STATISTICAL PHYSICS SEMINAR

FOKKER-PLANCK- AND LANGEVIN EQUATION

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NAME: DANIEL WALTER
SUPERVISOR: PROF. GEORG WOLSCHIN

Abstract

This report is associated with a presentation that I gave in the statistical physics seminar of Prof. Georg Wolschin at Heidelberg University in November 2021. It provides some background and derivations that were not presented in the talk and tries to give a basic introduction into the topic of Fokker-Planck- and Langevin equations.

We start out by deriving the Fokker-Planck equation from the general Master equation of time- and state-continuous Markov processes. In section 2 we give a (heuristic) introduction into the topic stochastic differential equations and Langevin equations and explore their connection to the Fokker-Planck equation. Finally we use some of the developed tools to investigate the Ornstein-Uhlenbeck process and classical Brownian motion.

Contents

1 THE FOKKER-PLANCK EQUATION

In his famous 1905 paper *"On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat"* [1], Albert Einstein manages to derive experimentally verifiable predictions about the diffusion rates of Brownian particles (that were later confirmed by Jean-Baptiste Perrin [3]). In his paper he uses the concept of state transition probabilities to derive a partial differential equation (the diffusion equation) that describes the probability density of the Brownian particle (or particle density if we are dealing with many particles) and solves the equation to conclude that the average deviation in position is given by $\lambda_x = \sqrt{2Dt}$. Where $D = \frac{RT}{N}\frac{1}{6\pi kP}$ is the diffusion coefficient.

In the following sections we are going to use a very similar reasoning to derive the Fokker-Planck equation. However, our derivation is going to be much more general and is going to include Einstein's diffusion equation as a special case.

1.1 Markov processes and the Master equation

We are going to derive the Fokker-Planck equation in the very general framework of Markov processes. To that end, let us first introduce the concept of a stochastic process.

A stochastic process is a family of random variables x_t over an index set $t \in I$. In our specific case we are going to assume that $x_t \in \mathbb{R}$ and $I = [0,\infty)$, that is to say we are assuming a state- and time-continuous process.

We say that a stochastic process has the *Markov property* if the probability of a future state is independent of the past history, given knowledge about the most recent state. Formally: Suppose we have past information about our system $(x_1,t_1),\dots,(x_n,t_n)$ such that $t_1 < \cdots < t_n \in [0,\infty)$ and suppose further that we want to predict the probability that our system is in state x at time t with $t > t_n$, then we have the conditional independence relation

$$p(x,t|x_n,t_n;\dots;x_1,t_1) = p(x,t|x_n,t_n). \tag{1}$$

A stochastic process that has the Markov property is called a *Markov process*. From here on we are going to assume that all processes in our discussion are state- and time-continuous Markov processes.

The dynamics of the probability density of such a Markov process can be described by a so-called *Master equation* (see e.g. [4])

$$\frac{\partial p(x,t)}{\partial t} = \int_{-\infty}^{+\infty} \left[W_t(x|x')\,p(x',t) - W_t(x'|x)\,p(x,t)\right] dx', \tag{2}$$

where $W_t(x|x')$ denotes the rate of state transitions from state x' to state x at time t. Notice that the left term on the RHS sums over all transitions from some state x' to the (fixed) state x. We can interpret this term as an "inflow" of states. The right term on the RHS sums over all transitions from the state x to some other state x', we can therefore think of this as an "outflow" of states. The Master equation can then be interpreted as follows: The change in probability density of our system occupying a state x is given by the difference between the inflow and outflow of this state.

1.2 The Kramers-Moyal expansion

The Master equation is an integro-differential equation that is generally hard to solve and in many cases somewhat inaccessible. We would like to transform the Master equation into a more accessible form, like a partial differential equation. As it turns out, this can be achieved via the so-called *Kramers-Moyal expansion* (named after the physicists Hendrik Anthony Kramers (1894-1952) and José Enrique Moyal (1910-1998)).

Before we discuss the expansion, let us slightly reformulate the Master equation as it is stated in eq. (2). The transition rates are given in terms of an initial state x' and a final state x. We are going to reparametrize the integral in terms of an initial state and a deviation from the initial state $r := x - x'$. Plugging this into eq. (2) and using $dx' = -dr$, we get:

$$\frac{\partial p(x,t)}{\partial t} = \int_{-\infty}^{+\infty} \left[W_t(x|x-r)\,p(x-r,t) - W_t(x-r|x)\,p(x,t) \right] dr. \tag{3}$$

Now let us define $W_t^*(r,x) := W_t(r+x|x)$, than $W_t^*(r,x)$ gives us the rates at which state x deviates by r from its initial position at time t. Putting this definition into eq. (3), we get:

$$\frac{\partial p(x,t)}{\partial t} = \int_{-\infty}^{+\infty} \left[W_t^*(r,x-r)\,p(x-r,t) - W_t^*(-r|x)\,p(x,t) \right] dr. \tag{4}$$

Notice that we can omit the minus sign in the transition rate function on the right since we are integrating over the region $(-\infty,+\infty)$ (changing the sign just mirrors the function, which results in the same definite integral):

$$\frac{\partial p(x,t)}{\partial t} = \int_{-\infty}^{+\infty} \left[W_t^*(r,x-r)\,p(x-r,t) - W_t^*(r|x)\,p(x,t) \right] dr. \tag{5}$$

The idea of the Kramers-Moyal expansion is now to Taylor expand the the function $W_t^*(r,x-$

$r)p(x-r,t)$ in the state argument around x. Then we get

$$= \int_{-\infty}^{+\infty} \left[W_t^*(r,x)p(x,t) + \sum_{k=1}^{\infty} \frac{(-r)^k}{k!} \frac{\partial^k}{\partial x^k} \left(W_t^*(r,x)p(x,t) \right) - W_t^*(r|x)p(x,t) \right] dr \tag{6}$$

$$= \int_{-\infty}^{+\infty} \sum_{k=1}^{\infty} \frac{(-r)^k}{k!} \frac{\partial^k}{\partial x^k} \left[W_t^*(r,x)p(x,t) \right] dr. \tag{7}$$

Notice that in general we will not be able to interchange the integral with the summation. We will have to make some implicit assumptions about the distribution W_t^*. In particular we are going to assume that

$$\int_{-\infty}^{+\infty} \sum_{k=1}^{\infty} \frac{|r|^k}{k!} \left| \frac{\partial^k}{\partial x^k} \left[W_t^*(r,x)p(x,t) \right] \right| dr < \infty. \tag{8}$$

Then by means of *Fubini's theorem* [5, sec. 5.6] we are allowed to interchange the integral with the sum:

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{-\infty}^{+\infty} r^k \frac{\partial^k}{\partial x^k} \left[W_t^*(r,x)p(x,t) \right] dr \tag{9}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial x^k} \left[\int_{-\infty}^{+\infty} r^k W_t^*(r,x) dr \cdot p(x,t) \right] \tag{10}$$

Defining

$$a_t^{(k)}(x) := \int_{-\infty}^{+\infty} r^k W_t^*(r,x) dr, \tag{11}$$

we finally get

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial x^k} \left[a_t^{(k)}(x) \cdot p(x,t) \right]. \tag{12}$$

This is the Kramers-Moyal expansion of the Master equation. $a_t^{(k)}(x)$ denotes the k-th statistical moment of the transition rate at time t and state x. Notice that the Kramers-Moyal expansion allows us to fully specify the dynamics of the probability density via the moments of the transition rates. In general this is of course only possible if the moments of the transition rates exist (remember the assumption for Fubini's theorem). Imagine for example that

W_t^* was Cauchy distributed (in r). Then of course non of the moments would exist and an expansion of this kind would not be possible.

The Kramers-Moyal expansion yields a partial differential equation of infinite order. To turn this into a PDE of finite order we are going to neglect all orders higher than 2 to get the *Fokker-Planck equation*:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[a_t^{(1)}(x)\cdot p(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[a_t^{(2)}(x)\cdot p(x,t)\right]. \tag{13}$$

In general this is of course only an approximation. However, we can justify the omission of higher orders in two ways: first, note that the higher orders are suppressed by a factor $\frac{1}{k!}$. We can therefore in many cases assume that the higher orders are only going to give small corrections to the Fokker-Planck equation. Secondly, *Pawula's theorem* [8, p. 70 ff.] states that there can only be three distinct cases. Either the expansion stops after the first term or it stops after the second term or it contains an infinite number of non-vanishing terms. In particular if $a_t^{(2k)}(x) = 0$ for some $k \in \mathbb{N}$, then $a_t^{(k)}(x) = 0 \ \ \forall k \geq 3$. And indeed the Fokker-Planck equation is exact in many cases.

2 THE LANGEVIN EQUATION

In 1908, three years after Einsteins Brownian motion paper, the french physicist Paul Langevin published his own paper on Brownian motion that retrieved the same results, using a completely different mathematical framework [7]. He himself deemed the approach "infinitely more simple" [1]. And in fact Langevin's method is much closer to the usual physics methodology. He modifies Newton's equation of motion by adding a randomly fluctuating force, accounting for the unpredictable "kicks" that the Brownian particle receives from the smaller fluid particles. The resulting equation is something that we would nowadays recognize as a *stochastic differential equation (SDE)*.

In the following sections we are going to give a quick (non-rigorous) introduction into the topic of SDE's (also referred to as Langevin equations) and explore their connection to the Fokker-Planck equation.

[1] translated by Anthony Gythiel; Department of History, Wichita State University

2.1 The Wiener process

Before we can discuss the theory of stochastic differential equations, we have to introduce the *Wiener process*, which is of fundamental importance for this subject.

Suppose $W_t \in \mathbb{R}, \quad t \in [0,\infty)$ is a stochastic process (do not confuse W_t with the transition rates of sec. 2). We call W_t a Wiener process if it fulfills the following four conditions [10]:

1. $W_0 = 0$, a Wiener process always starts at the origin.
2. A Wiener process is time-continuous.
3. For $s < t \in [0,\infty)$ we have $W_t - W_s \sim \mathcal{N}(0, t-s)$. That is to say, the deviations of a Wiener process are normally distributed with mean 0 and variance $t-s$.
4. For any $0 \leq s < t \leq u < v \in [0,\infty)$ we have that $W_t - W_s$ and $W_v - W_u$ are statistically independent. Intuitively this means that the random fluctuations of a Wiener process are independent from one another.

It can be shown that such a process exists and is unique [10]. Fig. 1 shows seven samples of the process that were created through a simple Monte-Carlo simulation. It is important to note, that all of the curves belong to the *same* process (namely the Wiener process). This is analogous to how we may sample different numbers from a normal distribution. The resulting numbers may be different, but they all come from the same underlying distribution. In sec. 3.1 we will see that the Wiener process can be directly used to describe Brownian motion.

Condition 3 tells us that the difference of the Wiener process over a time-interval $[t, t+\Delta t]$ is normally distributed and of magnitude $\sim \sqrt{\Delta t}$. Now imagine we make this time interval infinitesimally small, such that we have $W_{t+dt} - W_t$. Let us denote this quantity by

$$dW_t := W_{t+dt} - W_t \tag{14}$$

We can think of this as an infinitesimal fluctuation of the Wiener process at time t. Notice that condition 3 implies

$$\langle dW_t \rangle = 0 \quad \text{and} \quad \langle dW_t^2 \rangle = dt. \tag{15}$$

Mathematically, this quantity is of course only to be understood heuristically. dW_t has no well-defined meaning. Nevertheless it gives us the correct intuitions and is quite useful. In the next section we are going to use this quantity to generalize the notion of an ordinary differential equation to its stochastic analogue.

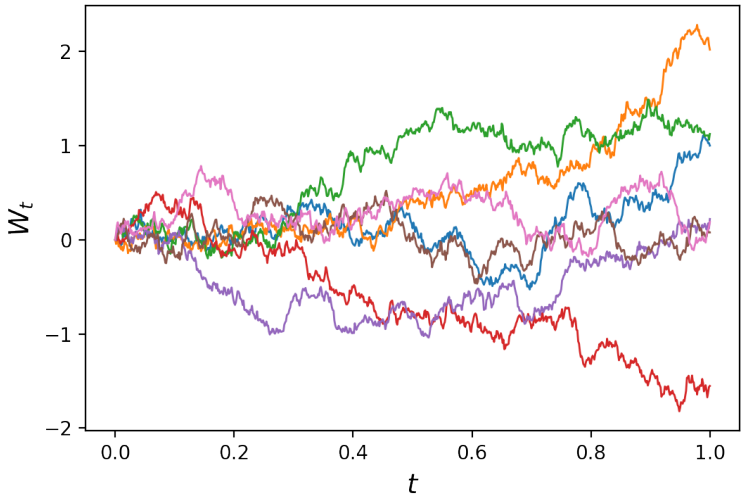

Figure 1: Monte-Carlo simulation of seven random instances of the Wiener process. Notice that the process always starts at $W_0 = 0$. [11]

2.2 Stochastic differential equations

Notice that almost all interesting ordinary differential equations (ODE's) of first order can be reformulated into the general form

$$\frac{dx}{dt} = A(x,t). \tag{16}$$

Where A denotes some general function of x and t. We can heuristically write this as an equation of differentials:

$$dx = A(x,t)\,dt. \tag{17}$$

Where we use the following interpretation: At every point in time t, a small change in x is given by a product of a small time interval dt and the function $A(x(t),t)$. The dynamics of such a system are of course completely deterministic.

To turn this into a *stochastic* differential equation, we add a small random term to the equation:

$$dx = A(x,t)\,dt + B(x,t)\,dW_t. \tag{18}$$

We are multiplying the stochastic noise by a function $B(x,t)$ to allow for varying fluctuations that may depend on the state x and time t. In physics we will often "divide" by dt to turn the

equation into a more familiar form:

$$\frac{dx}{dt} = A(x,t) + B(x,t)\frac{dW_t}{dt}. \tag{19}$$

This is the general *Langevin equation*. Note that one has to be careful when using this formalism. The Wiener process is not differentiable, the interpretation of the quantity $\frac{dW_t}{dt}$ is therefore unclear. This is one of the reasons why mathematicians tend to prefer the notation presented in eq. (18).

However, even eq. (18) is mathematically ill-defined. The rigorous formulation requires the concept of the *Itô integral* [10], the introduction of which would exceed the scope of this report. The SDE is then written in integral form as

$$x(t) - x(0) = \int_0^t dx = \int_0^t A(x,t)dt + \int_0^t B(x,t)dW_t, \tag{20}$$

where the rightmost term is interpreted as an Itô integral. This is then a mathematically precise expression.

As it turns out, these kinds of equations have somewhat unusual properties that deviate from the rules of classical calculus. *Itô calculus* extends the classical framework to the more general case of SDE's.

An example of this is *Itô's lemma*, which generalizes the change of variables formula to the stochastic case: suppose $x(t)$ is the solution of a SDE as given in eq. (18). And suppose further that $f : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ is an arbitrary function (that is twice differentiable in the first argument and differentiable in the second argument). Then Itô's lemma states [6] (without proof):

$$df(x,t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2, \tag{21}$$

where the stochastic differential dx^2 will generally not vanish. Notice the second derivative, which does not occur in the non-stochastic formula. In the next section we will use this lemma to derive a connection between the SDE's and the Fokker-Planck equation.

2.3 Connection to the Fokker-Planck equation

Looking at eq. (18) one notices that at every point in time, the next state $x + dx$ is only dependent on the current system state (and thereby conditionally independent of the previous states). From this we can deduce that every such process is a Markov process. In

addition, note that the transition rates are going to be reasonably well-behaved since we are assuming an underlying Wiener process. We can therefore deduce that there has to exist a Fokker-Planck equation that describes the dynamics of the probability density of x.

To see this, consider the stochastic process

$$dx = A(x,t)dt + B(x,t)dW_t \tag{22}$$

and some function $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ (differentiable as described above). Then from Itô's lemma we know that

$$df(x,t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 \tag{23}$$

$$= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\cdot(A(x,t)dt + B(x,t)dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\cdot(A(x,t)dt + B(x,t)dW_t)^2 \tag{24}$$

$$= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}A(x,t)\right)dt + \frac{\partial f}{\partial x}B(x,t)dW_t + \frac{\partial^2 f}{\partial x^2}A(x,t)B(x,t)dtdW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}B(x,t)^2dW_t^2, \tag{25}$$

where we assumed $dt^2 = 0$. Taking the expectation yields

$$\langle\frac{\partial f}{\partial x}B(x,t)dW_t\rangle = \langle\frac{\partial f}{\partial x}B(x,t)\rangle\langle dW_t\rangle = \langle\frac{\partial f}{\partial x}B(x,t)\rangle\cdot 0 = 0 \tag{26}$$

$$\langle\frac{\partial^2 f}{\partial x^2}A(x,t)B(x,t)dtdW_t\rangle = \langle\frac{\partial^2 f}{\partial x^2}A(x,t)B(x,t)dt\rangle\langle dW_t\rangle = 0 \tag{27}$$

$$\langle\frac{1}{2}\frac{\partial^2 f}{\partial x^2}B(x,t)^2dW_t^2\rangle = \langle\frac{1}{2}\frac{\partial^2 f}{\partial x^2}B(x,t)^2\rangle\langle dW_t^2\rangle = \langle\frac{1}{2}\frac{\partial^2 f}{\partial x^2}B(x,t)^2\rangle dt, \tag{28}$$

here we used eq. (15). Plugging this back into eq. (25) gives us

$$\langle df\rangle = \left[\langle\frac{\partial f}{\partial t}\rangle + \langle\frac{\partial f}{\partial x}A(x,t)\rangle + \langle\frac{1}{2}\frac{\partial^2 f}{\partial x^2}B(x,t)^2\rangle\right]dt \tag{29}$$

$$\implies \frac{d}{dt}\langle f\rangle = \frac{\langle df\rangle}{dt} = \langle\frac{\partial f}{\partial t}\rangle + \langle\frac{\partial f}{\partial x}A(x,t)\rangle + \langle\frac{1}{2}\frac{\partial^2 f}{\partial x^2}B(x,t)^2\rangle \tag{30}$$

$$= \int_{-\infty}^{+\infty}\frac{\partial f}{\partial t}p(x,t)dx + \int_{-\infty}^{+\infty}\frac{\partial f}{\partial x}A(x,t)p(x,t)dx + \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\partial^2 f}{\partial x^2}B(x,t)^2 p(x,t)dx. \tag{31}$$

On the other hand we have

$$\frac{d}{dt}\langle f\rangle = \frac{d}{dt}\int_{-\infty}^{+\infty} f(x,t)p(x,t)dx = \int_{-\infty}^{+\infty}\frac{d}{dt}\big(f(x,t)p(x,t)\big)\,dx \tag{32}$$

$$= \int_{-\infty}^{+\infty}\frac{\partial f}{\partial t}p(x,t)dx + \int_{-\infty}^{+\infty} f(x,t)\frac{\partial p}{\partial t}dx. \tag{33}$$

Setting eq. (33) equal to eq. (31) yields

$$\int_{-\infty}^{+\infty} f(x,t)\frac{\partial p}{\partial t}dx = \int_{-\infty}^{+\infty}\frac{\partial f}{\partial x}A(x,t)p(x,t)dx + \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\partial^2 f}{\partial x^2}B(x,t)^2 p(x,t)dx \tag{34}$$

$$\overset{\text{(part. integr.)}}{=} -\int_{-\infty}^{+\infty} f(x,t)\frac{\partial}{\partial x}\big(A(x,t)p(x,t)\big)\,dx + \frac{1}{2}\int_{-\infty}^{+\infty} f(x,t)\frac{\partial^2}{\partial x^2}\big(B(x,t)^2 p(x,t)\big)\,dx \tag{35}$$

Since this equation has to hold for all functions $f(x,t)$, we can conclude

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\big(A(x,t)p(x,t)\big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(B(x,t)^2 p(x,t),\big) \tag{36}$$

which is the Fokker-Planck equation. Notice that this gives us a relationship between the SDE coefficients $A(x,t)$ and $B(x,t)$ and the first and second moments of the transition rates:

$$\boxed{a_t^{(1)}(x) = A(x,t)} \tag{37a}$$

$$\boxed{a_t^{(2)}(x) = B(x,t)^2} \tag{37b}$$

We can conclude that every SDE/Langevin equation is equivalent to a Fokker-Planck equation.

3 EXAMPLES

In this section we are going to quickly discuss two of the most important processes that can be investigated with the tools developed above: the Ornstein-Uhlenbeck process and Brownian motion.

3.1 Ornstein-Uhlenbeck process

Consider a small particle submerged in a viscous fluid. If we assume that we are in a regime where Stokes law of drag [9] applies, then we can describe the motion of the particle via Newton's equations of motion:

$$m\frac{dv}{dt} = -\lambda v. \tag{38}$$

However, if the particle is small enough, we will have to take collisions with the fluid particles into account. Since there is no way for us to deterministically predict them, we think of them as a random force acting on the particle. Let us further assume, that the bounces are independent of time and velocity. Than we can turn Newton's equation into a Langevin equation:

$$m\frac{dv}{dt} = -\lambda v + \sigma\frac{dW_t}{dt}, \tag{39}$$

where σ is a constant that characterizes the strength of the stochastic bounces. We can rewrite this equation in the usual SDE notation:

$$dv = -\frac{\lambda}{m}v\,dt + \frac{\sigma}{m}dW_t, \tag{40}$$

from which we gather

$$A(v,t) = -\frac{\lambda}{m}v \quad \text{and} \quad B(v,t) = \frac{\sigma}{m} \tag{41}$$

$$\implies a_t^{(1)}(v) = A(v,t) = -\frac{\lambda}{m}v \quad \text{and} \quad a_t^{(2)}(v) = B(v,t)^2 = \frac{\sigma^2}{m^2}. \tag{42}$$

Plugging this into the Fokker-Planck equation yields

$$\frac{\partial p(v,t)}{\partial t} = \frac{\lambda}{m}\frac{\partial}{\partial v}\big(v \cdot p(v,t)\big) + \frac{\sigma^2}{2m^2}\frac{\partial^2}{\partial v^2}\big(p(v,t)\big). \tag{43}$$

Depending on the initial conditions, the system is going to exhibit complicated dynamics and we will not be able to find an analytic solution in the general case. However, it may be interesting to investigate the special case of a stationary solution (if such a solution exists). To that end let us assume $\frac{\partial p(v,t)}{\partial t} = 0$. Notice that this leads to an ODE of second order:

$$0 = \frac{\lambda}{m}\frac{\partial}{\partial v}\big(v \cdot p(v)\big) + \frac{\sigma^2}{2m^2}\frac{\partial^2}{\partial v^2}\big(p(v)\big). \tag{44}$$

Integrating on both sides yields

$$C = \frac{\lambda}{m} v \cdot p(v) + \frac{\sigma^2}{2m^2}\frac{\partial p(v)}{\partial v}, \tag{45}$$

where C is an integration constant. Not every solution to eq. (45) is going to be a probability density. We will only consider positive and normalized solutions:

$$p(v) \geq 0 \quad \forall v \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{+\infty} p(v)\,dv = 1. \tag{46}$$

It turns out that this forces $C = 0$. Rearranging eq. (45) gives us

$$\frac{\partial p(v)}{\partial v} = -\frac{2m\lambda}{\sigma^2} v \cdot p(v). \tag{47}$$

The solution to this first order ODE is a Gaussian:

$$p(v) = \frac{1}{\sqrt{2\pi\alpha^2}} \exp\left(-\frac{v^2}{2\alpha^2}\right) \quad \text{with} \quad \alpha := \frac{\sigma}{\sqrt{2m\lambda}}. \tag{48}$$

Notice that the standard deviation α scales linearly with the strength of the collisions σ and has an inverse relationship with the particles mass m and the magnitude of Stokes drag λ.

Fig. 2 shows Monte-Carlo simulations of several Ornstein-Uhlenbeck processes. Our particle starts with different initial velocities but quickly transitions into a stationary state. For large t the magnitude of fluctuations is going to be given by $\alpha = \frac{\sigma}{\sqrt{2m\lambda}}$.

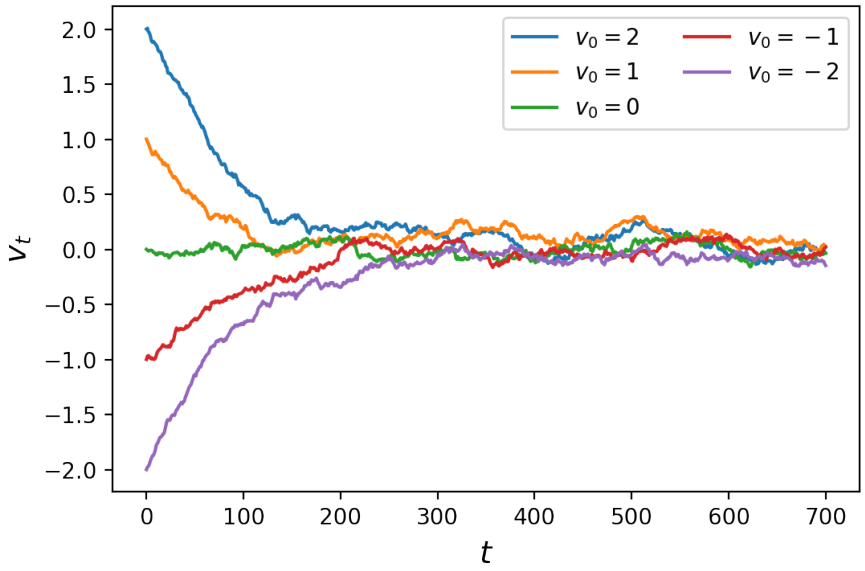

Figure 2: Monte-Carlo simulation of five Ornstein-Uhlenbeck processes with different initial conditions. [11]

3.2 Brownian motion

Suppose now that our process is dominated by the drag force term so that we can neglect the inertial term: $m\frac{dv}{dt} \approx 0$. We will then retrieve a SDE for the particle position:

$$\lambda \frac{dx}{dt} = \sigma \frac{dW_t}{dt} \tag{49}$$

$$\iff \quad dx = \frac{\sigma}{\lambda} dW_t \tag{50}$$

Notice that this implies, that our particles motion is a rescaled Wiener process. The coefficients take on a very simple form:

$$A(x,t) = 0 \quad \text{and} \quad B(x,t) = \frac{\sigma}{\lambda} \tag{51}$$

$$\implies a_t^{(1)}(x) = 0 \quad \text{and} \quad a_t^{(2)}(x) = \frac{\sigma^2}{\lambda^2}. \tag{52}$$

And the Fokker-Planck equation turns into the diffusion equation (sometimes referred to as heat equation):

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} \quad \text{with} \quad D := \frac{\sigma^2}{2\lambda^2} \tag{53}$$

It is a well-known result that the diffusion equation has an exact solution for an initially localized distribution (see eg. [2]). That is to say, assume $p_0(x) = \delta(x - x_0)$ as an initial condition, then

$$p(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{(x-x_o)^2}{4Dt}\right) \tag{54}$$

solves the Fokker-Planck equation of the above form (the so-called *fundamental solution*). Assuming that the drift d_x of the particle is proportional to the standard deviation of the (initially) localized solution, we get:

$$d_x(t) = \sqrt{2Dt} \tag{55}$$

It is this result that Einstein used for his empirical prediction of Brownian motion and that eventually lead to the confirmation of the existence of atoms [1].

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