuring of remaining radiation in spent fuel pools. The intensity of the characteristic blue Cherenkov light is proportional to the amount of nuclear disintegrations. This is presumably the only possibility to actually see the Cherenkov radiation, since the intensity of a single ray is too weak.
6 Sources

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  Tamm, Frank: „Coherent Visible Radiation of fast Electrons Passing Through Matter“
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