Phase Transitions and Critical Phenomena

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Abstract

The following report represents a short introduction to the topic of phase transitions and critical phenomena.
Matter can exist in several states differing in their respective properties. A change between those phases is characterised by critical behaviour in the thermodynamic quantities of the system. A classification for different transitions based on the Gibbs free energy is given.
Near the critical point the correlation length diverges leading to universal behaviour. The divergence of characteristic quantities will result in scaling laws from which parameters describing the behaviour can be obtained.
This report discusses only thermodynamic phenomena, although more general examples are briefly mentioned.

1 Introduction

The history of phase transitions is strongly interwoven with the procession of different materials like e. g. ceramics, which can be traced back 8000 years in the Near East. With the discovery of magnetism first meticulously described by THALES of Miletus and its useful applications thought of by PEREGRINUS of Maricourt the first ferromagnetic materials were introduced to the scientific world. It still needed another threehundred years until GALILEO developed the first thermometer later superseded by FAHRENHEIT's thermometer using alcohol until the foundation for the theory of thermodynamics was laid. In the following century the successive discoveries of the thermodynamic properties of gases by BOYLE, MARIOTTE, AMONTON, GAY-LUSSAC and AVOGADRO lead to the first formulation of an equation of state for a thermodynamic system. With the detailed analysis of thermodynamic processes by CARNOT for example the way was paved for our modern day understanding of thermodynamic systems and phase transitions. With the development of steam engines the first applications of thermodynamic processes were made in the 19. century. 1872 VAN DER WAALS first introduced a fluid undergoing a phase transition, albeit a second-order phase transition was already experimentally established by ANDREWS 1869 in CO_2. At the brink of the 20. century the loss of magnetisation due to heating of magnetic materials was demonstrated by Pierre Curie in his systematic investigations later pursued by WEISS. At the beginning of the twentieth century the technique to liquefying all gases allowed for intensive studies in the realm of low-temperature physics leading to discoveries like the superconductivity of mercury using liquid helium by ONNES in 1911. With the newborn possibilities of

studying microscopic structures with X-rays the detailed investigation of metal alloys lead to the hypothesis of an ordered phase in alloys in 1919 by TAMMANN. This was later confirmed in 1929 by JOHANNSEN and LINDE. TAMMAN demonstrated in 1926 the existence of a critical point in a bronze alloy. The introduction of the Ising model in 1915 due to ISING and ERNST a simple system developing ferromagnetic phase transitions under certain conditions lead to further fruitful development culminating in the solution of the two-dimensional Ising model by ONSAGER 1944. With the first classification schemes for phase transitions by EHRENFEST and LANDAU in the 1930s as well as the concept of long-range order by BRAGG and WILLIAMS the notion of an order parameter became manifest. The development of precision thermometry at the beginning of the fifties allowed for detailed experimental verification of the developed concepts as well as for the collection of data on which the concepts of scaling and universality were based in the late sixties by KADANOFF, DOMB and WIDOM. The introduction of the renormalisation group by WILSON at the same time opened up the possibilities of calculating the critical values for a multitude of systems as made explicit by FISHER in the 1970s. The growing need for new materials in industrial contexts stimulated a lot of further research on phase transitions.

A lot more can and probably should be said on phase transitions and critical phenomena, but due to the nature of this summary being only introductional, not all details and the complete mathematical rigour can be taken into account. Following the definition of phase transitions using the basic concepts of thermodynamics a closer description of the critical behaviour will be given. The concept of critical exponents, the scaling laws and universality interpretable as consequence of the before mentioned will be elaborated and emphasised using a particularly simple example. This report will make heavy use of thermodynamic notation. Our thermodynamic potential of choice will be the GIBBS free energy $G(T, J)$ from which the extensive state variables are directly derived via $\mathrm{d}G = -S\mathrm{d}T - m\mathrm{d}J$. The variable J corresponds to an external source like a pressure or a magnetic field. A notion of macroscopic structure is given by means of an order parameter m which corresponds to the response of the Gibbs free energy to a source

$$m = -\frac{\partial G}{\partial J}. \tag{1}$$

The entropy is obtained in a similar manner as (negative) T-derivative of the Gibbs free energy. In addition to the extensive state variables the response functions C, χ and α measure how those variables change due to a change in one of the intrinsic variables.

$$C_J = T \left(\frac{\partial S}{\partial T} \right)_J \tag{2}$$

$$\chi_T = \left(\frac{\partial m}{\partial J} \right)_T \tag{3}$$

$$\alpha_m = \left(\frac{\partial m}{\partial T} \right)_J. \tag{4}$$

Those variables should suffice for the purposes of this report. Due to implicit relations (e. g. an equation of state) of the intrinsic state variables, the response functions fulfill a variety of relations making them depend on each other. Also note
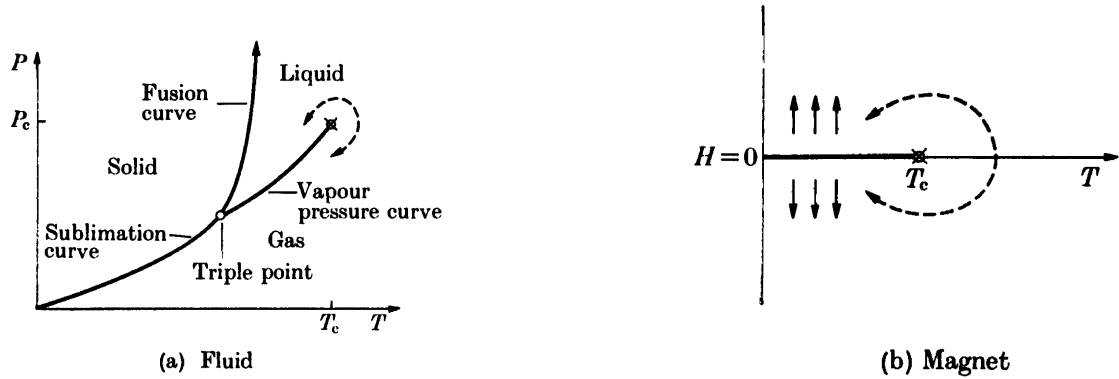
Figure 1: Phase diagram for a (non-anomalous) fluid and a magnetic system. At the critical point the phases can be continuously deformed into each other. Adapted from [1].

how by choosing the (negative) pressure as a source, the order parameter can be associated with the Volume or the density of the system and the response becomes the compressibility while by choosing a magnetic field the order is denoted by the magnetisation and the response becomes the susceptibility.

One remark should still be made. For a magnetic system the sign of C, χ and α can not easily be claimed. Nevertheless, it can be shown that a (magnetic) system with Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 - J\hat{m}, \tag{5}$$

still has a concave Gibbs free energy thus rendering the thermodynamic approach valid (GRIFFITH 1964).[1–3]

2   Phase Transitions

Since humanity (industrially) mastered materials the observation is made that matter can exist in different states differing in their properties. The most ubiquitous example being the solid, liquidous and gaseous states of ordinary substances found on earth. Transitions between phases are characterised by rapid changes in the properties of the system like the density rapidly decreasing during the vaporisation of a liquid. Later it was found that there is a greater variety of phase transitions for example regarding transitions between several crystalline configurations of a solid. A phase transition is induced by an external change in the intrinsic thermodynamic variables of the system of interest. To further analyse phase transitions of a system a phase diagram can be drawn plotting the state variables against each other. One example for this can be seen in Fig. 2. Each phase transition is characterised by a set of critical quantities indicating the threshhold for the transition most notably the *critical temperature* T_c.

There are different ways to distinguish phase transitions, all relying on the structure of the system's quantities near the transition. The two relevant classifications are by the degree of the corresponding derivatives of the thermodynamic potential (EHRENFEST 1933) or by the symmetry of the different phases which allow for the notion of an order parameter (LANDAU 1937).[1, 2]
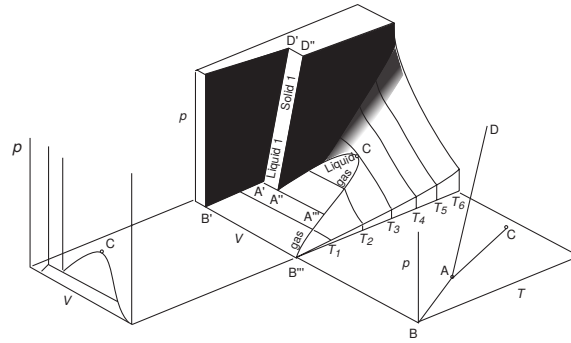
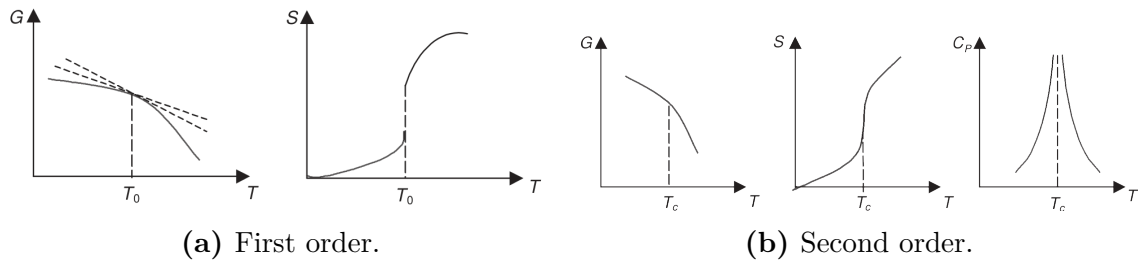**Figure 2:** Phase diagram for a pure substance, adapted from [2].

(a) First order. **(b)** Second order.

Figure 3: Phase transitions.

2.1 Ehrenfest classification

In this classification scheme proposed by EHRENFEST in 1933 the distinction is made between the order to which the critical thermodynamic quantity can be associated to a derivative of a thermodynamic potential. If an irreguarity occurs in one of the state variables (e. g. pressure, volume, magnetisation, temperature), the transition is of *first* order. Phase transitions of first order show jump discontinuities in their state variables. The size of this gap corresponds to the latent heat absorbed or being released in these processes. One example of a phase transition of first order would be a boiling fluid, where the volume increases drastically during the transition while the temperature stays constant. The latent heat involved here corresponds to the *enthalpy of vaporisation* of the fluid.

Phase transitions of *second* order are characterised by divergencies of the response functions of the system while the state variables stay finite at the transition. Since there is no latent heat (in a broader sense) involved in the process, both phases can be deformed into each other continuously. Such a response function can be the specific heat of a fluid or the susceptibility of a magnetic system. An example of this would be the Ising model, where the susceptibility near the critical temperature diverges as the magnetisation goes to zero (from below).

It is possible to associate higher order phase transitions in the Ehrenfest sense by identifying higher derivatives of the potentials of the system, albeit most phase transitions fall into one of the previous categories with the KOSTERLITZ-THOULESS transition being a famous exception.[1, 2]

2.2 Landau classification

In a different approach the symmetries of both phases involved in a transition can be compared to each other bearing the advantage of not being limited to a thermo-

dynamic description. This was first proposed by LANDAU in 1937.

If the symmetry groups of both phases can not be identified as one being contained in the other, no order parameter can be defined and thus the transition is without such a parameter. It can be shown that those transitions are always first-order in the Ehrenfest sense.

If an order parameter (often denoted m) can be defined, the symmetry group of the least symmetric phase is a subgroup of the symmetry group of the most symmetric phase and a distinction between the *unordered* and *ordered* phase is possible. During the transition *symmetry breaking* occurs, one example being the crystallisation of a liquid where the continuous translation symmetry of the liquidous phase is broken to a translational symmetry in a lattice that describes the spatial structure of the solid phase. Often, the order parameter can be defined in such a way that it is zero in the disordered phase and only nonzero below the critical point. If a phase transition with order parameter falls into one category due to Ehrenfest or the other is related to the behaviour of the parameter itself. A discontinuous order parameter hints to a first-order transition while it being continuous corresponds to a second-order transition.[2]

2.3   Correlation Length

To describe the formation of macroscopic order from the microscopic properties of a system the *correlation function* is defined as the two-point correlator between two sites of the system (e. g. lattice points):

$$G(\mathbf{r} - \mathbf{r}') = \langle m(\mathbf{r})m(\mathbf{r}')\rangle - \langle m\rangle^2. \tag{6}$$

This correlator can be split into a part due to self-interaction $\propto \delta(\mathbf{r} - \mathbf{r}')$ and the remaining interactions, denoted $\Gamma(\mathbf{r} - \mathbf{r}')$. Starting from this the structure function $\hat{S}(\mathbf{q})$ is defined as the Fourier transform of the correlation function and takes on the form:

$$\hat{S}(\mathbf{q} = n\left(1 + n\hat{\Gamma}(\mathbf{q})\right), \tag{7}$$

where the quantity n corresponds to the characteristic particle density of the system. There still remains an obstacle in the interactions governed by the term $\Gamma(\mathbf{r} - \mathbf{r}')$. It contains the direct interactions of site $\mathbf{r}$ with site $\mathbf{r}'$ as well as the interactions with a third site say $\mathbf{r}''$ which interacts with the site $\mathbf{r}'$. To extract the *direct* interaction without these cumulated contributions the direct correlation function $C(\mathbf{r})$ is needed. ORNSTEIN and ZERNIKE showed how it can be obtained via its Fourier transform as

$$\hat{C}(\mathbf{q}) = \frac{\hat{\Gamma}(\mathbf{q})}{1 + n\hat{\Gamma}(\mathbf{q})}. \tag{8}$$

From this the structure function follows as

$$\hat{S}(\mathbf{q}) = \frac{n}{1 - n\hat{C}(\mathbf{q})}. \tag{9}$$

Assuming isotropy of the direct correlation and expanding up to second order the structure function gains the Lorentz-like form

$$\hat{S}(q) \propto \frac{1}{\kappa^2 + q^2} \tag{10}$$

$$\Rightarrow \qquad G(r) \propto \frac{e^{-\kappa r}}{r^{d-2}}. \tag{11}$$

In the last line the inverse Fourier transform was applied. The quantity κ has dimensions of an inverse length and from it the correlation length ξ is defined as $\kappa = \xi^{-1}$. In the correlator, now behaving as $G \propto r^{-(d-2)}e^{-r/\xi}$, the correlation length provides a characteristic length scale for the decay of correlations. This exponential form is typical for macroscopically uncorrelated systems where nearest neighbours are still influencing each other on small scales.

More generally the correlator behaves as

$$G(r) \propto \frac{e^{-r/\xi}}{r^{\vartheta}}, \tag{12}$$

where the exponent ϑ accounts for other non-trivial contributions to the exact form of the interaction.

Near the critical point the correlation length diverges and the system loses its only relevant scale. This is one of the main reasons for the appearance of macroscopic structures because previous small scale fluctuations can now appear on every scale. This effect known as *critical opalescence* makes it quite difficult to observe systems near the critical point experimentally.

At criticality the divergence of the correlation length leads to an algebraically decaying correlator:

$$G(r) \propto \frac{1}{r^{d-2+\eta}}. \tag{13}$$

This indicates a correlation of the microscopic sites throughout the whole macroscopic system during a phase transition. It is of uttermost importance to note that the exponents η and ϑ can differ and are not trivially related.[1]

3   Critical Exponents

To give a proper description for phase transitions of second order, the divergent behaviour of the thermodynamic quantities must become comparable. Specifying a dimensionless quantity related to the vicinity to the critical point,

$$t \equiv \frac{T}{T_{\rm c}} - 1, \tag{14}$$

with $t = 0$ at $T = T_{\rm c}$, the (divergent) behaviour of a quantity $f(t)$ near the critical point is described by its critical exponent λ defined by:

$$\lambda \equiv \lim_{t \searrow 0} \frac{\ln |f(t)|}{\ln t}. \tag{15}$$

This implies that near $T_{\rm c}$ f behaves as $f(t) = At^{\lambda}\left(1 + Bt^x + \cdots\right)$. The critical exponent λ' from below $T_{\rm c}$ is defined in a similar manner and can differ from λ:

$$\lambda' \equiv \lim_{t \nearrow 0} \frac{\ln |f(t)|}{\ln t}. \tag{16}$$

(a) Logarithmic divergence.
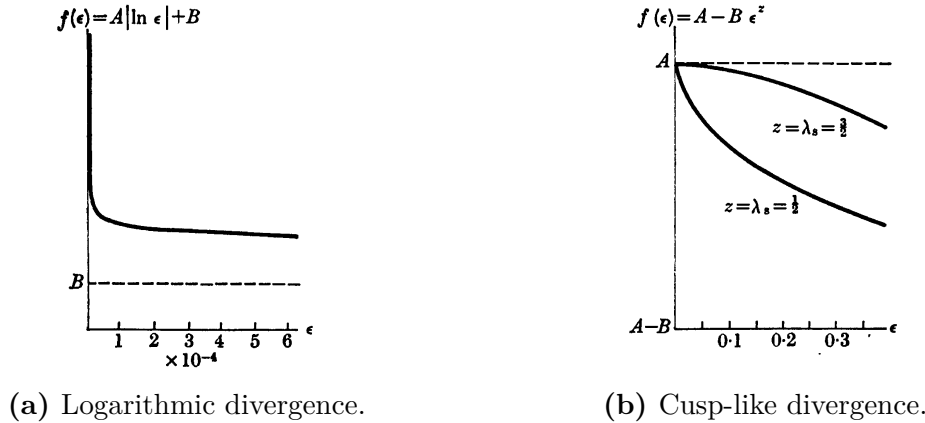

(b) Cusp-like divergence.

Figure 4: Divergencies with vanishing critical exponent, adapted from [1].

This behaviour is denoted as $f \sim t^{\lambda}$ from above and $f \sim (-t)^{\lambda'}$ from below. It must be emphasised that the case $\lambda = 0$ is not unique and can correspond to three different cases: a logarithmic divergence like $f(t) \propto \ln|t| + \cdots$, a cusp-like (but still finite) divergence $f(t) \propto A - Bt^x + \cdots$ or perfectly analytical behaviour up to maybe a jump discontinuity. This is seen in Fig. 4 and can be corrected[1] for, albeit such modifications are not that commonly used.

The most importand exponents $(\alpha^{()}, \beta, \gamma^{()}, \delta, \eta, \nu^{()})$ are defined as follows:

$$C_x \sim t^{-\alpha} \tag{18}$$

$$\sim (-t)^{-\alpha'} \tag{19}$$

$$m \sim (-t)^{\beta} \tag{20}$$

$$\frac{\partial m}{\partial J} \sim t^{-\gamma} \tag{21}$$

$$\sim (-t)^{-\gamma'} \tag{22}$$

$$\xi \sim t^{-\nu} \tag{23}$$

$$\sim (-t)^{-\nu'} \tag{24}$$

$$G \propto r^{-(d-2+\eta)} \tag{25}$$

$$m \propto |J|^{1/\delta}. \tag{26}$$

The last two exponents are only defined at the point of criticality, $t = 0$. Experimentally it can be hard to determine the exponents due to the often unknown exact critical point. To avoid this complication arbitrary powers of the measured quantities are fitted to a straight line [1, 2, 4].

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1   In this modification the extracted critical index is defined via

$$\lambda_s \equiv j + \lim_{t \to 0} \frac{\ln\left|f^{(j)}(t)\right|}{\ln|t|}, \tag{17}$$

where the integer j corresponds to the numer of successive derivations needed for the quantity to show a divergence at the critical point.

3.1   Example Calculation for the Ising model in the Mean Field Theory

Lets assume an Ising model on a lattice with sites i, spins s_i of magnitude μ pointing up or down, $\sigma_i = s_i/\mu \in \{\pm 1\}$, an external magnetic field H and a coupling J of nearest neighbours. The Hamiltonian for this model reads:

$$\mathcal{H}\left[\Omega\right] = -\mu H \sum_i \sigma_i - J \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \tag{27}$$

here $\langle i,j \rangle$ stands for all pairs i, j who belong to neighbouring sites. To solve this, spin fluctuations $\delta\sigma_i$ around their mean value $m = \langle \sigma \rangle$ are considered. Insertion into the Hamiltonian leads to:

$$\mathcal{H}\left[\Omega\right] = -\mu H \sum_i \sigma_i - J \sum_{\langle i,j \rangle} (m + \delta\sigma_i)(m + \delta\sigma_j) \tag{28}$$

$$= -\mu H \sum_i \sigma_i - zJ \sum_i \left(m^2 + m(\delta\sigma_i + \delta\sigma_j)\right) + \mathcal{O}\!\left(\delta\sigma^2\right) \tag{29}$$

$$\simeq -\mu H \sum_i \sigma_i - zJ \sum_i \left(m(\sigma_i + \sigma_j) - m^2\right) \tag{30}$$

$$= -\left[\mu H + zJm\right] \sum_i \sigma_i, \tag{31}$$

in the last line we introduced the coordination number z, counting the number of nearest neighbours, and evaluated the last two terms in the last sum, canceling each other. The factor in front of the sum can be expressed as an *effective* magnetic field H_{eff}, due to an external contribution H and a contribution due to the magnetisation m of the system. Evaluating the partition sum,

$$\ln \mathcal{Z}(\beta, H) = \ln \sum_{\Omega} e^{-\beta\mathcal{H}[\Omega]} \tag{32}$$

$$= \ln \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} e^{-\beta\mathcal{H}[\Omega]} \tag{33}$$

$$= N \ln 2 + N \ln \cosh \frac{\mu H + zJm}{kT}, \tag{34}$$

the magnetisation can be deduced to be:

$$m = \tanh \frac{\mu H + zJm}{kT}. \tag{35}$$

In the absence of an external field H a critical temperature $T_{\text{c}} = \frac{zJ}{k}$ can be found from simple graphical considerations. Therefore, this model shows a ferromagnetic phase transition at the critical temperature T_{c}. Near the critical point, eq. (35) can be expressed as

$$m = (1 + t)\,\text{artanh}\,m \tag{36}$$

$$\simeq (1 + t)\left(m + \frac{1}{3}m^3\right). \tag{37}$$

This can further be expanded to

$$m = \sqrt{\frac{-3t}{1+t}} \tag{38}$$

$$\simeq (-3t)^{1/2} \left[1 + \mathcal{O}(t)\right], \tag{39}$$

from which we can conclude $\beta = \frac{1}{2}$. Additionally we can calculate the susceptibility from eq. (35) by differentiating in H and reusing eq. (35) to obtain:

$$\chi = \frac{\mu}{zJ} \frac{1 - m^2}{t + m^2}. \tag{40}$$

From above the CURIE-WEISS law $\chi \propto t^{-1}$ can easily be obtained by sending m to 0. To see the behaviour from below eq. (39) can be inserted leading to the same exponent and thus we deduce $\gamma = \gamma' = 1$. Directly at the critical temperature, eq. (35) can be written as

$$\beta_{\mathrm{c}} \mu H = -m + \operatorname{artanh} m \tag{41}$$

$$\simeq \frac{1}{3} m^3 \left(1 + \mathcal{O}(m^2)\right), \tag{42}$$

from which it can be concluded that $m \propto |H|^{1/3}$ and thus $\delta = 3$. The other exponents can be found by similar arguments.

If one takes a closer look at this model one can see that using this approach even the Ising model in one dimension ($z = 2$) seems to show a phase transition when it was proven by Ising himself that that can not be the case. Comparison with the critical exponents in ONSAGERS famous solution shows that the values, calculated here, are off somehow. There is a deep reason behind this. In the above introduced approach by expanding around their mean value $\langle \sigma \rangle \propto \sum_i \sigma_i$, the sites nearest-neighbour interaction was promoted to an interaction throughout the whole lattice. This is called an infinite-dimensional interaction, because the number of neighbours grows infinitely by redefining the "distance" to *nearest* neighbours, while in the "ordinary" cases it grows obeying a power law with exponent given by the dimensionality of the model. The infinite-dimensional approach corresponds to viewing the mean field value as the expectation value of a free massless scalar field. This property is typical for classical approximations and can also be found in the theory of a VAN DER WAALS-fluid, where it will lead to the same critical exponents. In this context one can define a so called *lower critical dimension* from which on the solutions obtained using the mean field method are equivalent and thus lead to the same results as the exact solutions. Unfortunately, in most cases this critical dimension is 4 or 3. For example the Ising model has lower critical dimension 4. Up to today there are quite few analytically solved statistical models showing phase transitions that do not fall into the classical category and from which the critical exponents can be derived. This is sketched graphically in fig. 5 [2, 4].

3.2   Scaling Hypothesis

Near criticality due to the divergence of the correlation length the system loses its only relevant scale and thus some form of scale invariance should hold. From this DOMB, KADANOFF and WIDOM proposed the scaling hypothesis, stating that near
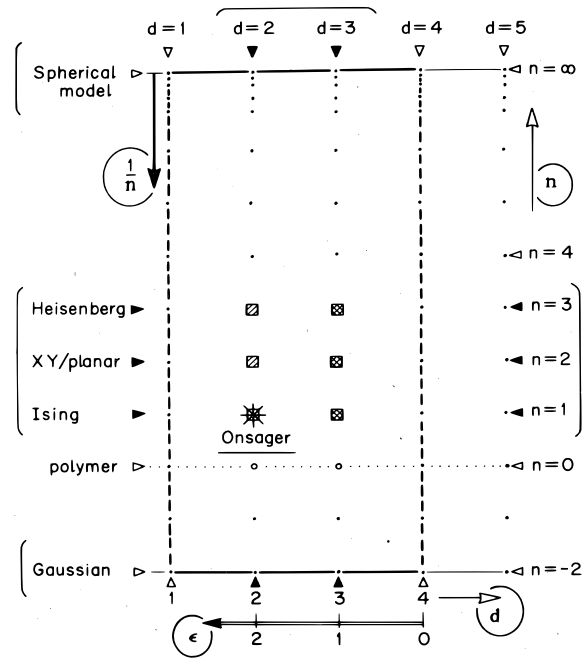
Figure 5: Diagram showing the positions of known statistical models due to their dimensionality d ($\varepsilon = 4 - d$) and the index of isotropy n. The only exact solution known in the inner region is Onsager's solution of the two-dimensional Ising model. The boundaries of the diagram correspond to classical solutions. Taken from [5].

the critical point the thermodynamic quantities obey *scaling laws*, meaning that the thermodynamic potentials, the extensive state variables and the response funtions become homogeneous functions of the intrinsic state variables. In this context it is sufficient to phrase this as[2]

$$G_{\mathrm{S}}(\lambda^p t, \lambda^q J) = \lambda G_{\mathrm{S}}(t, J), \tag{43}$$

here G_S corresponds to the part of the thermodynamic potential whose derivatives develop critical (and thus singular) behaviour at criticality, the analytical part has to be extracted resembling the fact, that the existance of a critical exponent does not imply a strictly power law behaviour of the corresponding quantity but a more complex form whose divergence is driven by a power law. This scale invariance allows to associate a thermodynamic system to a conformal field theory. There the scaling dimensions of the corresponding operators can be identified with the critical exponents of the system. With the development of renormalisation group technics due to WILSON 1971, critical exponents for a system became theoretically approachable and for the first time could be predicted outside the analytically solvable models as shown by FISHER 1974. The critical exponents only depend on the dimensionality d of the system and the degree of isotropy n indicating the inner symmetries of the system. With this at hand an expansion in the variables $\varepsilon = 4 - d$ and n^{-1} can be made to leave the classical realm [1, 4, 5].

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2   Note that choosing a power λ^r in front of G_{S} does not correspond to a more general form since by redefining $\lambda \to \lambda^{1/r}$ eq. (43) can be recovered.

Exponent	MFT	ISING 2D	ISING 3D	HEIS 3D
α	0	0	0.12	-0.14
β	1/2	1/8	0.31	0.3
γ	1	7/4	1.25	1.4
δ	3	15	5	[5.7]
ν	1/2	1	0.64	0.7
η	0	1/4	0.05	0.04

Table 1: Table with examples for critical exponents. Data taken from [6], value in brackets manually calculated from the relations.

3.3   Exponent Relations

From the scaling hypothesis and the relations between the intrinsic state variables (e. g. an equation of state) several relations regarding the critical exponents can be derived by thorough dimensional analysis. One example of this will be demonstrated here. Take eq. (43) and differentiate both sides with respect to J:

$$m(t, J) = \lambda^{q-1} m(\lambda^p t, \lambda^q J), \tag{44}$$

and analogously

$$S(t, J) = \lambda^{p-1} S(\lambda^p t, \lambda^q J). \tag{45}$$

In the case $J = 0$ using the definition of β and choosing $\lambda = t^{-1/p}$ eq. (44) implies $\beta = \frac{1-q}{p}$. Of course, the response functions are obtained in the same way

$$C(t, J) = \lambda^{2p-1} C(\lambda^p t, \lambda^q J) \tag{46}$$

$$\chi(t, J) = \lambda^{2q-1} \chi(\lambda^p t, \lambda^q J) \tag{47}$$

$$\alpha(t, J) = \lambda^{p+q-1} \alpha(\lambda^p t, \lambda^q J). \tag{48}$$

Similarly we can conclude $\gamma' = \frac{2q-1}{p}$. Therefore, via p and q all critical exponents can be expressed using the scaling hypothesis. This leaves only two of them independent. The most commonly used relations between the exponents are

$$2 = \alpha' + 2\beta + \gamma' \tag{49}$$

$$2 = \alpha' + \beta(1 + \delta) \tag{50}$$

$$\gamma = \beta(\delta - 1) \tag{51}$$

$$\nu = \nu' \tag{52}$$

$$\gamma = (2 - \eta)\nu \tag{53}$$

$$d\nu = 2 - \alpha. \tag{54}$$

The last expression made use of the dimensionality d of the system. A Few examples for critical exponents are given in tab. 1 [1, 4].

3.4   Universality

Experimentally it became clear early on that many different systems appear to have the same critical exponents and thus show the same behaviour at the critical point.

Rather wellknown examples of this include a VAN DER WAALS-fluid and the Ising model in dimensions greater three, or the liquid-gas transition of carbon dioxide or xenon and the three-dimensional Ising model (HOCKEN/MOLDOVER 1976). In the context of scaling this is no big surprise, since only the scale invariant properties of a system contribute to its critical behaviour and therefore the more delicate and subtle differences in the models seem to blur when approaching criticality. Only the broad symmetries leading to scaling dominate at the critical point implying that different models can be grouped in so-called *universality classes* by their critical exponents. The classes itself are defined by the broad symmetry groups leading to specified scaling behaviour.

It should be noted that while seeming strongly related the concepts of scaling and universality are not mutually inclusive, as there exist models featuring one of them but not the other. Take for example the 8-vertex model were the critical exponents vary continuously with the parameters in the Hamiltonian thus showing scaling but violating universality [1, 4].

4 Conclusion

We conclude this report about phase transitions and critical phenomena with a brief discussion of the fundamental concepts and an outlook an further applications.

In the previous sections we saw that phase transitions characterise the rapid changes in the properties of a system between distinct states called phases. Near those transitions the characterising quantities of the system show critical behaviour like divergencies or discontinuities being related to a change in symmetry or configuration of the system. For those quantities exponents describing the nature of the divergence can be defined and measured. Due to the divergence of

Figure 6: λ-transition of helium at $2.2\,\mathrm{K}$. The name is related to the shape of the curve. Adapted from [2].

quantities like the correlation length the system loses its relevant scale and shows universality. From the scaling behaviour of the system relations between the exponents can be derived. The universality allows to group different models and system showing the same behaviour at criticality into universality classes being defined by a set of critical exponents related to the broad symmetry groups. It also became apparent that there exist critical dimensions below which approximative solutions of thermodynamic models will lead to vastly different results compared to the measurements highlighting the need for the ability to determine the exponents using other methods.

With the developments regarding conformal field theory, the renormalisation group and the conformal bootstrap the tools were provided to calculate the critical exponents directly from the symmetries of the system, but there are still unsolved problems arising in the calculation of specific exponents with one famous example being the λ-transition of helium. The divergent behaviour of the specific heat of helium near its transition (as seen in fig. 6) to its superfluent phase is experimentally
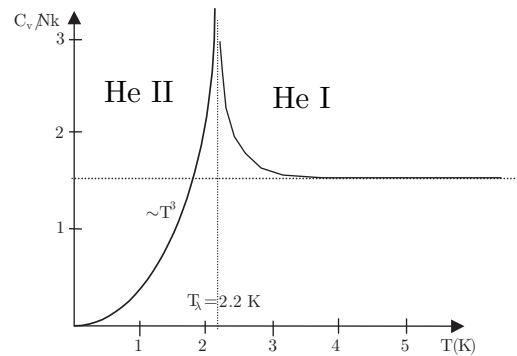
well established as $\alpha = -0.0127(3)$ [7], but still most theoretical models predict it with deviations outside the margin of error.

Most of the examples given here where of a thermodynamic origin, but that does not express a restriction. Actually, there exist a great variety of models making use of the thermodynamic limit but not bound to a thermodynamic point of view with examples being applications in neural networks [8] or even sociology describing human cooperation [9].

Finally it can be said that in the theory of phase transitions lies the foundation for our modern day standards including the processing of materials with progressively smaller structures for our industrial requirements as well as the theoretical descripten of macroscopic effects due to microscopic interactions. With its everyday applications and development a lot of future research will be dedicated to phase transitions and most definitely lead to further progress in humankinds understanding of the microscopic structure of nature.

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