

Topological Phase Transitions

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Seminar on Statistical Physics

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Summer term 2023

Abstract

The BKT transition is introduced by means of the two dimensional XY model whose correlation properties at low temperatures and at high temperatures are discussed. Vortices are introduced and their interaction energy is derived allowing for a qualitative argument supporting the idea of a vortex unbinding transition. Finally, a short introduction to the RG treatment of the model follows and the results are compared to experiments.

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1 Introduction

In 1966 Mermin and Wagner showed that there is no long range order (LRO) in $2d$ magnets with a continuous symmetry and short range interactions [17]. One year later Wegner applied the method of proof to exclude LRO in the two dimensional XY model [21]. Similar results were obtained for two dimensional crystals and Bose fluids [16, 9]. These findings are now a general result, known as the Mermin-Wagner(-Hohenberg) theorem in statistical physics: There is no phase transition due to the spontaneous breaking of a continuous symmetry for $T > 0$ in $d \leq 2$ dimensions. Back then the general conclusion was that there is no phase transition in this class of systems [13]. However, in the late 1960s both numerical and experimental findings shed doubt on this conclusion. Prominent examples include high temperature series expansions by Stanley [19] and measurements of the third sound velocity of superfluid Helium by Kagiwada [11]. The first step towards the solution of the apparent contradiction was done in 1971 when Berezinskii first recognizes the importance of vortex excitations in the XY model [3]. In 1973, indenpendently of Berezinskii, Kosterlitz and Thouless managed to fully describe a phase transition in the $2d$ XY model and related systems from a topologically ordered low temperature phase to a high temperature disordered phase [15].

2 The XY model

Consider a d -dimensional hyper cubic lattice of lattice spacing a . To each lattice site $i = 1, \dots, N$ a classical spin of unit length is attached that is confined to rotate in a plane

$$\mathbf{S}_i = (\cos \theta_i, \sin \theta_i) \quad (1)$$

where θ_i is the angle the i -th spin makes to an arbitrary axis. The XY model is defined by the Hamiltonian

$$H_{XY} = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \quad (2)$$

with coupling constant $J > 0$, that is we focus on the ferromagnetic case where all the spins are aligned in the zero temperature ground state ($T = 0$). The second equality (2) follows by inserting (1) and using a trigonometric identity. The partition function is given by

$$Z_{XY} = \prod_{i=1}^N \int_0^{2\pi} \frac{d\theta_i}{2\pi} e^{-\beta H_{XY}} \quad (3)$$

with $\beta = 1/T$. Note that the Hamiltonian (2) is invariant under $O(2)$ rotations whereas in the $T = 0$ ground state an arbitrary direction of the magnetization is singled out. In the following it is shown that the expected LRO due to spontaneous symmetry breaking for $T > 0$ is destroyed by long wave length fluctuations in $d \leq 2$ dimensions.

2.1 Low temperature approximation of the XY Hamiltonian

At low temperatures it is expected that slowly varying spin configurations dominate in the partition function (3). It is thus reasonable to expand the cosine in (2) up to second order. From here one can directly compute the spin-spin correlation function as done in [21]. The same result can be obtained by employing another approximation which consists in extending the integration range in the partition function (3) from $[0, 2\pi)$ to $(-\infty, \infty)$. The Hamiltonian and partition function of the XY model in the spin wave approximation [2] read

$$H' = \frac{J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2 \quad (4)$$

$$Z' = \prod_{i=1}^N \int_{-\infty}^{\infty} \frac{d\theta_i}{2\pi} e^{-\beta H'} \quad (5)$$

where to constant ground state energy has been dropped. The calculation of the correlation function from (4) and (5) can be found in [10]. As we are interested in the correlations on long length scales, however, it is sensible to take the continuum limit of (4) and (5), cf. [7, 5]. Sending $a \rightarrow 0$ and $N \rightarrow \infty$ in a finite volume $\Omega \sim L^d$ yields

$$H = \frac{Ja^{2-d}}{2} \int d^d r (\nabla \theta)^2 \quad (6)$$

$$Z = \int \mathcal{D}\theta \exp(-\beta H[\theta]) \quad (7)$$

where $\theta = \theta(\mathbf{r})$ is now a field defined on every point of the volume Ω and the integrals over the angular variables in (5) turned into a functional integral over all field configurations. Using (6) and (7) the magnetization $\langle S_x \rangle$ and the spin-spin correlation function $\langle \mathbf{S}(0) \cdot \mathbf{S}(\mathbf{r}) \rangle$ are easily estimated using the properties of Gaussian integration [7]. The result is that the magnetization vanishes in $d \leq 2$ dimensions whereas it stays finite for $d > 2$ in the thermodynamic limit $L \rightarrow \infty$ reproducing the statement of the Mermin-Wagner theorem. The asymptotic behaviour $|\mathbf{r}| \rightarrow \infty$ of the spin-spin correlation function amounts to exponential decay in $d = 1$ dimension and a constant but finite value in $d > 2$. In exactly $d = 2$ dimensions one finds an algebraic decay of correlations

$$\langle \mathbf{S}(0) \cdot \mathbf{S}(\mathbf{r}) \rangle \simeq \left(\frac{r}{L} \right)^{-\eta}, \quad \eta = \frac{T}{2\pi J} \quad (8)$$

Kosterlitz and Thouless identified this behaviour as a new type of order that they called topological order [15]. It is also known as quasi long range order (QLRO). This behaviour has to be compared with the high temperature properties of the XY model.

2.2 High temperature properties of the XY model

In order to estimate $\langle S_x \rangle$ and $\langle \mathbf{S}(0) \cdot \mathbf{S}(\mathbf{r}) \rangle$ at high temperatures the partition function (3) is expanded in powers of $K = J/T$ [1].

$$Z_{XY} = \prod_{i=1}^N \int_0^{2\pi} \frac{d\theta_i}{2\pi} \prod_{\langle ij \rangle} (1 + K \cos(\theta_i - \theta_j) + \mathcal{O}(K^2)) \quad (9)$$

Using the properties of integrals over periodic functions and keeping only the lowest power of K one finds that the magnetization vanishes and the correlation function decays exponentially [1].

In order to understand the cross over from a low temperature phase with QLRO to a high temperature disordered phase in two dimensions, topological excitations need to be taken into account. Because of the vanishing order parameter both at low and at high temperatures the transition cannot be explained by spontaneous symmetry breaking as in $d > 2$ dimensions. In one dimension correlations decay exponentially both at low and at high temperature and hence there is no possibility for a transition. Thus, from now on we implicitly assume $d = 2$.

3 Vortices

When assuming the angular variables to follow a Gaussian distribution in (5) their periodic nature is being neglected. This excludes so called vortex excitations examples of which are shown in fig. 1. When going around a closed contour encircling the center of a vortex in the upper line of fig. 1 the angle changes by an amount of 2π so that the spin (1) at a specific site is well defined. Similarly, the angle changes by -2π in the lower half of fig. 1 and one speaks of an anti-vortex. This condition is encapsulated in the following equation.

$$\oint_{\mathcal{C}} \frac{d\theta}{ds} ds = 2\pi k \quad (10)$$

where $k \in \mathbb{Z}$ is called the charge of the vortex and measures the number of turns the spin vector does along the curve $\mathcal{C}$. $k = 0$ either means that there is no vortex or that the curve $\mathcal{C}$ does not encircle the center of the vortex. For a given winding number k there exist infinitely many vortex configurations that can be smoothly transformed into one another. Such a map is called a homotopy. Underneath some examples of vortex configurations with winding number k .

$$\theta(\mathbf{r}) = k\phi + \theta_0 \quad (11)$$

$$\theta(\mathbf{r}) = k\phi + \sin \phi \quad (12)$$

$$\theta(\mathbf{r}) = k\phi + \tanh(r \cos \phi) \cos 7\phi \quad (13)$$

On the other hand, there exists no homotopy that transforms vortices of different charges into one another.

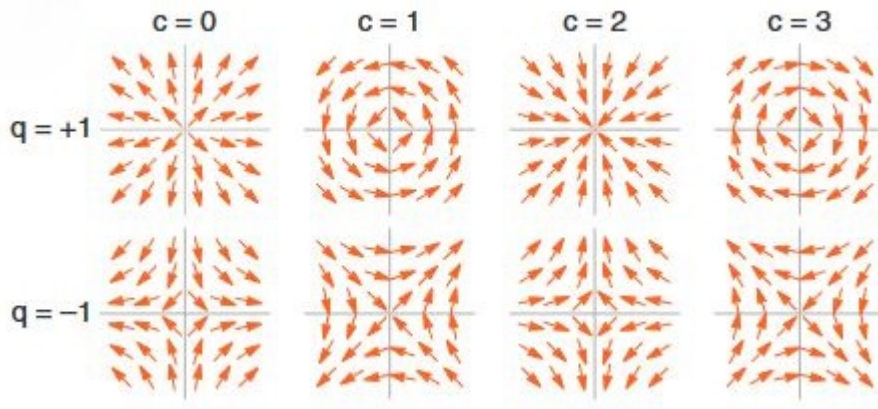


Figure 1: Examples of vortex configurations in the $2d$ XY model of the form $\theta(\mathbf{r}) = q\phi + c\frac{\pi}{2}$ for different values of q and c . ϕ is the polar angle for $\mathbf{r} = (r, \phi)$. The configurations with $c = +1$ are called vortices, the ones with $c = -1$ are called anti-vortices. Adapted from [12].

Vortices are energetically stable spin configurations in the sense that they are local minima of the energy functional (6) and thus obey [5]

$$\nabla^2\theta = 0 \quad (14)$$

They are topologically stable spin configurations as well meaning that there is no homotopy connecting them to the fully aligned ground state configuration. Every attempt to smoothly transform a vortex configuration into the ground state necessitates a energetically disfavoured intermediate state. More remarks on homotopies and the stability of vortices can be found in [5].

3.1 Energy of a single vortex

To estimate the energy of a single vortex we start from configuration (11) the gradient of which is given by

$$\nabla\theta = \frac{k}{r}\mathbf{e}_\phi \quad (15)$$

where $\mathbf{e}_\phi$ is the unit vector perpendicular to lines of constant polar angle in our polar coordinate system. The gradient (15) shows a divergence for $r = 0$ reflecting the fact that the continuum approximation of the original lattice model is not valid for distances smaller than the lattice spacing a . Hence the energy of a single vortex consists of two parts

$$E_v = E_{c,k} + E_{el} \quad (16)$$

the core contribution $E_{c,k}$ presumably depending on the charge k of the vortex and the elastic contribution E_{el} . The latter can be obtained by plugging (15)

into (6) where the integral runs from $a < r < L$ in polar coordinates. The lattice spacing a is used here as an estimate for the radius of the vortex core. The result is

$$E_v - E_{c,k} = \pi k^2 J \ln \left(\frac{L}{a} \right) \quad (17)$$

The energy of a vortex diverges logarithmically with the linear system size L . Thus, one might falsely come to the conclusion that vortices may be neglected in the thermodynamic limit. In the following it is shown that they cannot be neglected as the energy of vortex anti-vortex pairs stays finite even for $L \rightarrow \infty$.

3.2 Interaction energy of vortices

The aim of this section is to incorporate the above introduced vortex degrees of freedom into our low temperature approximation (6, 7). For notational convenience define the distortion field

$$\mathbf{u} = \nabla \theta \quad (18)$$

Assuming an arbitrary number of vortices of charges $\{k_i\}$ at positions $\{\mathbf{r}_i\}$ in our system we find

$$\oint_C \frac{d\theta}{ds} ds = \oint_C \mathbf{u} \cdot d\mathbf{s} = \int d^2 r \mathbf{e}_z \cdot \nabla \times \mathbf{u} = 2\pi \sum_i k_i \quad (19)$$

where Stokes theorem was applied in the second step. Eq. (19) leads to the identification

$$\nabla \times \mathbf{u} = 2\pi \mathbf{e}_z \sum_i k_i \delta^{(2)}(\mathbf{r}_i - \mathbf{r}) \quad (20)$$

Consider the ansatz

$$\mathbf{u} = \mathbf{u}_{sw} + \mathbf{u}_v \quad (21)$$

$$\mathbf{u}_{sw} = \nabla \phi \quad (22)$$

$$\mathbf{u}_v = -\nabla \times (\mathbf{e}_z \psi) \quad (23)$$

ψ is a smooth scalar field describing the spin wave degrees of freedom in our system. Consequently $\nabla \times \mathbf{u}_0 = 0$ and $\mathbf{u}_0$ is solution to the homogeneous equation of (20). Plugging (23) into (20) and using

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \quad (24)$$

for a vector field $\mathbf{v}$ yields

$$\nabla^2 \psi = 2\pi \sum_i k_i \delta^{(2)}(\mathbf{r}_i - \mathbf{r}) \quad (25)$$

Eq. (25) is the Poisson equation in two dimensions. The vortices in our system are thus described by the 2d Coulomb potential for charges $\{2\pi k_i\}$ at positions $\{\mathbf{r}_i\}$.

$$\psi(\mathbf{r}) = \sum_i k_i \ln \left| \frac{\mathbf{r}_i - \mathbf{r}}{a} \right| \quad (26)$$

Eqs. (21) to (23) together with (26) are inserted into the Hamiltonian (6) and one obtains

$$\beta H = \beta H_{sw} + \beta H_v \quad (27)$$

$$\beta H_{sw} = \frac{K}{2} \int d^2 r (\nabla \phi)^2 \quad (28)$$

$$\beta H_v = \sum_i \beta E_{c,k_i} - 2\pi K \sum_{i < j} k_i k_j \ln \left| \frac{\mathbf{r}_i - \mathbf{r}_j}{a} \right| \quad (29)$$

In the derivation of (27) the equation of motion (14) was used resulting in a decoupling of spin waves and vortices. Furthermore, the boundary term of a partial integration has been put to zero by imposing

$$\sum_i k_i = 0 \quad (30)$$

as otherwise the vortex interaction energy would diverge as $\ln L$. Finally, a UV divergence has been regularized by the core energies E_{c,k_i} of the vortices. In summary we find that the energy of the system decomposes into a Gaussian spin wave contribution and into the energy of a neutral 2d Coulomb gas which stays finite in the thermodynamic limit. Further details on the derivation of eqs. (27) to (29) can be found in [1] and [5].

As the core energy of the vortices scale like $\sim k_i^2$, at low temperatures only vortices of charge $k = \pm 1$ need to be considered. This way we arrive at the following low temperature approximation for the partition function of the XY model

$$Z = Z_{sw} Z_v \quad (31)$$

$$Z_{sw} = \int \mathcal{D}\phi e^{-\frac{K}{2} \int d^2 r (\nabla \phi)^2} \quad (32)$$

$$Z_v = \sum_{N=0}^{\infty} \frac{y_0^{2N}}{(N!)^2} \int \left(\prod_{i=1}^{2N} \frac{d^2 r_i}{a^2} \right) \exp \left(2\pi K \sum_{i < j} k_i k_j \ln(|\mathbf{r}_i - \mathbf{r}_j|) \right) \quad (33)$$

The factor $1/(N!)^2$ in (33) accounts for the indistinguishability of the N vortices and anti-vortices respectively. Furthermore the fugacity has been defined

$$y_0 = e^{-\frac{E_c}{T}} \quad , \quad E_c = E_{c,1} = E_{c,-1} \quad (34)$$

and the integrals over r_i fulfill the constraint $|r_i - r_j| > a$ for $i, j = 1, \dots, 2N$.

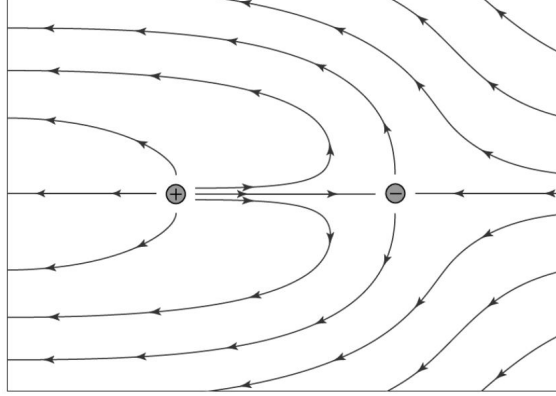


Figure 2: Spin configuration of a vortex anti-vortex pair with charges $k = \pm 1$. The distortions $\nabla\theta$ caused by the individual vortices cancel each other at large distances from the pair resulting in a finite energy. Graphic from [1].

3.3 Energy of a vortex anti-vortex pair

The fact that the energy of neutral vortex configurations (29) stays finite in the limit $L \rightarrow \infty$ as opposed to the energy of a single vortex (16) can be understood intuitively. Consider to this end a single vortex anti-vortex pair with charges $k = \pm 1$ at positions $\mathbf{r}_+$ and $\mathbf{r}_-$ as shown in fig. 2. The energy of this configuration easily follows from (29)

$$V(\mathbf{r}_+ - \mathbf{r}_-) = 2E_c + 2\pi J \ln(|\mathbf{r}_+ - \mathbf{r}_-|/a) \quad (35)$$

From eq. (15) we get the distortion from the vortex and antivortex respectively

$$\nabla\theta_{\pm}(\mathbf{r}) = \frac{\pm 1}{|\mathbf{r} - \mathbf{r}_{\pm}|} \mathbf{e}_{\phi} \quad (36)$$

Adding the two contributions and taylor expanding yields

$$\nabla\theta = \nabla\theta_+ + \nabla\theta_- \sim \frac{d}{r^2} \quad (37)$$

with $d = |\mathbf{r}_+ - \mathbf{r}_-|$. The distortions caused by the individual vortices cancel each other at large distances from the pair resulting in a finite energy. This behaviour can be seen in fig. 2.

Taking into account that the spin configuration depicted in fig. 2 is topologically equivalent to the ground state configuration we conclude that vortex anti-vortex pairs are important low temperature excitations of the XY model that cannot be neglected [5].

4 Heuristic argument for a vortex unbinding transition

We are now in a position to understand Kosterlitz and Thouless' qualitative argument in favour of a vortex unbinding transition in the two dimensional XY model [15]. The probability to find the system in a state with a single vortex excitation is proportional to $e^{-\beta F}$ with the free energy

$$F = E_v - TS_v \quad (38)$$

The energy of a single vortex E_v is given by (16). The entropy for having a single vortex in the system is estimated as

$$S_v = \ln \left(\frac{L^2}{a^2} \right) \quad (39)$$

since there are approximately L^2/a^2 ways to distribute a vortex of size a^2 on a square lattice of size L^2 . Inserting (16) with $k^2 = 1$ and (39) into (38) we obtain

$$F = E_c + (\pi J - 2T) \ln \left(\frac{L}{a} \right) \quad (40)$$

Considering the thermodynamic limit $L \rightarrow \infty$ we conclude from (40) that a single vortex in the system is favoured for $T > \pi J/2$ whereas it is disfavoured below some transition temperature $T < \pi J/2$. Combining this insight with the results obtained in sections 2 and 3 a transition from a low temperature phase showing tightly bound vortex anti-vortex pairs and QLRO to a high temperature disordered phase containing free vortices is expected at a finite temperature. In the Coulomb gas analogy this corresponds to a transition from an insulating low temperature phase to a conducting high temperature phase.

Note that the logarithmic dependence of the vortex energy (16) on the linear system size is a crucial ingredient for the above argument. A divergence faster than logarithmic would imply in the thermodynamic limit that vortices are disfavoured at all temperature whereas a slower divergence favours vortices at all temperatures.

In order to improve the prediction of the transition temperature, vortex fluctuations need to be taken into account. As vortices disorder the phase field we expect them to reduce the coupling constant which describes the energetic cost of neighbouring spins pointing in different directions. Consequently, we can already guess that vortex fluctuations lower the transition temperature.

5 Renormalization group treatment

5.1 Coupling renormalization

There are different approaches to integrating out the vortex degrees of freedom described by the partition function (33). The approach taken in ref. [10] consists in directly calculating the spin-spin correlation function in the presence of vortices modifying the result (8) to

$$\langle \mathbf{S}(0) \cdot \mathbf{S}(\mathbf{r}) \rangle \simeq \left(\frac{r}{L} \right)^{-\eta}, \quad \eta = \frac{T}{2\pi J_R} \quad (41)$$

J_R is the effective coupling constant, renormalized by vortex fluctuations.

Another approach is to calculate the difference in free energy of a system with a non vanishing average phase gradient $\mathbf{v} = \Omega^{-1} \int d^2r \langle \nabla \theta \rangle$ and a system with vanishing average gradient in the presence of vortices.

$$F(\mathbf{v}) - F(0) = \frac{1}{2} \Omega J_R v^2 \quad (42)$$

Details on this derivation can be found in ref. [5].

Finally it is also possible to obtain the renormalized coupling by looking at the interaction between a vortex and an anti-vortex at positions $\mathbf{r}_+$ and $\mathbf{r}_-$ respectively

$$e^{-\beta V_{eff}(\mathbf{r}_+ - \mathbf{r}_-)} = \langle e^{-\beta V(\mathbf{r}_+ - \mathbf{r}_-)} \rangle_v \quad (43)$$

In (43) $V(\mathbf{r}_+ - \mathbf{r}_-)$ is the interaction between two vortices (35) modulus two times the core energy and the average is taken with respect to (33). The calculation can be found in ref. [1].

In all three approaches the effective coupling is obtained as a power series in the fugacity $y_0 = e^{-E_c/T}$ which represents a measure for the number of vortices in the system. At low temperatures the fugacity is small and only the lowest order of y_0 needs to be taken into account in the partition function (33). The result is

$$K_R^{-1} = K_0^{-1} + 4\pi^3 y_0^2 \int_a^\infty \frac{dr}{a} \left(\frac{r}{a} \right)^{3-2\pi K_0} + \mathcal{O}(y_0^4) \quad (44)$$

$K_0 \equiv J/T$ is the reduced coupling of the original lattice model. As expected the coupling is lowered by the vortex excitations which decrease the energetic cost of spin waves (42) and screen the interaction of vortices separated by larger distances (43) in analogy to the case of an electric field applied to a polarizable medium. For $K_0 < 2/\pi$ the integral in (44) diverges marking the breakdown of the theory. In order to extract more information out of eq. (44) a renormalization procedure first introduced by José et al. [10] needs to be applied.

5.2 A renormalization group procedure

The idea is to split the integral in eq. (44) into two parts

$$\int_a^\infty \rightarrow \int_a^{ae^l} + \int_{ae^l}^\infty \quad (45)$$

The contribution from the first integral is used to define a new coupling at length scale l

$$K^{-1} = K_0^{-1} + 4\pi^3 y_0^2 \int_a^{ae^l} \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K_0} \quad (46)$$

In the second term of (45) the integration variable is rescaled $r \rightarrow re^{-l}$ and a new fugacity on length scale l is defined

$$y = y_0 e^{l(2-\pi K_0)} \quad (47)$$

With (46) and (47) the inverse renormalized coupling takes the following form

$$K_R^{-1} = K^{-1} + 4\pi^3 y^2 \int_a^\infty \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K} + \mathcal{O}(y^4) \quad (48)$$

The coupling K_0 in the exponent of the integrand of eq. (44) may be replaced by K as we are only working to order y^2 [6]. Comparing (44) with (48) we find that the functional dependence of the renormalized coupling K_R on the couplings K and y is the same on all length scales.

$$K_R[K_0, y_0] = K_R[K(l), y(l)] \quad (49)$$

Performing an infinitesimal RG step $l \rightarrow l + \delta l$ one obtains the Kosterlitz recursion relations describing the flow of the effective parameters as $l \rightarrow \infty$.

$$\frac{dK^{-1}}{dl} = 4\pi^3 y^2 + \mathcal{O}(y^4) \quad (50)$$

$$\frac{dy}{dl} = (2 - \pi K)y + \mathcal{O}(y^3) \quad (51)$$

The flow equations (50) and (51) were originally derived by Kosterlitz in 1974 [14] by directly integrating out the vortex fluctuations on small distances from the partition function (33) which is much more complicated than the approach schemed above [13]. The flow equations may also be derived by mapping the two dimensional Coulomb gas model to the Sine-Gordon model and applying the usual RG methods in momentum space [20].

5.3 Interpretation of the flow equations

Increasing the parameter a to ae^l amounts to increasing the distance between vortices by integrating out the vortex pairs separated by smaller distances than

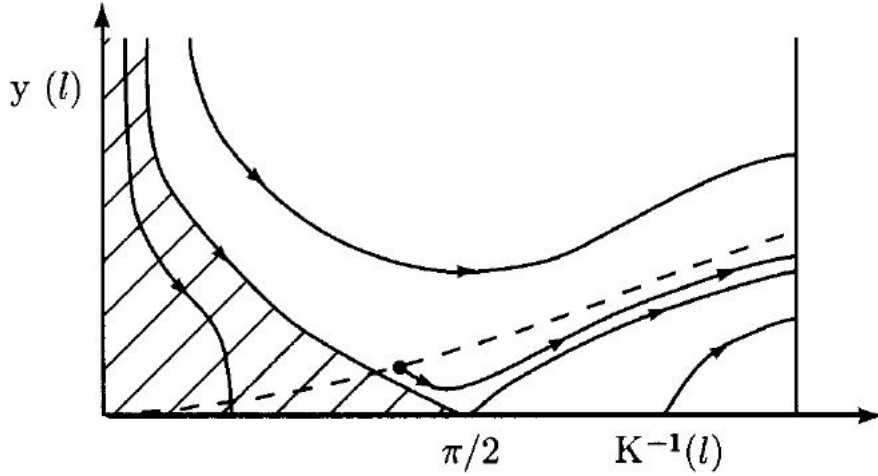


Figure 3: Phase diagram from the Kosterlitz recursion relations. Adapted from [5]

ae^l . The latter reduce the effective coupling on length scale l as explained above. That is why K^{-1} is always increasing under the RG flow (50) which is depicted in fig. 3. Next we observe that $y = 0$ is a fixed line that is stable for $\pi K > 2$ and unstable for $\pi K < 2$. In the first case the fugacity decreases as $l \rightarrow \infty$ and the flow terminates on the fixed line at

$$K(l) = K_R \geq 2/\pi \quad (52)$$

which we infer from eq. (48). Since the fugacity measures the presence of vortices in the system this means that they show no effect on large length scales. As discussed this is the case for bound vortex anti-vortex pairs, cf. fig. 2. Therefore this regime is interpreted as a low temperature phase of tightly bound vortex anti-vortex pairs which corresponds to the shaded area in fig. 3. The vanishing of the fugacity on longest length scales justifies the calculation of the spin-spin correlation function for far apart spins in the approximation of section 2.1 that completely neglects vortices. Indeed the only effect of vortices is an adjustment of the exponent η (41).

For $\pi K < 2$ both couplings flow to infinity and the theory breaks down as we expanded in small y . The increase in the fugacity is interpreted as a proliferation of vortices and thus this part of the phase diagram 3 corresponds to a high temperature phase of free vortices that destroy the algebraic correlations present below the transition temperature.

The critical line that separates the two regimes ends at the critical point ($K_R^{-1} = \pi/2, y = 0$). The dashed line in fig. 3 represents initial conditions of the flow as

a function of temperature. The intercept of this line with the separatrix marks the critical temperature which is identified with the largest temperature such that the flow ends on the fixed line $y = 0$. From eq. (51) it follows that

$$\lim_{l \rightarrow \infty} (2 - \pi K(l)) = 0 \quad (53)$$

and therefore

$$K_R^{-1} = \frac{\pi}{2} \quad (54)$$

$$J_R(T_c^-) = \frac{2}{\pi} T_c \quad (55)$$

This is an important prediction of the Kosterlitz-Thouless theory known as the universal jump of the superfluid density.

In order to further investigate the critical behaviour of the XY model, eq. (50) may be linearized about the fixed point $(K_R^{-1} = \pi/2, y = 0)$, a detailed description of which may be found e.g. in ref. [13] and [1].

6 Experimental confirmation of the BKT theory

6.1 Universal jump in the superfluid density

The order parameter of a two dimensional film of superfluid helium may be expressed as

$$\psi = \sqrt{\rho_s^0} e^{i\theta} \quad (56)$$

with the superfluid density ρ_s and the phase of the condensate wave function θ which is related to the velocity of the superfluid by

$$\mathbf{v}_s = \frac{1}{m} \nabla \theta \quad (57)$$

In the low temperature phase the important fluctuations are only in the phase θ of the superfluid so that the order parameter is described by a single angular variable as in the case of the XY model. Hence the two models are members of the same universality class and their critical behaviour may be described by the same effective theory [18]. All there is left to do is to replace the coupling constant in the XY model (6) by

$$J \rightarrow \frac{\rho_s^0}{m^2} \quad (58)$$

Relation (55) then translates to

$$\rho_s(T_c^-) = \frac{2}{\pi} m^2 T_c \quad (59)$$

which has been confirmed in the experiment of ref. [4] the result of which is shown in fig. 4.

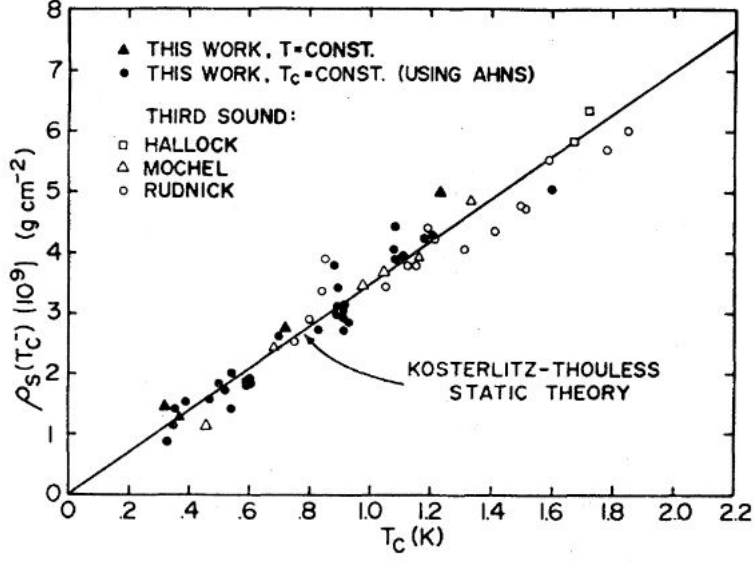


Figure 4: Density of superfluid ^4He at the transition as a function of the transition temperature. The solid line corresponds to the predicted universal jump of the Kosterlitz-Thouless theory. Figure from [4].

6.2 Uncovering the microscopic mechanism of the BKT transition in a $2d$ Bose gas

In 2006 Hadzibabic et al. [8] managed to quantitatively relate the proliferation of vortices in a quasi two dimensional Bose gas with the loss of QLRO. The experiment starts with a three dimensional Bose Einstein condensate that is split into two clouds of $\sim 10^5$ atoms each which are compressed into the $2d$ regime using a one dimensional optical lattice. The experimental setup is depicted in fig. 5. The two clouds are allowed to equilibrate for 200 ms before all trapping potentials are turned off and they expand for 20 ms primarily in the z direction adapting the coordinate system of fig. 5a. The resulting matter wave interference pattern is recorded onto a CCD camera using the absorption of a resonant probe laser. Fig. 5c shows an interference pattern for low temperatures and fig. 5d for high temperatures. The waviness of the interference fringes at high temperature is attributed to the increasing presence of phase fluctuations in the $2d$ clouds. At high temperatures Hadzibabic et al. occasionally observed sharp dislocations in the interference pattern as shown in fig. 6. The group also managed to extract information on the phase correlations from the interference patterns and studied the temperature dependence of the system. They observed that the onset of the presence of free vortices coincides with the loss of QLRO thus providing conclusive evidence for a BKT transition in the $2d$ Bose fluid.

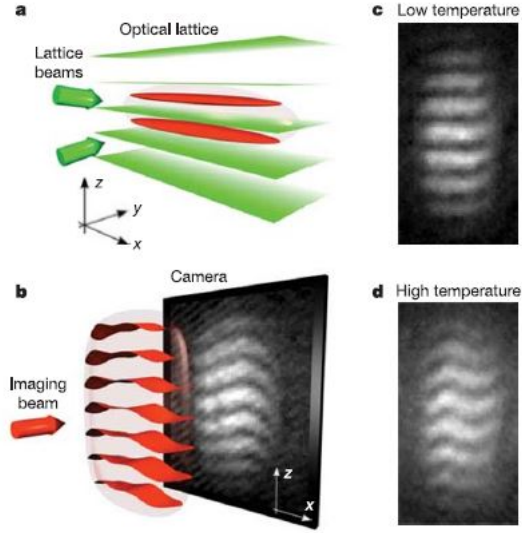


Figure 5: **a**, A $3d$ gas is split into two quasi $2d$ clouds. **b**, All trapping potentials are switched off, the clouds expand primarily in the z direction and the matter wave interference pattern is recorded onto a CCD camera by means of the absorption of a resonant probe laser. **c**, **d** Examples of interference pattern at low and high temperature. Figure from [8].

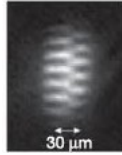


Figure 6: Example of an interference pattern with an isolated sharp dislocation that is explained by the presence of a free vortex in one of the overlapping clouds. Adapted from [8].

References

- [1] A. Altland and B. D. Simons. *Condensed Matter Field Theory*. Cambridge University Press, 2010.
- [2] M. N. Barber. “Phase transitions in two dimensions”. In: *Physics Reports* 59.4 (1980), pp. 375–409. ISSN: 0370-1573. DOI: [https://doi.org/10.1016/0370-1573\(80\)90026-5](https://doi.org/10.1016/0370-1573(80)90026-5). URL: <https://www.sciencedirect.com/science/article/pii/0370157380900265>.
- [3] V. L. Berezinsky. “Destruction of long range order in one-dimensional and two-dimensional systems having a continuous symmetry group. I. Classical systems”. In: *Sov. Phys. JETP* 32 (1971), pp. 493–500.
- [4] D. J. Bishop and J. D. Reppy. “Study of the superfluid transition in two-dimensional ^4He films”. In: *Phys. Rev. B* 22 (11 Dec. 1980), pp. 5171–5185. DOI: 10.1103/PhysRevB.22.5171. URL: <https://link.aps.org/doi/10.1103/PhysRevB.22.5171>.
- [5] P. M. Chaikin and T. C. Lubensky. *Principles of Condensed Matter Physics*. Cambridge: Cambridge University Press, 1995.
- [6] C. Domb. *Phase transitions and critical phenomena*. Vol. 7. Elsevier, 2000.
- [7] J. Fröhlich and T. Spencer. “The Kosterlitz-Thouless transition”. In: Jan. 1970, pp. 15–20. ISBN: 978-3-540-11192-4. DOI: 10.1007/3-540-11192-1_5.
- [8] Z. Hadzibabic et al. “Berezinskii–Kosterlitz–Thouless crossover in a trapped atomic gas”. In: *Nature* 441 (2006), pp. 1118–1121. DOI: <https://doi.org/10.1038/nature04851>.
- [9] P. C. Hohenberg. “Existence of Long-Range Order in One and Two Dimensions”. In: *Phys. Rev.* 158 (2 June 1967), pp. 383–386. DOI: 10.1103/PhysRev.158.383. URL: <https://link.aps.org/doi/10.1103/PhysRev.158.383>.
- [10] J. V. José et al. “Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model”. In: *Phys. Rev. B* 16 (3 Aug. 1977), pp. 1217–1241. DOI: 10.1103/PhysRevB.16.1217. URL: <https://link.aps.org/doi/10.1103/PhysRevB.16.1217>.
- [11] R. S. Kagiwada et al. “Superflow in Helium Films: Third-Sound Measurements”. In: *Phys. Rev. Lett.* 22 (8 Feb. 1969), pp. 338–342. DOI: 10.1103/PhysRevLett.22.338. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.22.338>.
- [12] J.-V. Kim. “Role of nonlinear anisotropic damping in the magnetization dynamics of topological solitons”. In: *Physical Review B* 92 (July 2015), p. 014418. DOI: 10.1103/PhysRevB.92.014418.

- [13] J. M. Kosterlitz. “Kosterlitz–Thouless physics: a review of key issues”. In: *Reports on Progress in Physics* 79.2 (Jan. 2016), p. 026001. DOI: 10.1088/0034-4885/79/2/026001. URL: <https://dx.doi.org/10.1088/0034-4885/79/2/026001>.
- [14] J. M. Kosterlitz. “The critical properties of the two-dimensional xy model”. In: *Journal of Physics C: Solid State Physics* 7.6 (Mar. 1974), p. 1046. DOI: 10.1088/0022-3719/7/6/005. URL: <https://dx.doi.org/10.1088/0022-3719/7/6/005>.
- [15] J. M. Kosterlitz and D. J. Thouless. “Ordering, metastability and phase transitions in two-dimensional systems”. In: *Journal of Physics C: Solid State Physics* 6.7 (Apr. 1973), p. 1181. DOI: 10.1088/0022-3719/6/7/010. URL: <https://dx.doi.org/10.1088/0022-3719/6/7/010>.
- [16] N. D. Mermin. “Crystalline Order in Two Dimensions”. In: *Phys. Rev.* 176 (1 Dec. 1968), pp. 250–254. DOI: 10.1103/PhysRev.176.250. URL: <https://link.aps.org/doi/10.1103/PhysRev.176.250>.
- [17] N. D. Mermin and H. Wagner. “Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models”. In: *Phys. Rev. Lett.* 17 (22 Nov. 1966), pp. 1133–1136. DOI: 10.1103/PhysRevLett.17.1133. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.17.1133>.
- [18] Advanced information. NobelPrize.org. Nobel Prize Outreach. *Scientific background: Topological phase transitions and topological phases of matter*. 2023. URL: <https://www.nobelprize.org/prizes/physics/2016/advanced-information/> (visited on 04/05/2023).
- [19] H. E. Stanley. “Dependence of Critical Properties on Dimensionality of Spins”. In: *Phys. Rev. Lett.* 20 (12 Mar. 1968), pp. 589–592. DOI: 10.1103/PhysRevLett.20.589. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.20.589>.
- [20] D. Tong. *Statistical Field Theory*. <https://www.damtp.cam.ac.uk/user/tong/sft.html>. Online; accessed 29 January 2014. 2017.
- [21] F. Wegner. “Spin-ordering in a planar classical Heisenberg model”. In: *Z. Physik* 206 (1967), pp. 465–470. DOI: <https://doi.org/10.1007/BF01325702>.