Markovian, Non Markovian process and Master equation

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Abstract

A Markov process is a stochastic process Y_n in which a future state Y_{i+1} is only determined by the present state Y_i, and all the past states are ignored. This simplifies calculations for any series of random variables that follows this law. In this report we are going to talk about the various cases where it is effective, and will go in detail for the probability evolution of this process, which is the master equation.

1 Definitions

1.1 Stochastic Process

A **Stochastic process** is a time series of correlated random variables y_i where $P(y_n, t_n)$ gives the probability of a system being in state y_n at time t_n, and this probability is determined by [9]

$$P(y_n, t_n) = P_{1|n-1}\left(y_n, t_n \mid y_1, t_1; \ldots; y_{n-1}, t_{n-1}\right) \tag{1}$$

1.2 Markov Process

When we talk about a **Markov process** all the previous probabilities are ignored and only the probability $P(y_{n-1}, t_{n-1})$ is taken into consideration for the probability to be in $P(y_n, t_n)$.[10]

$$P_{1|n-1}\left(y_n, t_n \mid y_1, t_1; \ldots; y_{n-1}, t_{n-1}\right) = P_{1|1}\left(y_n, t_n \mid y_{n-1}, t_{n-1}\right) \tag{2}$$

$P_{1|1}$ is the transition probability and it gives the transition from one state to another .

1.2.1 Brownian Motion

An example of this is giving by the brownian motion of a random particle. The particle is in a homogeneous fluid which experiences equal forces from all sides. Here the x_{k+1} position only depends on the x_k and not on $(x_{k-1}, \ldots, x_{k-n})$ [11]

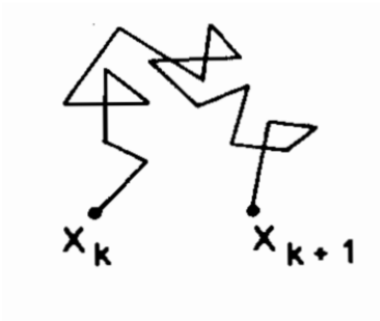
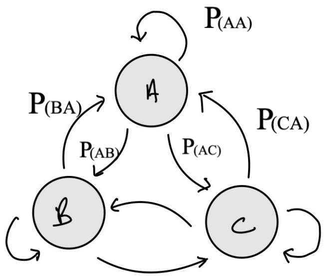


Figure 1: Brownian motion
[10]

1.3 Markov Chain and Transition Probability



Consider three states A,B and C. The system can be in either of the three states. The probability of a system of being in a particular state A is give as P_A.
The transition probability from the state A to B is given as P_{AB}.
Since there are 3 states and particle has to be in either of them, we get the equation.

$$P_{AA} + P_{AB} + P_{AC} = 1 \tag{3}$$

These transition probabilities are defined using a matrix called as the transition matrix.

$$\begin{bmatrix} P_{AA} & P_{AB} & P_{AC} \\ P_{BA} & P_{BB} & P_{BC} \\ P_{CA} & P_{CB} & P_{CC} \end{bmatrix}$$

Here the rows always add up to 1 and the diagonal elements can be 0.

2 The Chapman Kolmogorov Equation

The Chapman Kolmogorov equation gives the two step transition probability of a Markov process.

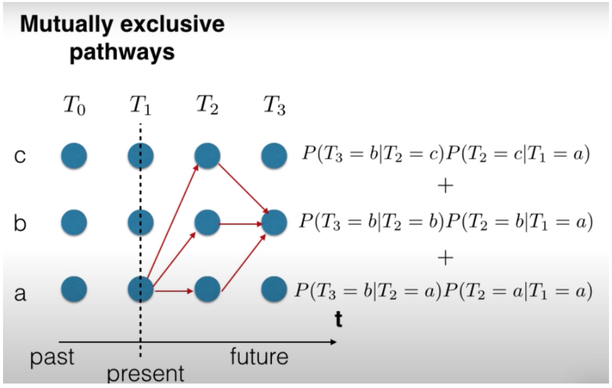


Figure 2: Chapman Kolmogorov
[8]

In the above figure we talk about the probability of transition from state y_a at time t_1 to state y_c at time t_3 through states y_a, y_b and y_c at time t_2.
The particle can take various paths in the intermediate time frame and reach the final state. We integrate over all the possible paths .
The particle probability is given as $P_1(y_a, t_1)$. The transition probability from state at time t_1 to t_2 is given by[8]

$$P_{1|1}(y_a, t_1; y_a, t_2) = P_{y_a,t_2} \mid P_{y_a,t_1} \tag{4}$$

$$P_{1|1}(y_a, t_2; y_a, t_3) = P_{y_c,t_3} \mid P_{y_a,t_2} \tag{5}$$

Similarly we can give two step transition probability

$$\begin{aligned} P_{1|1}(y_a, t_1|y_c, t_3) &= P_{1|1}(y_a, t_2|y_a, t_1)P_{1|1}(y_c, t_3|y_a, t_2) && (6)\\ &+ P_{1|1}(y_b, t_2|y_a, t_1)P_{1|1}(y_c t_3|y_b t_2) && (7)\\ &+ P_{1|1}(y_c, t_2|y_a, t_1)P_{1|1}(y_c t_3|y_c t_2) && (8) \end{aligned}$$

We took a discrete sum over all the possible paths the particle could take and as we can see it is mutually exclusive.

Now we shall properly derive the relation for Chapman Kolmogorov equation.

$$P_2\left(y_1, t_1; y_3, t_3\right) = P_1\left(y_1, t_1\right) \int P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) dy_2 \tag{9}$$

In the above equation $P_1(y_1, t_1)$ is an absolute probability of particle being in an initial state.

$$P_{1|1}(y_2, t_2|y_1, t_1) = \frac{P_2(y_2, t_2; y_1, t_1)}{P_1(y_1, t_1)} \tag{10}$$

Using the Bayes theorem we give a relation of conditional probability and joint probability distribution.

$$P_{1|1}\left(y_3, t_3|y_1, t_1\right) P_1\left(y_1 t_1\right) = P_1((y_1 t_1) \int P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) dy_2 \tag{11}$$

then we divide both sides by $P_1(y_1, t_1)$

$$P_{1|1}\left(y_3, t_3 \mid y_1, t_1\right) = \int P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) dy_2 \tag{12}$$

Hence we derive the Chapman Kolmogorov relation, as we can see this is a two step transition relation and it will help us solve any equation with this condition. It is also non linear, but can be converted to a linear form, called the Master Equation.

2.1 Stationary Markov Process

A **Stationary Markov Process** Y_t is a special type of Markov process where the moments $P(Y(t_n)|Y(t_i), \ldots, Y(t_j))$ are time translation invariant, that is

$$P(Y(t_n)|Y(t_i), \ldots, Y(t_j)) = P(Y(t_n + \tau)|Y(t_i + \tau), \ldots, Y(t_j + \tau) \tag{13}$$

Here the origin of time does not matter since it is a function of elapsed time. It is extremely useful to define equilibrium fluctuations.
Here the transition probability $P_{1|1}$ does not depend on two times but only on the time interval and hence the transition probability becomes T_τ.[10]

$$P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) = T_\tau\left(y_2 \mid y_1\right) \tag{14}$$

$$\text{We define } \tau = t_2 - t_1, \tau' = t_3 - t_2 \quad \text{and} \, (\tau, \tau' > 0) \tag{15}$$

Using the Chapman Kolmogorov Equation

$$P_{1|1}\left(y_3, t_3 \mid y_1, t_1\right)=\int P_{1|1}\left(y_3, t_3 \mid y_2, t_2\right) P_{1|1}\left(y_2, t_2 \mid y_1, t_1\right) d y_2 \tag{16}$$

Substituting eq 11 in eq 13

$$T_{\tau+\tau^{\prime}}\left(y_3 \mid y_1\right)=\int T_{\tau^{\prime}}\left(y_3 \mid y_2\right) T_\tau\left(y_2 \mid y_1\right) \mathrm{d} y_2 \tag{17}$$

The above equation can be read as a product of integral kernels.

$$T_{\tau+\tau^{\prime}}=T_{\tau^{\prime}} T_\tau \tag{18}$$

Now we define an important identity for all stationary Markov Process. This can be directly used in case of equilibrium.

$$P_2\left(y_1, t_1 ; y_2, t_2\right)=T_\tau\left(y_2 \mid y_1\right) P_1\left(y_1\right) \tag{19}$$

The joint Probability distribution P_2 is symmetric,

$$T_\tau(y_2|y_1) P_1(y_1)=T_{-\tau}(y_1|y_2) P_1(y_2) \tag{20}$$

Another important result of stationary Markov Process is that, as $t \rightarrow \infty$ the process saturates to $P_1(y(2))$

$$P_2\left(y_2, t_2-t_1 \mid y_1\right)=P_1\left(y_2\right) \tag{21}$$

2.2 Homogeneous Markov Process

A markov Process where all other probabilities become stationary because the conditional probability is stationary.
A stationary process $Y(t)$ exists such that $P_1\left(y_1\right)$ and $T_\tau\left(y_2 \mid y_1\right)$ takes place . We define a fixed

time t_0 and a fixed value y_0. $Y^*(t)$ is a non stationary markov process for $y \geqslant t_0$ [10]

$$P_1^*(y_1, t_1) = T_{t_1 - t_0}(y_1 \mid y_0) \tag{22}$$

$$P_{1|1}^*(y_2, t_2 \mid y_1, t_1) = T_{t_2 - t_1}(y_2 \mid y_1) \tag{23}$$

One can extract a process $Y(0)$ at t_0 that the values of $Y_(t_0)$ are distributed according to $P(y_0)$, this results into[10]

$$P_1^*(y_1, t_1) = \int T_{t_1 - t_0}(y_1 \mid y_0)\, p(y_0)\, dy_0 \tag{24}$$

$$P_{1|1}^*(y_2, t_2 \mid y_1, t_1) = T_{t_2 - t_1}(y_2 \mid y_1) \tag{25}$$

Since these processes are non stationary but their transition probability depends on time difference T_τ,hence it is Homogeneous Markov process
It was shown above that if we have a non stationary and stationary markov process and if we can define the non stationary markov process with the time shift function for transitional probability T_τ then we can classify that as a homogeneous process.

3 Master equation

They are differential equations that describe the evolution of the probabilities with respect to time. For systems that jump from one to other state in a continuous time. In this sense they are the continuous time version of the recurrence relations for Markov chains.[12]
In order to derive the Master Equation we start from the Chapman Kolmogorov Equation.

$$P_{1|1}(y_3, t_3 \mid y_1, t_1) = \int P_{1|1}(y_3, t_3 \mid y_2, t_2)\, P_{1|1}(y_2, t_2 \mid y_1, t_1)\, dy_2 \tag{26}$$

Using the definition of Stationary Markov Process we shall define transition probability T_τ as [12]

$$T_\tau(y_2, t_2 - t_1 | y_1) \rightarrow \delta(y_2 - y_1) \text{ for } \tau \rightarrow 0 \tag{27}$$

Expanding the transition probability w.r.t τ as

$$T_{\tau'}(y_2|y_1) = (1 - a_0\tau')\delta(y_2 - y_1) + \tau' W(y_2|y_1) + O(\tau') \tag{28}$$

In eq 28 it is shown that for small τ' the transition probability $T_{\tau'}(y_2|y_1)$ has the following form.[4]

Using the normalization condition for $T_{\tau'}(y_2|y_1)$
$W(y_2|y_1)$ is the transition probability per unit time from y_1 to y_2 and therefore $W(y_2|y_1) \geq 0$ a_0

$$\int dyT_{\tau'}(y_2|y_1) = 1 \tag{29}$$

Using Eq 28

$$= 1 - a_0(y_1)\tau' + \tau' \int dyW(y_2|y_1) + O(\tau') \tag{30}$$

Hence we get

$$a_0(y_1) = \int W(y_2|y_1)dy_2 \tag{31}$$

The coefficient $a_0\tau'$is the probability that transitions take place during τ'
Using the Chapman Kolmogorov equation and Eq 28

$$T_{\tau+\tau'}(y_3|y_1) = \tau' \int W(y_3|y_2)T_\tau(y_2|y_1)dy_2 + [1 - a_0(y_3)\tau']T_\tau(y_3|y_1) \tag{32}$$

$$T_\tau(y_3|y_1)T_{\tau'}(y_3|y_1) = \tau' \int W(y_3|y_2)T_\tau(y_2|y_1)dy_2 - \tau' \int W(y_2|y_3)T_\tau(y_3|y_1)dy_2. \tag{33}$$

Dividing both sides by τ' and the limit of $\tau' \to 0$

$$\frac{\partial T_\tau(y_3|y_1)}{\partial t} = \int W(y_3|y_2)T_\tau(y_2|y_1)dy_2 - \int W(y_2|y_3)T_\tau(y_3|y_1))dy_2 \tag{34}$$

To write it in a more simple form,

$$\frac{\partial P(y,t)}{\partial t} = \int W(y|y')P(y',t)dy' - \int W(y'|y)P(y,t)dy' \tag{35}$$

Eq 35 is the Master equation for the range of Y which is continuous. It can be transformed for discrete set of states as

$$\frac{dp_n(t)}{dt} = \sum_n (W_{nn'}p_{n'}(t) - W_{n'n}P_n(t)) \tag{36}$$

The probability evolution of the state n with t consists of gain-loss terms. The gain term is gives a transition from $(n' \to n)$ and the loss term is the transition from $(n \to n')$. $W_{nn'}$ defines the transition rates where $W_{nn'} \geq 0$ and $n \neq n'$.

Solution of Master Equation

Starting from the Master Equation which is a first order differential equation in time,

$$\frac{d\rho_n(t)}{dt} = \sum_n \{W_{nn'}\rho_{n'}(t) - W_{n'n}\rho_n(t)\} \tag{37}$$

Transforming the above master equation into a different form of matrix $\mathcal{W}$:

$$\mathcal{W}_{nn'} = W_{nn'} - \delta_{nn'}(\sum_{n"} W_{n"n}) \tag{38}$$

Then Eq 38 becomes,

$$\frac{\partial\rho_n(t)}{\partial t} = \sum_{n'} \mathcal{W}_{nn'}\rho_{n'}(t) \tag{39}$$

$$\frac{\partial\rho_n(t)}{\partial t} = \mathcal{W}\rho(t) \tag{40}$$

ρ is a vector with vector components ρ_n given as $\begin{pmatrix} \rho_{1,t} \\ \rho_{2,t} \\ \dots \\ \dots \\ \rho_{n,t} \end{pmatrix}$

The solution of Eq 39 with initial $P_n(0)$ is

$$\rho(t) = e^{iW}\rho 0 \tag{41}$$

It is not certain that $\mathcal{W}$ is symmetric, so all solutions obtained need not be superpositions of these eigensolutions.
The solutions however need to follow the following properties.

$$\mathcal{W}_{nn'} >= 0 \quad \text{for} \neq n' \tag{42}$$

$$\sum_n \mathcal{W}_{nn'} = 0 \quad \text{for each} \quad n' \tag{43}$$

An important property of the Master Equation is that at $t \to \infty$ all solutions tend to **Stationary Solutions**.

Example for master equation

Considering a radioactive decay, since the next state of the particle only depends on the previous state, it can be said that the following example follows the Markov assumption.
Starting at some initial time t_0, $P_1(t)$ and $P_2(t)$ are probabilities that the particle is in state 1 and 2, respectively, at time t.[5]. The following relation must be satisfied.

$$P_1(t) + P_2(t) = 1. \tag{44}$$

State 1 is before the decay of the element
State 2 is completely after decay.
$w(1 \to 2)$ is the rate at which one particle jumps from state 1 to state 2.Now we are going to derive a differential relation for $P_1(t)$ by relating the probabilities at t and $t + dt$.
Assuming transitions occur uniformly and randomly at the constant rate.
$w(1 \to 2)dt$ is the transition probability that the particle jumps from 1 to 2 in the time $(t, t+dt)$.

$$P_1(t + dt) = P_1(t) \quad \text{Prob(staying in 1)} + P_2(t) \quad \text{Prob(jumping from 2 to 1)} \tag{45}$$

The probability of jumping from 2 to 1 in the time interval (t, t + dt) is $w(2 \to 1)dt$ where as the probability of staying in state 1 is $(1 - w(1 \to 2)dt)$.

$$P_1(t + dt) = P_1(t)[1 - w(1 \to 2)dt] + P_2(t)[w(2 \to 1)dt] + O(dt^2) \tag{46}$$

Where $O(dt^2) = w(1 \to 2)dt \times w(2 \to 1)dt$ but with limit $dt \to 0$ all higher order vanish.
The Master equation for this process is given as

$$\frac{dP_1(t)}{dt} = -\omega(1 \to 2)P_1(t) + \omega(2 \to 1)P_2(t) \tag{47}$$

Similarly we get

$$\frac{dP_2(t)}{dt} = -\omega(2 \to 1)P_2(t) + \omega(1 \to 2)P_1(t) \tag{48}$$

Using the condition in Eq 44 and differentiating it with respect to t.

$$\frac{d(P_1(t) + P_2(t))}{dt} = 0 \tag{49}$$

In order to obtain the solution and to solve the equation further, we shall define probability current.

$$J_1(t) = \frac{dP_1(t)}{dt} \quad \text{and} \quad J_2(t) = \frac{dP_2(t)}{dt} \tag{50}$$

$J(1 \to 2)$ gives the probability current from state 1 to 2, hence we can write.

$$J(1 \to 2) = -\omega(1 \to 2)P_1(t) + \omega(2 \to 1)P_2(t). \tag{51}$$

At the start we had defined that it is a homogeneous process with constant rates, we can use this to find the solution of Eq 47 and Eq 48.
We define $\omega(1 \to 2) + \omega(2 \to 1) = \alpha$

$$P_1(t) = P_1(t_0)\left(\frac{\alpha e^-(\alpha)(t - t_0)}{\alpha}\right) + P_2(t_0)\left(\frac{\omega_{2\to1}(1 - e^-(\alpha)(t - t_0)}{\alpha}\right) \tag{52}$$

$$P_2(t) = P_2(t_0)\left(\frac{\alpha e^-(\alpha)(t - t_0)}{\alpha}\right) + P_1(t_0)\left(\frac{\omega_{1\to2}(1 - e^-(\alpha)(t - t_0)}{\alpha}\right) \tag{53}$$

Eventually all Master Equations saturate to stationary solution and this applies to radioactive decay as well, hence for $t \to \infty$ the exponential term will vanish and the equation simplifies to:

$$P_1 = \frac{\omega 2 \to 1}{\alpha} \tag{54}$$

$$P_2 = \frac{\omega 1 \to 2}{\alpha} \tag{55}$$

If the stationary distribution satisfies the following relation then we get a balanced condition,this means that the transition rate from state $(1 \to 2)$ is same as the transition rate from state $(2 \to 1)$

and the decay of the element comes to a stable point.

$$\omega(1 \to 2)P_1 = \omega(2 \to 1)P_2 \tag{56}$$

Hence it is shown that any process that can be described by the time evolution of the probability distribution and satisfies the Master equation can be easily simplified with the expected results of the Master Equation.

3.1 Quantum master equation

A **Quantum Master Equation** describes the time evolution of the quantum systems.Here we use the density matrix ρ such that $\rho = |\Psi\rangle \langle\Psi|$, but every system ends up interacting with the environment to some extent.
Since it is an open system,we have to define the interaction Hamiltonian.A Hamiltonian H comprises of H_s as Hamiltonian of the quantum system and H_b is the Hamiltonian of the heat bath. H_{sb} is the interacting hamiltonian.

$$H = H_s + H_b + H_{sb}$$

3.2 Markovian Quantum Master Equation

The **Markovian** assumption states that the system does not have any memory of the previous states. This approximation is justified when the system in question has enough time to relax to equilibrium before being perturbed again by interactions with its environment.If the interaction between the system and its environment is weak, then any changes to the combined system over time can be approximated as originating from only the system in question.[2]

The most general type of the master equation is the **Gorini–Kossakowski–Sudarshan–Lindblad** equation or GKSL equation.

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H_s, \rho_s] + \mathcal{L}_D(\rho_s) \tag{57}$$

H_s is the hermitian Hamiltonian and $\mathcal{L}_D$:

$$\mathcal{L}_D(\rho_s) = \sum_n \left(V_n \rho_{\mathrm{S}} V_n^\dagger - \frac{1}{2}\left(\rho_{\mathrm{S}} V_n^\dagger V_n + V_n^\dagger V_n \rho_{\mathrm{S}}\right)\right) \tag{58}$$

This describes the dissipative part through system operators V_n the influence of the bath on the system. Markov property assumes that system and bath are uncorrelated.
The GKSL equation leads any initial state ρ_s to a steady state solution. $\rho_s(t \to \infty) = 0$

4 Fermi's Golden Rule

In this section we will show the proof that for open systems **weakly coupled** to heat baths the transition probabilities per unit time obtained using Markov approximations are equal to the transition probabilities of Fermi's Golden rule.[1]

$$P_{fi} = 2\pi |\langle f|V|i\rangle|^2 \delta\left(\epsilon_i - \epsilon_f\right) \tag{59}$$

P_{fi} gives the transition probability per unit time from initial state $|i\rangle$ to final state $|f\rangle$.
Hamiltonian of the system has the form as $H = H_0 + V$, where H_0 is system without interaction. Now we talk about the interaction between weakly coupled system and the heat bath.

$$H^\lambda = H_S + H_R + \lambda V \tag{60}$$

and the interaction term is defined as

$$\lambda V = \lambda S \otimes R \tag{61}$$

In order to satisfy the Markovian assumption we take $\lambda \to 0$ and rescale time factor $\tau = \lambda^2 t$. we use Liouville's theorem where $\mathcal{K}^0$ is the Krauss operator.

$$\frac{d\rho(\tau)}{d\tau} = K^0 \rho(\tau)\rho(\tau) = \Lambda(\tau)\rho, \quad \rho \in D\left(K^0\right) \tag{62}$$

The systems S and R obey the Krauss operator relation

$$K^0 = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} e^{-i\hat{H}Sx} K e^{i\hat{H}Sx} dx \tag{63}$$

$$\hat{H}S = [HS_1 \cdot] \tag{64}$$

$$K\rho = \int_0^\infty dt \left\{ -\operatorname{tr}\left(R_t R \sigma_0\right) S_t S \rho + \operatorname{tr}\left(R_t R \sigma_0\right) S \rho S_\tau \right. \tag{65}$$

$$\left. + \operatorname{tr}\left(R R_\tau \sigma_0\right) S_\tau \rho S - \operatorname{tr}\left(R R_\tau \sigma_0\right) \rho S S_\tau \right\} \tag{66}$$

$$S_{\mathrm{r}} = e^{iH_{gt} S e^{-iH_{St}}}, \quad R_t = e^{iH_{S^T} R_e - iH_{St}}$$

We took the trace with respect to the environment and substituted it in the following form. At the end we rewrite

$$h(t) = \mathrm{tr}\left[R_t R \sigma_0\right] \tag{67}$$

Using the Pauli Master equation we write the transition probability as

$$\begin{aligned} \frac{dp_n(\tau)}{d\tau} &= \sum_m a_{nm} p_m(\tau) - a_{nm} p_n(\tau) \\ a_{nm} &= |\langle n|S|m\rangle|^2 \hat{h}\left(\epsilon_m - \epsilon_m\right) \end{aligned} \tag{68}$$

We define composite system as $|m\rangle \otimes |E, \gamma\rangle$ and $|n\rangle \otimes |E', \gamma'\rangle$ and they are orthonormal set of eigenvectors.
Substituting this in the transition probability

$$\begin{aligned} P_{m,E,\gamma;n,E',\gamma'} =& 2\pi\lambda^2 |\langle m|S|n\rangle|^2 \left|\langle E, \gamma|R|\gamma', E'\rangle\right|^2 \\ & \times \delta\left(\epsilon_m + E' - \epsilon_n - E'\right) \end{aligned} \tag{69}$$

Then using

$$\langle E\gamma|\, \sigma_0 \,|E'\gamma'\rangle = \sigma_0(E, \gamma) \times \delta\left(E - E'\right) \delta_{\gamma\gamma'} \tag{70}$$

and using the delta function relation we get

$$P_{nm} = \lambda^2 |(n|S|m)|^2 \int_{-\infty}^{+\infty} dt e^{i(\varepsilon_m - \epsilon_n)t}\, \mathrm{tr}\left(\sigma_0 R_t R\right) \tag{71}$$

$$P_{nm} = \lambda^2 |\langle n|S|m\rangle|^2 \hat{h}\left(\epsilon_m - \epsilon_n\right) \tag{72}$$

Thus we define the new transition probability with rescaled time $\tau = \lambda^2 t$ as

$$\tilde{P}nm = Pnm\lambda^2 = a_{nm} \tag{73}$$

Thus proving that Markovian assumptions work for weakly coupled systems.[1]

5 Non Markovian Processes

A **Non Markovian Process** is a stochastic process where the future state depends on the past states and not just the present state.

$$P_k(y_{n+k}, t_{n+k}; ...; y_{n+1}, t_{n+1}|y_n, t_n, ...y_1 t_1); \text{n=1,2..k}$$

Here the above equation gives the joint probability relation.
The probability of transitioning to the final state for Non Markovian assumption is given using the conditional probability below.

$$P_{1|1}(y_{n+1}, t_{n+1}|y_n, t_n; ......; y_1, t_1)$$

Markovian Process is a special case while the Non Markovian process comprises of every other stochastic process.Different techniques have been derived to solve the Non Markovian processes where a term called as the Memory Kernel will take into account all the previous states.

6 Non Markovian Master Equation

The processes described by generalized master equations (GME), derived from the Liouville equation on the basis of different conditions, we have a Markovian or Non-Markovian. It mainly depends on explicit time integration.

6.1 Nakajima-Zwangzig Equation

In order to describe open quantum systems where model follows the Non Markovian dynamics we use the Nakajima-Zwanzig Equation.This provides the counterpart to the Lindblad structure of Markovian model.[6] It makes use of the memory kernel to describe the previous states. It also uses the Projection operators P and Q. P describes the relevant part of the density matrix ρ and Q describes the irrelevant part.[6]

We start defining the memory kernel of the equation by using **Liouville-von Neumann Equation**.

$$\frac{\partial P\rho(t)}{\partial t} = P\frac{\partial \rho(t)}{\partial t} \tag{74}$$

$$\frac{\partial P\rho(t)}{\partial t} = P[\alpha L(t)\rho(t)] = \alpha PL(t)\rho(t) \tag{75}$$

Where α is a constant
L is the Liouville operator acting on density matrix ρ. we will use the identity $P + Q = I$

$$\frac{\partial P\rho(t)}{\partial t} = \alpha PL(t)I\rho(t) = \alpha PL(t)(P+Q)\rho(t) \tag{76}$$

$$\frac{\partial P\rho(t)}{\partial t} = \alpha PL(t)P\rho(t) + \alpha PL(t)Q\rho(t) \tag{77}$$

similarly we get

$$\frac{\partial Q\rho(t)}{\partial t} = \alpha QL(t)P\rho(t) + \alpha QL(t)Q\rho(t) \tag{78}$$

We have to derive a closed form for the relevant part $P\rho(t)$ for the reduced density matrix $\rho_s = tr_B\rho(t)$ of the open system . We will look for a solution from eq (73) to give $\rho(t_0)$ for some initial time t_0

$$Q\rho(t) = K(t,t_0)Q\rho(t_0 + \alpha \int_{t_0}^{t} dsK(t,s)QL(s)P\rho(s) \tag{79}$$

$K(t,s) = T^{\leftarrow}\exp[\alpha \int_s^t ds'QL(s')]$is the propagator and $T^{\leftarrow}$ is time ordering. The propagator satisfies the differential equation:

$$\frac{\partial K(t,s)}{\partial t} = \alpha QL(t)K(t,s) \tag{80}$$

substituting Eq 79 in Eq 76

$$\frac{\partial P\rho(t)}{\partial t} = \alpha PL(t)P\rho(t) + \alpha PL(t)\left[K(t,t_0)Q\rho t_0 + \alpha \int_{t_0}^{t} dsK(t,s)QL(s)P\rho(s)\right] \tag{81}$$

$$\frac{\partial P\rho(t)}{\partial t} = \alpha PL(t)P\rho(t) + \alpha PL(t)K(t,t_0)Q\rho t_0 + \alpha^2 \int_{t_0}^{t} dsPL(t)K(t,s)QL(s)P\rho(s) \tag{82}$$

The above equation is the **Nakajima-Zwangzig** equation. It describes the time evolution of the relevant part of the density matrix.
There is an integral from $[t_0, t]$ and this consists of all the history of the system, hence we can say that the above equation follows a **Non Markovian** assumption.

We shall define a **Memory kernel** $\mathcal{K}(t, s)$ as:

$$(\mathcal{K})(t, s) = \alpha^2 PL(t)K(t, s)QL(s)P \tag{83}$$

Using the memory kernel definition we can rewrite the Nakajima-Zwanzig equation as

$$\frac{\partial P\rho(t)}{\partial t} = \int_{t_0}^{t} ds(\mathcal{K})(t, s)P\rho(s) \tag{84}$$

Thus we have derived a Non Markovian Master equation for open quantum systems. [3]

Two-level system

As an example we again consider a two level system interacting with a reservoir.In order to describe the strong coupling effects between the system and the reservoir we make the Non Markovian assumption.
I shall provide a summary of the derivation [7],

$$H = H_s + H_E + H_I \tag{85}$$

Then we define the density matrix $\rho(t)$ as the trace with respect to the environment.

$$\rho(t) = Tr_E[U(t)\rho(0) \otimes \rho_E U^{+}(t)] \quad \text{U(t) gives the unitary evolution of the system} \tag{86}$$

$$\rho(t) = \Phi(0) \tag{87}$$

The general expansion in terms of memory kernel $\mathcal{K}(t)$ is given as:

$$\frac{\partial \rho(t)}{\partial t} = \mathcal{K}(t)\rho(t) \tag{88}$$

We looked in detail at the derivation of the Nakajima-Zwanzig master equation and also had an overview about where it can be used. This equation can also be used in order to describe Gaussian white noise or the velocity of a Brownian motion [6].

7 Summary

Stochastic processes comprise of both Markovian and Non - Markovian processes. Non-Markovian processes comprise a broad variety of problems whereas Markovian is a special case. A Markovian process is a memoryless process and the probability of attaining a future state only depends on the present state. We can write a two step Markovian process by combining the conditional probability of past - present and present - future states. This relation is given by the Chapman-Kolmogorov equation. The first issue we notice is that it is a non-linear equation, so in order to make calculations easier we convert it into a first order linear differential equation. This is a Master equation, also called the gain loss equation, which gives the evolution of probability over time. As we saw earlier, the master equation described the radioactive decay process.This proved that the calculations became extremely easier with the markov assumption. Introducing the Quantum Master equations we noticed that the Markovian assumptions hold when the system interacts weakly with the surrounding. We shift to non Markovian to describe strong coupling or when the system and the environment interacts over a longer time.

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