One- and Two-Dimensional Ising Model

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Abstract

Statistical Physics is a field of physics which faces all the various types of phenomena with a stochastic nature. Under this concept, a fundamental role is taken by the Ising model. It is one the simplest models to describe large interacting systems that reveals the appearance of phase transitions. This report provides a short summary of the talk *One- and Two-Dimensional Ising Model* given in the Statistical Physics seminar by Prof. Georg Wolschin in the summer term 2023. Using a transfer matrix approach, it will be shown that in one dimension no phase transition can occur. In two dimensions, however, the existence of a phase transition can be derived by a simple geometrical argument. The outline will be an application of the Ising model for nowadays challenges.

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1 Introduction

1.1 The Model

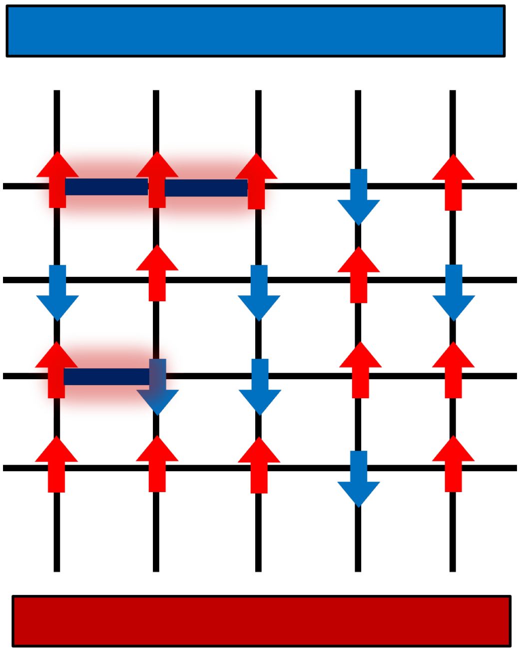


Figure 1: The Ising model with interacting spins, red (up) and blue (down), in an external magnetic field illustrated by the boxes

In the Ising model (figure 1), we consider a d-dimensional lattice with a total of N lattice sites. At each lattice site, a spin variable S_i is placed with either the value $S_i = +1$ (up) or $S_i = -1$ (down). The spins can interact with their nearest neighbours, and the interaction strength can be described by a coupling constant J. A positive coupling means that neighbouring spins tend to align in the same direction (ferromagnetic case), while a negative coupling favours an opposite alignment of spins (antiferromagnetic case). Furthermore, the whole lattice shall be under the influence of an external magnetic field B, which forces the spins to take a preferred direction. And, we will consider the system in the regime of the canonical ensemble such that the Ising lattice can occupy different states depending on the temperature T.

The Ising model was firstly developed to answer the question of spontaneous magnetization by ferromagnetic elements. This happens if the system goes from a disordered state to an ordered state at finite temperature (in the thermodynamic limit $N \to \infty$). To find out if the Ising model undergoes such phase transition, we can describe the dynamics with following Hamilton function:

$$H = -J \sum_{<ij>} S_i S_j - B\mu \sum_i S_i \tag{1}$$

where $< ij >$ are all nearest neighbours and μ is the magnetic moments of the spins. The first term in (1) is the interaction energy and the second one is the energy arising from the action by the external field. With this, we can analyze the properties of the Ising model in $d = 1$ and $d = 2$ dimensions.

1.2 Historical Background

The origin of the Ising model goes back to the early 20th century. At this time, Pierre-Ernest Weiss discovered the existence of magnetic domains in ferromagnetic elements and tried to give a theory for ferromagnetism [2, 3]. However, Weiss was not able to explain why certain elements show the phenomenon of spontaneous magnetization.

To address this problem, Wilhem Lenz, a German professor at Hamburg University, invented an own idea for ferromagnetism [4], the Ising model, and gave this model to his doctorate student Ernst Ising, who published a solution for it in 1925 [5]. Ising solved the model in one dimension and showed there exists no phase transition at finite temperature. He, however, wrongfully adapted this assumption also to the case of two and three dimensions.

Figure 2: Ernst Ising [1]

Due to this misleading result, Werner Heisenberg developed his own theory for describing ferromagnetism in 1928 [6] in which he generalized the Ising problem with a more complex spin-spin interaction. Scientists were trying to explore this model first when they went back to investigate the simpler Ising model.

After eight years, the physicist Rudolf Peierls was able to justify the existence of a phase transition in the two dimensional model without an external field using a geometrical argument [7]. Even though the original approach by Peierls contained an incorrect step as it has been discovered later by Robert Griffiths [8], the idea still had a valid importance.

In 1941, a first quantitative number for the critical temperature has been found out by Hans Kramers and Gregory Wannier using a duality argument that is based on the symmetry of the high and low temperature expansion of the partition function [9, 10]. Kramers and Wannier also gave the motivation for the so called *transfer matrix* method which laid the foundations for solving the 2D model exactly by Lars Onsager in 1944 [11].

Since then, no analytical solution for the 2D model with external magnetic field or the 3D Model has been found, yet. Nonetheless, a lot of information can be gained by numerical simulations, like Monte Carlo simulations [12]. Moreover, the Ising model is involved in a broad range of concepts in physics, e.g. mean-field theory or renormalization-group theory (to learn more about such theories in statistical physics, consider [13], [14]). In this sense, the Ising model has a great importance.

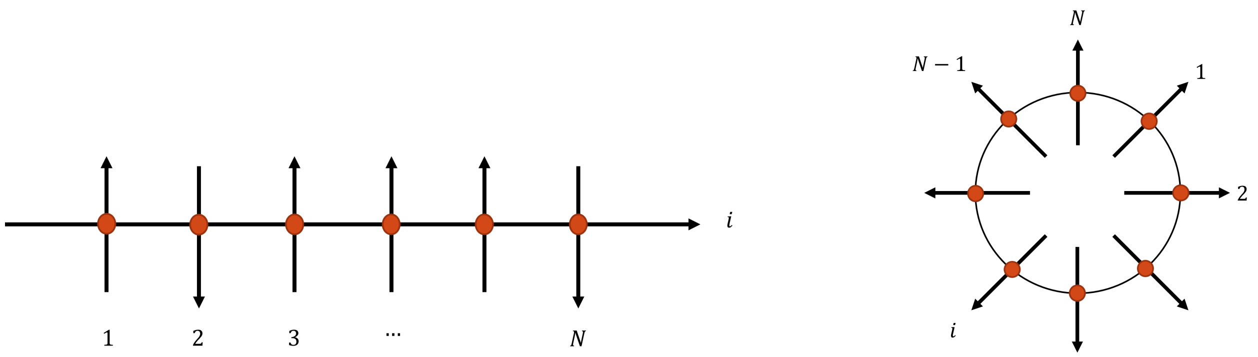
Further historical facts about the Ising model and the life of Ernst Ising can be found in [1] and [15].

We will now go on investigating the 1D and 2D problem.

2 1D Ising Model

We will solve the Ising model with the transfer matrix approach and follow the proof explained as in [16] or [17].

2.1 Transfer Matrix Method



(a) Linear Chain **(b)** Ring

Figure 3: The Ising model in one dimension

In the 1D case, the Ising lattice is a linear chain (figure 3a). If we apply periodic boundary conditions to this chain (3b), the Hamiltonian (1) reduces to:

$$H = -J\sum_{i=1}^{N} S_i S_{i+1} - B\mu \sum_{i=1}^{N} S_i. \tag{2}$$

In the canonical ensemble, we are interested in the partition function

$$Z = \sum_{\{S_i\}} e^{-\beta H} \tag{3}$$

where $\{S_i\}$ denotes the sum over all possible configurations and $\beta = \frac{1}{k_B T}$ is the inverse temperature with Boltzmann factor k_B. The partition function contains all information about the relevant thermodynamic quantities. To separate between order and disorder, we further need an order parameter. In our case, the natural choice is the magnetization:

$$M = \frac{1}{Z}\sum_{i=1}^{N} \mu S_i e^{-\beta H}. \tag{4}$$

This can be rewritten as :

$$M = \frac{\partial}{\partial(\beta B)} \ln Z. \tag{5}$$

Our goal is now to find an expression for Z such that we can derive the magnetization from it and determine if the system shows a phase transition.

We start by rewriting the Hamiltonian in a slightly different way:

$$H = \sum_{i=1}^{N} \left(-JS_i S_{i+1} - \frac{B\mu}{2}(S_i + S_{i+1}) \right). \tag{6}$$

With this, we have extracted the summation to the left and split the sum in the magnetic field term, which will turn out to be useful later.
Plugging this into the partition function (3) yields:

$$Z = \sum_{\{S_i\}} \exp \left[-\beta \sum_{i=1}^{N} \left(-JS_i S_{i+1} - \frac{B\mu}{2} (S_i + S_{i+1}) \right) \right]. \tag{7}$$

By writing the sum in the exponential as a product, we get:

$$Z = \sum_{\{S_i\}} \prod_{i=1}^{N} \exp \left[\beta \left(JS_i S_{i+1} + \frac{B\mu}{2} (S_i + S_{i+1}) \right) \right] \tag{8}$$

$$= \sum_{\{S_i\}} \prod_{i=1}^{N} T_{i,i+1} \tag{9}$$

where we have defined the so called *transfer function*

$$T_{i,i+1} := \exp \left[\beta \left(JS_i S_{i+1} + \frac{B\mu}{2} (S_i + S_{i+1}) \right) \right]. \tag{10}$$

The transfer function $T_{i,i+1}$ has the special property that it only depends on the spin values of S_i and S_{i+1}. We can now write out the possible outcomes for the transfer function:

$$T_{i,i+1} = \begin{cases} e^{\beta J + \mu B} & S_i = +1, S_{i+1} = +1 \\ e^{-\beta J} & S_i = +1, S_{i+1} = -1 \\ e^{-\beta J} & S_i = -1, S_{i+1} = +1 \\ e^{\beta J - \mu B} & S_i = -1, S_{i+1} = -1 \end{cases}. \tag{11}$$

Using the quantum mechanical notation for spin variables, and identifying each spin state as a vector:

$$|S_i = +1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{12}$$

$$|S_i = -1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{13}$$

the transfer function can be expressed in a more compact way, namely:

$$T_{i,i+1} = \langle S_i | T | S_{i+1} \rangle \tag{14}$$

with the definition of the *transfer matrix* T:

$$T = \begin{pmatrix} e^{\beta(J+\mu B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu B)} \end{pmatrix}. \tag{15}$$

Putting this back into the partition function (9), we gain;

$$Z = \sum_{\{S_i\}} \prod_{i=1}^{N} \langle S_i|T|S_{i+1}\rangle = \sum_{S_1=\pm1} \dots \sum_{S_N=\pm1} \langle S_1|T|S_2\rangle \langle S_2|T|S_3\rangle \dots \langle S_N|T|S_1\rangle. \tag{16}$$

where we have explicitly written out the sum over all configurations.

The crucial point is now that in (16) between each transfer matrix there is a product $|S_i\rangle\langle S_i|$ which, when performing the sum $\sum_{S_i=\pm1}$, yields an identity matrix. Thus, each spin sum except the sum over the first spin will vanish and we get the simplified form:

$$Z = \sum_{S_1=\pm1} \left\langle S_1 \middle| T^N \middle| S_1 \right\rangle = tr(T^N). \tag{17}$$

If the Eigenvalues λ_1, λ_2 of T exist, the partition function gets the form:

$$Z = \lambda_1^N + \lambda_2^N. \tag{18}$$

So, we are left with solving an Eigenvalue problem for T:

$$\det T - \lambda \mathbb{1} = \begin{vmatrix} e^{\beta(J+\mu B)} - \lambda & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu B)} - \lambda \end{vmatrix} \stackrel{!}{=} 0. \tag{19}$$

From that we obtain:

$$\lambda_{1,2} = e^{\beta J} \left[\cosh \beta\mu B \pm \sqrt{\cosh^2 \beta\mu B - 2e^{-2\beta J} \sinh 2\beta J} \right]. \tag{20}$$

In the thermodynamic limit, only the larger eigenvalue λ_1 becomes relevant:

$$\lim_{N\to\infty} Z = \lim_{N\to\infty} \lambda_1^N \left(1 + \frac{\lambda_2^N}{\lambda_1^N} \right) \approx \lambda_1^N. \tag{21}$$

With this, we get a final expression for the partition function:

$$Z = \left(e^{\beta J} \left[\cosh \beta\mu B + \sqrt{\cosh^2 \beta\mu B - 2e^{-2\beta J} \sinh 2\beta J} \right] \right)^N. \tag{22}$$

Using (5) we derive the magnetization:

$$M(T,B) = \frac{N\mu \sinh \beta\mu B}{\sqrt{\cosh^2 \beta\mu B - 2e^{-2\beta J} \sinh 2\beta J}}. \tag{23}$$

In figure 23 a plot of the magnetization for different temperatures is given. The curve of the magnetization has a S-shape. For large values B, the magnetization will either go to the limit of μN or μN. As we can see, if there is no magnetic field, $B = 0$, the magnetization is always 0 for finite temperature. If the temperature gets less, the S-shape of the curve will get steeper. At the absolute temperature of $T = 0$, the magnetization will turn into a step function. At this point, $M \neq 0$ and has one of the discrete values $\pm\mu N$. So, only at this point, the symmetry of our system gets broken and a phase transition occurs. Otherwise a phase transition is not existent at finite temperature.

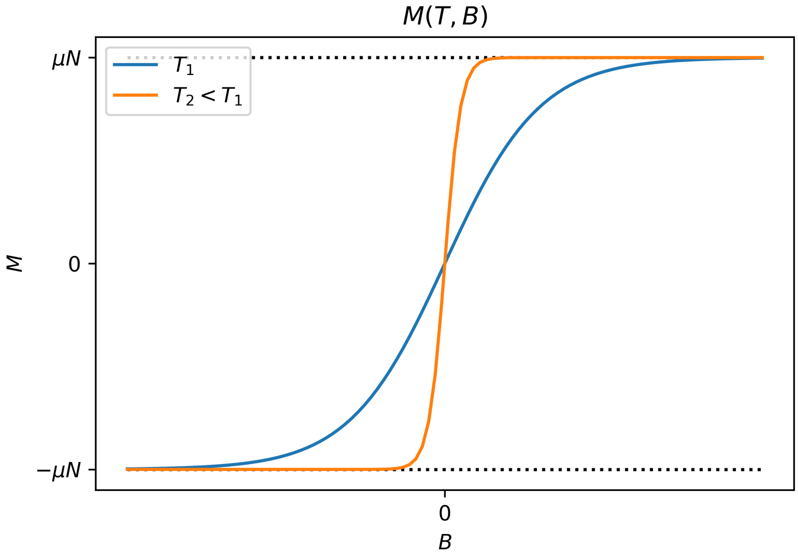
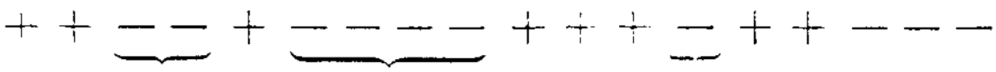


Figure 4: Magnetization of the Ising chain

2.2 Ising's Original Approach

The fact that there exist no phase transition in one dimension was already discovered by Ising. He used a different approach than the transfer matrix method to calculate the partition function. We will give a short overview of his solution.



(a) Ising chain

$$Z = \sum \left[\binom{\nu_1 - 1}{s} \binom{\nu_2 - 1}{s + \delta - 1} + \binom{\nu_2 - 1}{s} \binom{\nu_1 - 1}{s + \delta - 1} \right] e^{-(2s + \delta)\frac{\varepsilon}{kT} + (\nu_1 - \nu_2) \cdot \alpha}$$

(b) Ising's partition sum, $\alpha = \beta\mu B$

$$\mathfrak{J} = \mathfrak{m} \cdot n \cdot \frac{\mathfrak{Sin}\,\alpha}{\sqrt{\mathfrak{Sin}^2 \alpha + e^{-\frac{2\varepsilon}{kT}}}}$$

(c) Ising's magnetization, $m \widehat{=} \mu$, $n \widehat{=} N$

Figure 5: Excerpts from Ising's original solution [5]

First of all, he associated the spins with plus and minus signs (figure 5a) and considered a chain with solely plus spins, which is a ground state of the system. By fixing the first sign (left) with a positive spin, he counted the possible ways of placing groups of negative spins into this chain because only the borders of the arising gaps, where a plus sign meets a minus sign, would yield a relevant contribution to the total energy (2) (assuming positive coupling J).
If s denotes the number of these gaps and ν_1 shall be the total number of positive spins in a configuration, then there are $\binom{\nu_1-1}{s}$ ways of placing such gaps into the chain. In each of these possibilities there are $\binom{\nu_2-1}{s+\delta-1}$ ways of placing the total number ν_2 of negative spins into these gaps depending on the value δ of the last spin in the chain where we define $\delta := 0,1$ if this spin is positive or negative. By the interchange of $\nu_1 \to \nu_2$, we get the cases where the chain starts with a negative spin.
If the energy of the ground state is set to zero, then the total energy E of the system for a configuration depending on ν_1,ν_2,s and δ is:

$$E = (2s+\delta)\epsilon + (\nu_1 - \nu_2)\mu B \tag{24}$$

where $\epsilon \sim J$ is a rescaled coupling constant (due to the zero energy condition).

Hence, Ising got the partition sum (compare figure 5b):

$$Z = \sum_{\nu_1,\nu_2,s,\delta} \left(\binom{\nu_1-1}{s}\binom{\nu_2-1}{s+\delta-1} + \binom{\nu_2-1}{s}\binom{\nu_1-1}{s+\delta-1} \right) e^{-\beta\left((2s+\delta)\epsilon+(\nu_1-\nu_2)\mu B\right)}. \tag{25}$$

He then performed the summation explicitly (the details of the calculation can be found in his dissertation). In this way, he was also able to get an expression for the magnetization (compare figure 5c):

$$M = \mu N \frac{\sinh\alpha}{\sqrt{\sinh^2\alpha + e^{-2\beta\epsilon}}} \tag{26}$$

with $\alpha := \beta\mu B$. This is the same expression we got by the transfer matrix method if we identify $\epsilon = 2J$

3 2D Ising Model

In contrast to the one dimensional case, for the two dimensional Ising lattice without a magnetic field a phase transition can be found. We will show this using Peierls argument and give a small insight to Onsager's exact solution

3.1 Peierls Argument

Because the original Peierls argument was not quite rigorous, we will, based on Griffiths argumentation [8], follow the proof for the existence of a phase transition given in [18] and [19] (where also the Peierls argument in higher dimensions was discussed).
First, we define the average magnetization per site as:

$$m = \frac{M}{N} = \frac{\langle N_+ \rangle - \langle N_- \rangle}{N} \tag{27}$$

with the thermal average of the number of positive $\langle N_+ \rangle$ and negative $\langle N_- \rangle$ spins. Note, due to the absence of an external magnetic field, there is a symmetry in the system when flipping a spin from $+1 \to -1$ and vice versa. So, the average magnetization per site in the system would always be zero when summing over all possible configurations since each configuration is equally probable to its flipped counterpart. Therefore, we have to consider m in the limit of a vanishing magnetic field $B \to 0$ if we want to determine a phase transition. For this purpose, we consider a 2D square lattice with $N = n \times n$ lattice sites and positive spins placed on the boundary of this lattice which simulate the influence of such a vanishing magnetic field for $N \to \infty$. Next, we rewrite (27) by using $N = N_+ + N_-$:

$$m = 1 - 2\frac{\langle N_- \rangle}{N}. \tag{28}$$

In this formula for m, we can see that if the fraction $\frac{\langle N_- \rangle}{N}$ is smaller than $\frac{1}{2}$ at some finite temperature, then the average magnetization is not zero anymore, which indicates the presence of a phase transition. The key idea of Peierls argument is now to find such a limit for $\frac{\langle N_- \rangle}{N}$. We will find this limit by introducing *domain walls* γ^L.
Domain walls are continuous lines drawn in the 2D square lattice. They are defined in the sense that they lie between two spins and always have a negative spin to their right and a positive spin to their left. If there is an ambiguity, the domain wall will tend to the right. In this manner, no two domain walls will cross, and, due to our boundary condition, every domain wall is a closed loop. The length L of a domain wall is the number of lattice segments it contains. In figure 6 some examples of domain walls are drawn.
Each domain wall encloses a certain number of negative spins. This number is given by the area $A(\gamma^L)$. In order to decide if a domain wall is present in a particular configuration, we define a configuration parameter $X(\gamma^L)$ which takes either the value $X = 1$ if the domain wall occurs or $X = 0$ if not. The total number of negative spins N_- in a given configuration can then be estimated by:

$$N_- \leq \sum_L \sum_{i=1}^{\#L} A\left(\gamma_i^L\right) X(\gamma_i^L) \tag{29}$$

where we sum over all possible values of L, and the number $\#L$ of different domain walls at each length.

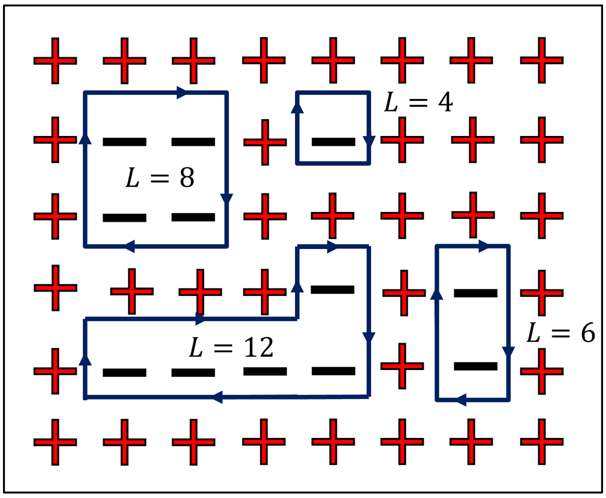


Figure 6: Domain walls of different lengths L

The sum in (29) depends on the shape of the domain walls. However, we can show that this sum can be expressed in terms of solely L.

Estimation for the area

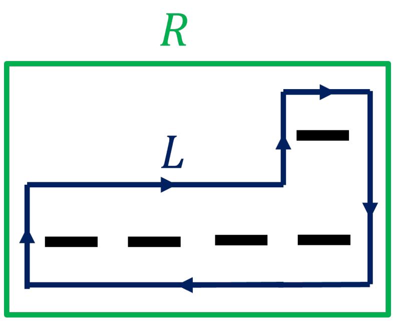


Figure 7: Domain wall γ_i^L enclosed by a rectangle of size R

First, we consider the area of a domain wall. For each domain wall γ^L, we can find a smallest possible rectangle of perimeter R which contains the domain wall (see figure 7). From our definition, it is $R \leq L$. The area $A(R)$ of the rectangle builds an upper limit for the area $A(\gamma^L)$ of the domain wall. The maximum area of the rectangle with a maximum perimeter L is $\left(\frac{L}{4}\right)^2$. Hence, following relation holds:

$$A\left(\gamma_i^L\right) \leq A\left(R\right) \leq \left(\frac{L}{4}\right)^2 =: A(L). \tag{30}$$

This turns (29) to:

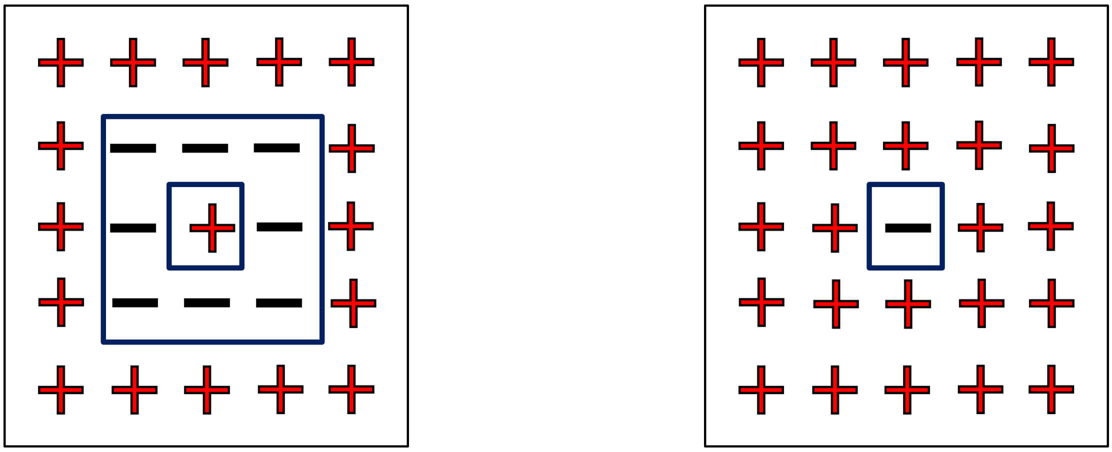
$$N_- \leq \sum_L A(L) \sum_{i=1}^{\#L} X(\gamma_i^L). \tag{31}$$

Estimation for the configuration parameter

Now, we switch to the thermal average $\langle N_{-} \rangle$ as we would consider this for (28). The averaged configuration parameter $\langle X(\gamma_i^L) \rangle$ has then an upper bound. It can be found by defining $\mathscr{B}$ as the set of all configurations with the positive boundary, and $\mathscr{C}$ as the set of the configurations which contain the domain wall γ_i^L. Thus, we can write:

$$\left\langle X(\gamma_i^L) \right\rangle = \frac{\sum_{c \in \mathscr{C}} e^{-\beta E(c)}}{\sum_{c \in \mathscr{B}} e^{-\beta E(c)}} \tag{32}$$

with the energy $E(c)$ of a configuration c.



(a) c **(b)** $\bar{c}$

Figure 8: Configuration with spins inside γ_i^L unflipped **a** and flipped **b**

Let $\bar{\mathscr{C}} = \{\bar{c} | c \in \mathscr{C}\}$ be the set of all configurations $\bar{c}$ which result from c by flipping the spins inside the domain wall γ_i^L (see figure 8). The energy $E(\bar{c})$ is related to $E(c)$ via:

$$E(\bar{c}) = E(c) - 2JL \tag{33}$$

because similar to the 1D Ising solution, the L spins at the domain wall yield a contribution to the energy difference, which demonstrates the significance of domain walls.
Since $\bar{\mathscr{C}}$ is a subset of $\mathscr{B}$ as the boundary of the lattice remains the same, we have

$$\sum_{\bar{c} \in \bar{\mathscr{C}}} e^{-\beta E(\bar{c})} \leq \sum_{c \in \mathscr{B}} e^{-\beta E(c)}. \tag{34}$$

This means for (32):

$$\left\langle X(\gamma_i^L) \right\rangle \leq \frac{\sum_{c \in \mathscr{C}} e^{-\beta E(c)}}{\sum_{c \in \bar{\mathscr{C}}} e^{-\beta E(\bar{c})}} = e^{-\beta 2LJ} =: X(L) \tag{35}$$

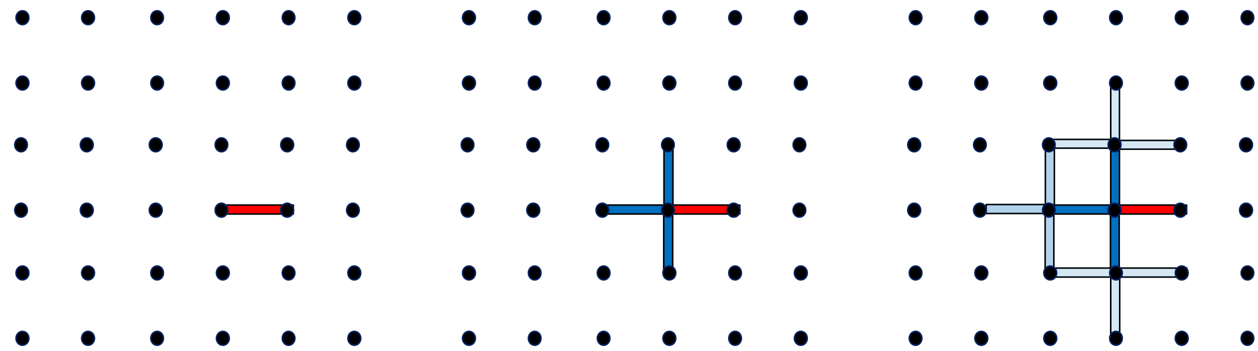
which simplifies (31) to:

$$\langle N_{-} \rangle \leq \sum_{L} A(L)\, X(L)\, \#L. \tag{36}$$

So, we are left with finding the number $\#L$ of possible domain walls at fixed L.

Estimation for the number of domain walls

This is a combinatorial problem and $\#L$ can be approximated with following procedure (compare figure 9).



(a) First edge **(b)** Second edge **(c)** Third edge

Figure 9: Counting the possible number of $\#L$ domain walls

There are at most 2^N possibilities of placing the first edge of a domain wall (9a). In the next step, there are 3 possible ways of placing the second edge (9b). This one again has 3 choices for the upcoming edge (9c). We can iterate through this $L-2$ times as the last edge must close the loop of the domain wall. Furthermore, we can start a domain wall from each of its edges. As a consequence, we have a degeneracy of L. Summing up all possibilities, the number of domain walls given L is limited to:

$$\#L \leq \frac{2N3^{L-2}}{L}. \tag{37}$$

Putting this back into (36), we finally get:

$$\langle N_- \rangle \leq \sum_L \left(\frac{L}{4}\right)^2 \cdot e^{-\beta 2LJ} \cdot \frac{2N3^{L-2}}{L} \tag{38}$$

which, if we notice that L must be even and starts with $L = 4$, can be expressed as:

$$\langle N_- \rangle \leq \frac{N}{72} \sum_{L=4,6,...} L\left(3e^{-2\beta J}\right)^L. \tag{39}$$

This is a geometric series, which converges if $3e^{-2\beta J} \leq 1$. This happens at sufficiently low, but finite temperature.
Performing the sum, we get:

$$\sum_{L=4,6,...} L\left(3e^{-2\beta J}\right)^L = \frac{N}{36} x^2 \frac{2-x}{1-x^2} \tag{40}$$

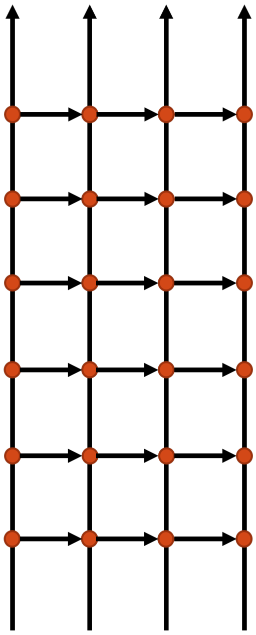
with

$$x = \left(3e^{-2\beta J}\right)^2. \tag{41}$$

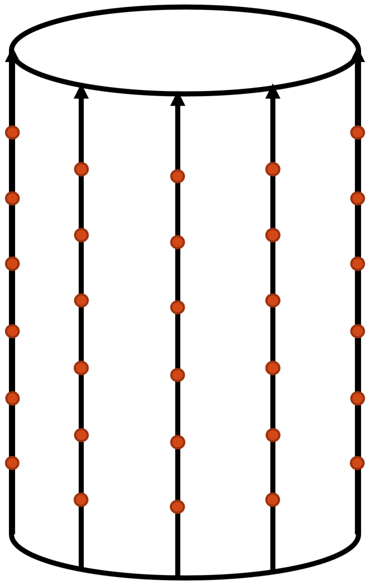
And indeed, we have shown for low temperatures, the fraction $\frac{\langle N_- \rangle}{N}$ will be less than $\frac{1}{2}$ and the magnetization m takes a value larger than zero (e.g. choose $x = \frac{1}{4}$, then $m \approx 0.97$). Thereby, we have proven the existence of a phase transition in the 2D sqaure lattice.

3.2 Onsager's Exact solution

To get a more concrete value for the critical temperature T_c at which the phase transition occcurs, we will have a short view on Onsager's exact solution.



(a) Coupled linear chains



(b) Cylinder of 1D chains

Figure 10: 2D square lattice Ising model

Note first that the 2D Ising model is nothing less than an arrangement of several 1D Ising chains and each of this chains interact with its nearest neighbour (figure 10a). Applying periodic boundary conditions will yield a cylindrical lattice (figure 10b). The fundamental idea is to use again the transfer matrices. While in the linear chain the transfer matrix described the interaction energy when going from one spin to the next one (compare (14)), the transfer matrix for the square lattice illustrates the interaction energy between two neighbouring Ising chains. If we write Ω_k for the configuration of one chain and Ω_{k+1} for the neighbour chain, then the transfer matrix P is defined as:

$$\langle \Omega_k | P | \Omega_{k+1} \rangle = \exp[-\beta (E(\Omega_k, \Omega_{k+1}) + E(\Omega_\text{k}))] \tag{42}$$

with the chain interaction energy $E(\Omega_k, \Omega_{k+1})$ and the energy of a single chain $E(\Omega_k)$
Similar to (16) and (17), we find the partition function:

$$Z(T) = \sum_{\Omega_1} \langle \Omega_1 | P^n | \Omega_1 \rangle = Tr(P^n). \tag{43}$$

So one more time, one has to solve the Eigenvalue equation for P. This is not a simple task as the dimension of P is $2^n \times 2^n$ (with n the size of one Ising chain). A simplification of this problem is in the large N limit. Here, it can be shown that only the largest Eigenvalue λ_{max} is relevant and:

$$\lim_{N\to\infty} \frac{1}{N} \ln Z(T) = \lim_{n\to\infty} \frac{1}{n} \ln \lambda_{max}. \tag{44}$$

Onsager has found the appropriate Eigenvalue. Without further details on the exact solution (see his publication [11] for more information), we state here the partition function he obtained expressed as the free energy per site $F = -\frac{1}{N\beta}\ln Z$ (in the limit $N \to \infty$):

$$-\beta F = \ln 2 - \frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi} \ln\left[\sinh^2 2\beta J - \sinh 2\beta J\left(\cos\omega - \cos\phi\right) + 1\right] d\omega d\phi \tag{45}$$

The free energy gets a singularity at the critical temperature T_c. This corresponds to the case when the logarithm in (45) is zero. Thus, we find that following relation must hold:

$$\sinh 2\beta J \stackrel{!}{=} 1. \tag{46}$$

The critical temperature reads then:

$$T_c = \frac{2J}{\ln\left(1+\sqrt{2}\right)k_B} \approx 2.269\frac{J}{k_B}. \tag{47}$$

Note that this critical temperature T_c can be verified numerically (e.g. in [20] the value $T_c\left[\frac{J}{k_B}\right] = 2.269 \pm 0.002$ was obtained with Monte Carlo simulations of the 2D Ising model).
The magnetization per site derived from the above partition function is:

$$m = \left(1 - \sinh^{-4}(2\beta J)\right)^{\frac{1}{8}}. \tag{48}$$

In figure 11 a plot of the magnetization is given. We can see that above the critical value T_c, m is always zero. Going lower than (47), spontaneous symmetry breaking takes place and the magnetization will either tend to positive or negative values. Since the magnetization is a continuous function, this is an example of a second order phase transition, which has a critical exponent of $\frac{1}{8}$. At zero temperature, the magnetization is either ± 1.

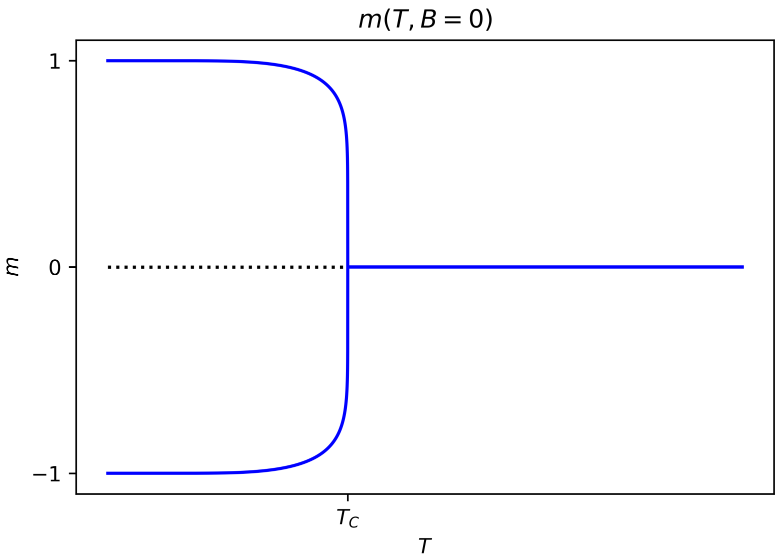


Figure 11: Magnetization of the 2D square lattice

4 Applications of the Ising Model

The Ising model does not only have a big significance in physics, but also in other fields like politics [21],[22], biology [23] or epidemiology [24]. Especially in a recent study by Mello et al. [25], the spread of Covid-19 was analyzed based on the Ising model. In this article, they have associated the spin variables $S_i = \pm 1$ of the Ising model with the status of a person being infected or not. Moreover, they have used the coupling constant $\delta\epsilon$ (equal to J in the ferromagnetic consideration) to describe the social interaction between people and modelled Gaussian curves depending on $\delta\epsilon$ for estimating the number of infected people over a period of time (figure 12).
They have also considered the Ising model on a Bethe lattice [26], which is like a constrained 2D Ising model, to calculate the probability of a person in the Bethe lattice getting infected by a person situated on the center, the zeroth shell, of the lattice (figure 13).
To sum up, this paper has displayed different ways on embedding the Ising model in solutions for nowadays challenges like Covid, and illustrated the diverse applications of the Ising model.

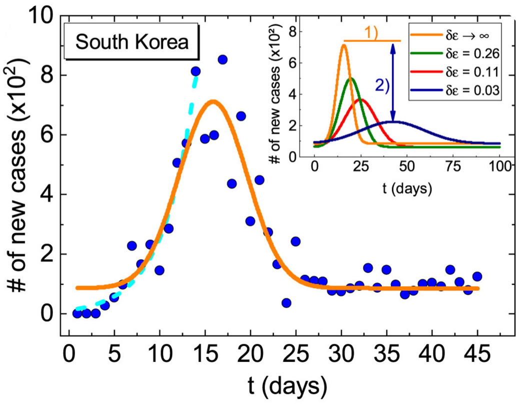


Figure 12: Gaussian fit to infection cases in South Korea. Depending on $\delta\epsilon$ the Gaussian gets narrowed (strong interaction) or flattened (weak interaction) (figure 3.(a) in [25])

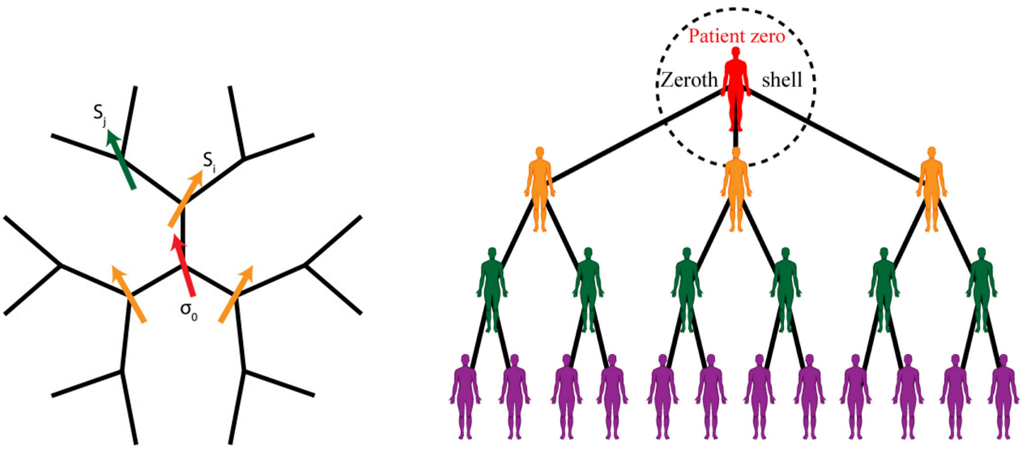


Figure 13: The Ising chain on a Bethe lattice. At each level of this tree, the number of people who can get infected by one human is the same (figure 8.(a) and 9 in [25])

5 Conclusion

In this report, we have reviewed the Ising model in the one- and two-dimesional case. By the transfer matrix method, we have derived an expression for the magnetization of the linear system and proved that there is no phase transition at finite temperature. Our result matched up with Ising's original approach. Then, we have considered the square lattice Ising model. We gave a rigorous argument for the existence of a phase transition in this dimension. From Onsager's exact solution, which also involved the transfer matrices, we adapted the value $T_c \approx 2.269 \frac{J}{k_B}$ as the critical temperature and have seen the spontaneous symmetry breaking of the magnetization. Finally, we have looked on an application of the Ising model for estimating the spread of Covid 19.

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