Markovian and non-Markovian processes and Master Equation

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Motivation

A Markov process assumes that the next state of a process only depends on the present state and not the past states.

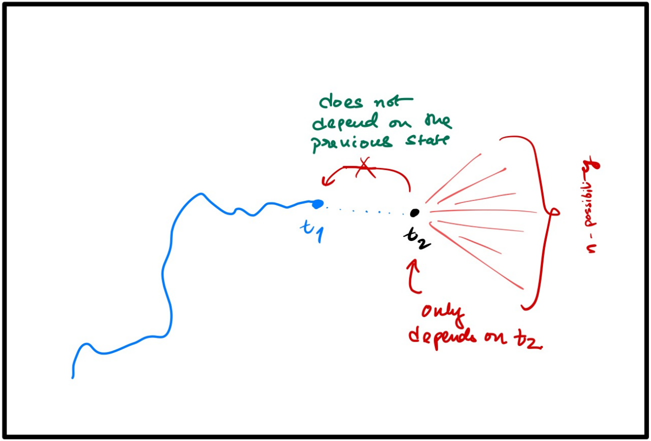

Table of contents

Markov process

Chapman Kolmogorov

Master Equation

Fermi's golden rule

Non markovian process

Lindblad's equation

Origin of Markov

The **weak law of large numbers** states that if you have a sample of independent and identically distributed random variables, as the sample size grows larger, the sample mean will tend toward the population mean. .
Nekrasow's claim - independence is a necessary condition for the weak law of large numbers.

Markov's modification of Bernoulli's experiment

Markov proved that dependence of variables on previous state can also lead to convergence.

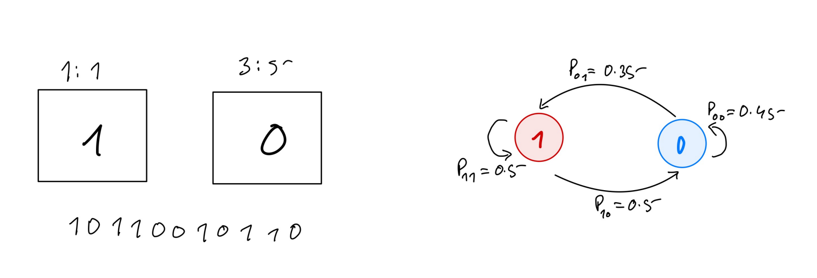

Figure 2: An example of a stochastic process with dependence on the last example

Markov process

Definition: A stochastic process where the future state only depends on the present state and all the past states are eliminated.

$$P_{1|n-1}\left(y_n, t_n \mid y_1, t_1; \ldots; y_{n-1}, t_{n-1}\right) = P_{1|1}\left(y_n, t_n \mid y_{n-1}, t_{n-1}\right)$$

$$\begin{aligned} P_3\left(y_1, t_1; y_2, t_2; y_3, t_3\right) &= P_2\left(y_1, t_1; y_2, t_2\right) P_{1|2}\left(y_3, t_3 \mid y_1, t_1; y_2, t_2\right) \\ &= P_1\left(y_1, t_1\right) P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) \end{aligned}$$

where $t_1 < t_2 < t_3$

Chapman Kolmogorov

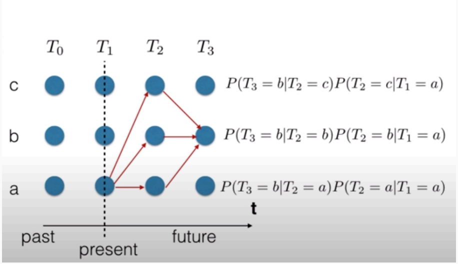

Figure 3: Two step transition probability

[3]

The Chapman–Kolmogorov equation provides the two step transition probability

$$P_2\left(y_1, t_1; y_3, t_3\right) = P_1\left(y_1, t_1\right) \int P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) dy_2$$

Chapman Kolmogorov

$$P_2\left(y_1, t_1; y_3, t_3\right) = P_1\left(y_1, t_1\right) \int P_{2|1}\left(y_2, t_2 \mid y_1, t_1\right) P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) dy_2$$

Use Bayes theorem
[5] Divide both sides by $P_1(y_1, t_1)$

$P_{11}\left(y_3 \left|t_3\right| y_1, t_1\right) P_1\left(y_1 t_1\right) = \int P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) dy_2$

$$P_{1|1}\left(y_3, t_3 \mid y_1, t_1\right) = \int P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) dy_2$$

Stationary Markov process

When the states of the system $P_1(t)$ is not affected by a time shift , then $P_1(t)$ is a stationary Markov process . origin of time does not matter since it is a function of elapsed time .[4]

Transition probability depends on time interval between two states

$$P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) = T_\tau\left(y_2 \mid y_1\right) \quad \text{with } \tau = t_2 - t_1, \tau' = t_3 - t_2$$

Stationary Markov process

Chapman-kolmogorov equation is
$P_{1|1}(y_3, t_3 \mid y_1, t_1) = \int P_{1|1}(y_3, t_3 \mid y_2, t_2) P_{1|1}(y_2, t_2 \mid y_1, t_1) dy_2$
Chapman-Kolmogorov equation then becomes $(\tau, \tau' > 0)$

$$T_{\tau+\tau'}(y_3 \mid y_1) = \int T_{\tau'}(y_3 \mid y_2) T_\tau(y_2 \mid y_1) \mathrm{d}y_2$$

since it is already product of two matrices

$$T_{\tau+\tau'} = T_{\tau'} T_\tau \quad (\tau, \tau' > 0)$$

$$P_2(y_1, t_1; y_2, t_2) = T_\tau(y_2 \mid y_1) P_1(y_1)$$

As $t \to \infty$ $P_2(y_2, t_2 - t_1 \mid y_1) = P_1(y_2)$

Homogeneous Markov Process

A stationary process $Y(t)$ such that $P_1(y_1)$ and $T_\tau(y_2 \mid y_1)$ is given. t_0 and y_0 is a fixed time and fixed value. $Y^*(t)$ is a non stationary markov process for $y \geqslant t_0$ [4]

$$P_1^*(y_1, t_1) = T_{t_1 - t_0}(y_1 \mid y_0)$$
$$P_{1|1}^*(y_2, t_2 \mid y_1, t_1) = T_{t_2 - t_1}(y_2 \mid y_1)$$

Homogeneous Markov Process

One can extract a process$Y(0)$ at $t_0$from $P(0)$ and are distributed according to $P(y_0)$

$$P_1^*(y_1, t_1) = \int T_{t_1 - t_0}(y_1 \mid y_0)\, p(y_0)\, dy_0$$

$$P_{1|1}^*(y_2, t_2 \mid y_1, t_1) = T_{t_2 - t_1}(y_2 \mid y_1)$$

These processes being non stationary but their transition probability depends on time difference,hence it is homogenous markov process

Markov chain

We have three states A,B and C where the particle can travel . P_{AB} is the probability of a particle to go from state A to state B.

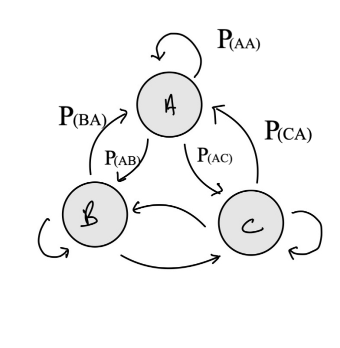

Figure 4: Markov chain

Transition Probability matrix

The transition probability $T_\tau\left(y_2 \mid y_1\right)$ is a $N \times N$ matrix . The Transition Probability matrix is given as

$$\begin{bmatrix} P_{AA} & P_{AB} & P_{AC} \\ P_{BA} & P_{BB} & P_{BC} \\ P_{CA} & P_{CB} & P_{CC} \end{bmatrix}$$

$$P_{AA} + P_{AB} + P_{AC} = 1 \quad \text{where all rows add up to 1}$$

The Markovian Master equation

$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \sum_n \{W_{nn'} p_{n'}(t) - W_{n'n} p_n(t)\}$

They are differential equations that describe the evoultion of probability with respect to time . The master equation is a gain loss equation for the probabilities of seperate states n.[5]

Long Time Limit

As $t \to \infty$ all solutions tend to the stationary solution.

For a closed isolated system in equilibrium , all transitions per unit time into any state n must be balanced by transitions into n'.[5]
$W_{nn'} p^{\mathrm{e}}_{n'} = W_{n'n} p^{\mathrm{e}}_n$

Non Markovian Processes

Any process that depends on all the past states is a non Markovian process, which implies that the memory of the previously visited sites changes the distribution.

Lindblad Master Equation

Describes a non-unitary evolution of the density operator ρ

$\rho = |\psi\rangle\langle\psi|$

For shorter time interaction with the environment compared to the internal time scale of system, we use the Markovian assumption[1]

$\frac{\partial\rho(t)}{\partial t} = \mathcal{L}[\rho(t)]$

H is the Hamiltonian of the system and L_μ are called Lindblad operators. The superoperator $\mathcal{L}$ is called Liouvilian.

$$\mathcal{L}[\rho] = -\mathrm{i}[H, \rho] + \sum_\mu \left[L_\mu \rho L_\mu^\dagger - \frac{1}{2}\left\{L_{ji}^\uparrow L_\mu, \rho\right\}\right]$$

The Markovian approximation is not accurate when the interactions inside the environment have comparable strengths to the interactions inside the system.

Nakajima Zwangzig master equation

NZ is used as non Markovian master equation[1]

$$\frac{\partial \rho(t)}{\partial t} = -\mathrm{i}[H, \rho(t)] + \int_0^t \mathrm{d}s \mathcal{K}_{\mathrm{r}-s}^{-\mathrm{N}z}[\rho(s)] + I(t)$$

$\mathcal{K}_t^{\mathrm{NZ}}$ is memory kernel the above equation describes non-Markovian processes, because the state at time t $+\mathrm{dt}$ depends not only $\rho(t)$ but also on the states $\rho(s)$ for $s < t$

$$\frac{\partial \rho(t)}{\partial t} = -i[H, \rho(t)] + \int_0^t \mathrm{ds} N_{t\to -1}^{-2(x)}[\rho(t)] + I(t)$$

Summary

1. Defined Markov Prcoess, stationary and homogenous process
2. Derived Chapman Kolmogorov and Master Equation
3. Proved that system weakly coupled by the heat bath gives similar transition probability to Fermi's Golden rule
4. Non Markov process defined
5. Lindblad's Master Equation
6. Nakajima Zwangzig master equation defined

References

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