

Phase Transitions and Critical Phenomena

Statistical Physics Seminar SS2023

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Seminar talk summary for the seminar of
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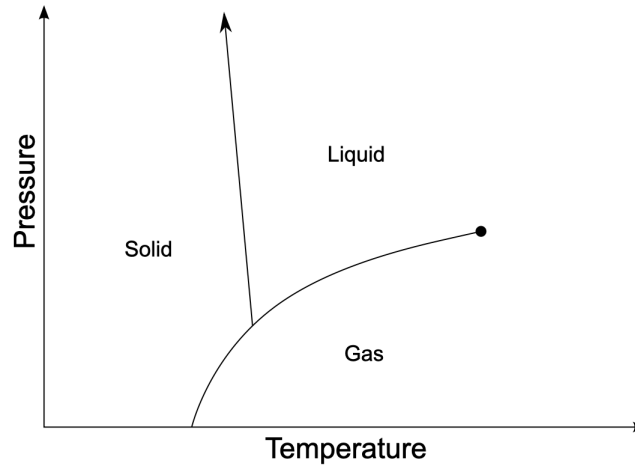


Figure 1: phase diagram of water, by [3]

Abstract

This is a summary of my introductory talk for the seminar of statistical physics SS23, held by Professor Wolschin, Heidelberg University. This summary begins with a short introduction to phase transitions and then precedes to cover a brief history of phase transitions. After the brief history, two definitions for phase transitions are introduced, the Ehrenfest definition and a more modern definition nowadays commonly used. With these definitions, the most important critical phenomena are discussed; Correlations and Fluctuations, Symmetries, Order Parameters, Critical Exponents, Universality and static Scaling Laws. Consequently there will be a short section dedicated to the Landau theory, which was advocated during the talk to calculate some mean field critical exponents for the Lenz-Ising model. In the end a short outlook of current applications beyond physics is given.

1 Introduction

Phase transitions can be defined by an abrupt, non-analytical behaviour of the function describing characteristics of the system. I here deliberately avoided the terminology of the change of the state of "matter" since modern research regarding phase transitions exceeds condensed matter applications by far. In fact, the fields medal of mathematics 2022 was awarded to Hugo Duminil-Copin due to his work in the connection between symmetries, probability theory, group theory and phase transitions [2]. Despite the familiarity of phase transitions in our everyday life, physically they are extraordinary! Usually things in physics happen gradually, which is of no surprise when considering that analytical physics came along with the development of calculus. As we know, i.e. the movement of a particle is described by a continuous function $x(t)$. Yet, thermodynamics and statistical physics tells us, that collectively particles behave very differently. Describing such a system thermodynamically, suddenly discontinuities appear, which as it turns out, can be related to phase transitions of a certain kind. Looking at Figure 1, moving along the so called coexistence curve (the black lines drawn in this diagram) the two phases coexist. However, keeping i.e. temperature constant and slightly increasing pressure, the system suddenly is in the solid or liquid phase. Along this coexistence curves, so called first order phase transitions happen. Looking closer at this diagram one sees a black dot at the right sight, which is the so called critical point. Beyond

this critical point, there is no clear notion of a distinct phase between the liquid and gas phase anymore. The system is in a new phase, the so called supercritical fluid phase. At this critical point a second order phase transition happens. Closer theoretical and experimental investigations of phase transitions uncover all sorts of interesting behaviour; different states compete with each other, infinities and discontinuities appear and some sort of universal behaviour is going on. The goal of my talk and this summary is to give a brief introduction to the rich field of phase transitions, explain the most important concepts like what is a first or second order phase transition and give an overview of critical phenomena. Beyond that, phase transitions are one of the most fascinating aspects of modern physics and bridge the field of physics to biology, mathematics, material sciences engineering, neuroscience and many more. Thus it's far more than interesting to study phase transitions in detail! I'd also like to convey this fascination.

2 A brief History of Phase Transitions

The history of phase transitions comes hand in hand with the history of material sciences and recently with the developments in thermodynamics and the renormalization group... but lets start at the beginning.

As soon as humans were able to produce fire, solid materials were made from phase transitions, like ceramics or metals. Ceramics were known since the eight millennium before Christ and they were first produced by some sort of clay in an oven. Bronz and Iron appeared a bit later on the timeline, approximately in the second millennium before Christ since they required a mastery of high temperatures. Magnetic materials were known in the antique. Thales of Miletus described the behaviour of magnetic material more than 2500 years ago. As science evolved, the first real theoretical contact with phase transitions was possible with the development of a notion of temperature. This notion came along with the invention of Galileos air thermometer in 1592 and with the later invention of Fahrenheits mercury thermometer between 1708 and 1724. Scientists like Bacon, Descartes, Galileio and Boyle believed that heat originated in a change of state of the system, like melting. With the industrialisation in the 18th century the thermal machines began to emerge. With J. Black, a Scottish Physisist, the notion of specific heat and latent heat was introduced. Other important contributions to thermodynamic were made by people like Carnot, Joule, Lord Kelvin and Clausius. I'm sure if you studied thermodynamics before, you have encountered these names. At the end of the 19th century thermodynamic functions, or state variables such as entropy or internal energy were formalised and tools for systematically studying phase transition phenomena were available. Especially important in this area were the works of J.W. Gibbs and P. Duhem. The existence of second order phase transition was observed in CO_2 by Andrews in 1869 and shortly after Van der Waals introduced a theoretical framework to explain these results by the state equation for a real gas. Studies of magnetism followed by P.Curie, where he investigated high temperature ferromagnetic properties. Pursuing the path of studying ferromagnetism P. Weiss advanced the theories. At the beginning of the 20th century, with the discovery of X-rays, the more precise determination of structures was permitted. The idea for a unified approach of phase transitions only arose in the middle of the 20th century with the work of P.Ehrenfest and L.Landau, who introduced the concept of order parameters and highlighted the importance of symmetries. Experimental methods for investing phase transitions advanced and 1944 L. Onsager was able to solve the two-dimensional Ising model exactly, which was developed by Lenz and Ising. At the end of the 1960s, stimulated by experiments, the concept of second order phase transitions was renewed. Studies of L.P. Kardanoff and M.E.Fisher and many others led to notions of scaling and universality laws. Using the tools of the renormalization group a unified approach to second-order phase transitions was allowed. Since then phase transitions have been studied extensively and the field flourishes with applications all over physics. Just to give some examples for physics applications of phase transitions (in the Conclusion and Outlook section I'll try to give applications beyond the usual physics applications), one can look at the following two very recent papers [1] , [9].

3 Definitions of Phase Transitions

A physical phase of a system is a state with distinct macroscopic properties, usually occurring as equilibrium state. States of matter come with a stability regions given by a set of parameters, visualised by a phase diagram. The properties of the state change at the phase boundary as seen in Figure 1. Generally one can distinguish between phase transitions with the generation of latent heat and without the generation of latent heat, but this is a characterisation rather than a definition.

Even though some authors might use different terminology, the following two definitions are the most common found in literature, the former one in older literature, the latter one in modern literature, i.e. [11], [7], [5].

3.1 Ehrenfest Definition

P. Ehrenfest [4] proposed the following definition for phase transitions after the 1932 discovery of a surprising new phase transition in liquid helium, the so called “lambda transition”, discovered by W. H. Keesom and coworkers in Leiden:

An n -th order phase transition is a discontinuity in the n -th order derivative of the free energy with respect to any argument of the free energy. Furthermore he proposed to schematically divide phase transitions into those with the generation of latent heat and those without generation of latent heat.

Even though a good first definition of phase transitions, it does not cover the whole story. As it turns out, phase transitions do not only come with discontinuities but also divergences can appear. It even turns out, that the lambda transition in Helium is a divergence rather than a discontinuity and here is a nice paper pointing out the irony [6]. Nevertheless it was an important conceptual step by Ehrenfest .

3.2 Modern Definition

A more modern definition found in many textbooks, i.e.[11],[5],[7], extends the definition of Ehrenfest the following way:

A first order transition can be described by a discontinuity in the derivative of an appropriate thermodynamics potential, exactly like Ehrenfest proposed (with latent heat). However, a second order phase transition, nowadays commonly called continuous phase transition, is continuous in its first derivative and exhibits discontinuous or divergent behaviour in its second derivatives (without latent heat).

Note the similarities between these two definitions.

4 Critical Phenomena

Now that we have come to terms with the definition(s) of phase transitions, let us examine characteristic phenomena of phase transitions, so called critical phenomena. These phenomena are distinct features of phase transitions and can be found over almost all different phase transitions. In fact they are so characteristic for phase transitions, one could equally as well use them to define different types of phase transitions. The following critical phenomena should not be viewed to be independent, they are all closely intertwined and often just a different side of the same (hyperdimensional) coin.

4.1 Symmetries and Order Parameters

4.1.1 Symmetries

Symmetries play an important role in physics and one can find them all over the place, also in phase transitions. Different phases of matter, or more generally of a dynamical system, can be characterised by symmetries. One can think about a crystal with cubic lattice structure, which has comparably less symmetry than a liquid since it breaks the continuous rotational and translations invariant and only offers a discrete \mathbf{Z}_n set of translations and rotational compared to the liquid. Perhaps a bit surprising but liquids and gases break no such symmetries and thus, by this criterion, could be classified as the same phase of matter. This characterisation is very helpful, since by simple symmetry considerations many of the systems properties can be inferred. A very recent breakthrough regarding symmetries is the fields medal 2022 work of Duminil-Copin and others in which they were able to prove rotational invariance for a large set of 2d lattice models close to criticality [2]. Symmetries are also one of the key components of the so called Landau theory, which is a qualitative theory to investigate phase transitions.

4.1.2 Order Parameters

Closely linked to the concept of symmetries, and also a key component of the Landau theory, is the concept of order parameters. Landau associated changes in symmetries of a system with a specific parameter, the so called order parameter. In general an order parameter η is a physical quantity of (often extensive) character which is zero in the most symmetric (or most disordered) phase and non-zero in the least symmetric (or ordered) phase. Thus, generally speaking, it is a parameter controlling the order of a system. This very neat concept however, turns out to be rather complicated in practise. The reason being that it is not always very obvious which order parameter to choose. Here is a short list of common order parameters

Transition	order Parameter
liquid-gas transition	$\rho_G - \rho_L$
ferromagnetic transition	magnetization M
superfluid transition	condensate wave function $\langle \Psi \rangle$
antiferromagnetic transition	sublattice magnetization Π

Table 1: short list of order parameters, a extensive list can be found in [11] p.21

Using the concept of order parameters P. Papon et al [7] proposed the following two types of phase transitions

- transitions with no order parameter for which the symmetry group of the two phases are such that non is strictly included in the other; they are first order in the Ehrenfest definition, with latent heat.
- transitions for which an order parameter can be defined and for which the symmetry group of the least symmetric phase is a subgroup of the symmetry group of the most symmetric phase. If the order parameter is discontinuous at the transition, it is first order in the Ehrenfest classification, if it is continuous it is second order (without latent heat).

4.2 Fluctuations and Correlations

Correlations are yet another ways to characterise phases and understand phase transitions In this way it is natural to look at correlators and investigate their behaviour in vicinity of a phase transition. By doing so we will encounter the concept of the correlation length ζ . The concept of the correlation length is the length scale over which the fluctuations of the microscopic properties of a system are correlated. The larger $\zeta(T)$, the more the local fluctuations are connected with each other, meaning the local fluctuations in one part of the system influence the local fluctuations in another part of the system. For first order phase transitions, in the sense of Ehrenfest discontinuities, the correlation length remains finite. For continuous phase transitions, the correlation length diverges when approaching the phase transition. Fluctuations become correlated over all distances, which forces the whole system to be in a unique, critical, phase.

To begin we will define the autocorrelation function $G(r)$

$$G(r, r') = \langle \delta n(r) \delta n(r') \rangle \quad (1)$$

measuring the correlation between local density fluctuations in the local particle density $n(r)$ with $\Delta n(r) = n(r) - \langle n(r) \rangle$ ¹. Being in the grand canonical ensemble one can write

$$\langle (N - \langle N \rangle)^2 \rangle = \frac{1}{\beta} \left(\frac{\partial^2 \ln Z}{\partial \mu^2} \right) \quad (2)$$

with Z the grand canonical partition function and N being the particle number. Using the definition of $G(r)$ one furthermore finds

$$\langle \Delta N^2 \rangle = V \int dr G(r) \quad (3)$$

¹this case can easily be extend to magnetic systems by replacing local densities with local spins and the mean density with the magnetization.

by simply plugging in the definition of $G(r)$ and assuming spatial translation in variance, $G(r, r') = G(r - r') = G(r'')$. Considering the scattering of electromagnetic radiation one is led to the definition

$$\frac{1}{n} \int dr \exp(-iqr) G(r) \equiv \frac{1}{n} S(q) \quad (4)$$

with $S(q)$ being the structure function. Assuming the following functional relationship

$$G(r - r') = n\eta(r - r') + n^2\Gamma(r - r') \quad (5)$$

with $\eta(r - r')$ the correlation of particles with one self and $\Gamma(r - r')$ the correlation of different particles with each other. Defining

$$C(q) = \int dr C(r) \exp(-iqr), \quad C(q) \equiv \frac{\Gamma(q)}{1 + n\Gamma(q)} \quad (6)$$

with $C(r)$ the direct correlation function and $\Gamma(q)$ the Fourier transformation of $\Gamma(r)$, one can now Taylor expand $C(q)$ for small q , which is often called the Ornstein-Zernik assumption. Using the functional relationship

$$\frac{n}{S(q)} = 1 - nC(q) \quad (7)$$

and Taylor expanding $C(q)$ while truncating at the 4th order one arrives at

$$\frac{n}{S(q)} = R^2 (\kappa_1^2 + q^2 + \dots) \quad (8)$$

with $R^2 \equiv -nc_2(n, t)$ and c_2 the second order coefficient of the Taylor expansion of $C(q)$. Now rearranging equation (8) such that

$$\frac{S(q)}{n} = \frac{R^{-2}}{\kappa_1^2 + q^2} \quad (9)$$

and after Fourier transforming it yet again, one finally obtains

$$G(r) \propto \frac{\exp(-\kappa_1 r)}{r} \quad (10)$$

with $\kappa_1 \equiv \kappa = \zeta^{-1}$. Thus for $\kappa_1 \rightarrow 0$ as $T \rightarrow T_C$, $\zeta = \frac{1}{\kappa_1} \rightarrow \infty$, ζ being the correlation length. Furthermore, near the critical temperature T_C , one can show that

$$S(k) \approx \zeta^2. \quad (11)$$

So with the correlation length ζ we found a way to look at correlations of fluctuations across the system. $S(q)$, the structure function expresses the same property, just in Fourier space. This so called correlation length ζ in general diverges for second order phase transitions but remains finite for first order phase transitions. Further development of this Ornstein-Zernik assumption for general dimensions regarding

$$G(r) = \int S(q) \exp(-iqr) dq \quad (12)$$

led Fischer to the following result [11]

$$G(r) \propto \begin{cases} \ln(r) \exp(-\kappa r) & d = 2 \\ \frac{\exp(-\kappa r)}{r} & d = 3 \\ \frac{\exp(-\kappa r)}{r^{d-2}} & d > 3 \end{cases} \quad (13)$$

with d denoting the dimension, for fixed r and $T \rightarrow T_C^+$. The phenomenological dependence on dimensionality d is one of the recurring features of phase transitions. One famous example would be the 1d Ising model, where no phase transition occurs at finite temperature while in the 2d case, there is a phase transition at finite temperature. As a final remark for this section, the phenomena of increasing fluctuations in vicinity of a phase transition can be nicely demonstrated with the critical opalescence phenomena.

4.3 Critical Behaviour

4.3.1 Critical Exponents

The modern area of phase transitions began with the emergence and application of the renormalization group to the study of phase transitions and the systematic study of so called critical exponents. Critical exponents are a set of indices with values in \mathbb{R} . They describe the behaviour of the quantity of interest near the critical point, the point at which the phase transition occurs and many people think of these critical exponents as being fundamental constants of physics and mathematics like π or e . Defining these exponents mathematically, with f a function describing the system near the phase transition $f(\epsilon)$, $\epsilon = \frac{T-T_C}{T}$ one gets

$$\lambda \equiv \lim_{\epsilon \searrow 0} \frac{\ln f(\epsilon)}{\ln(\epsilon)} \quad (14)$$

$$\lambda' \equiv \lim_{\epsilon \nearrow 0} \frac{\ln f(\epsilon)}{\ln(\epsilon)} \quad (15)$$

where, λ and λ' define the critical exponents, depending on how the critical temperature T_C is approached, from above or from below.

Notationally one often finds $f(\epsilon) \sim \epsilon^\lambda$, one however should be aware that this corresponds to $f(\epsilon) = A\epsilon^\lambda(1 + B\epsilon^y + \dots)$, $y > 0$, meaning higher order correction terms generally exist. To provide some intuition, we will look in some detail at different values of λ . Suppose $\lambda < 0$, this means with our definition at hand, that our function diverges to infinity at the critical point. For $\lambda > 0$ one finds that our function $f(\epsilon)$ approaches zero. The intermediate case of $\lambda = 0$ corresponds to non-unique behaviour. The function can either diverge, contain a cusp-like singularity or even exhibit analytical behaviour. To further distinguish the $\lambda = 0$ case, H. Stanley [11] proposed another definition of critical exponents

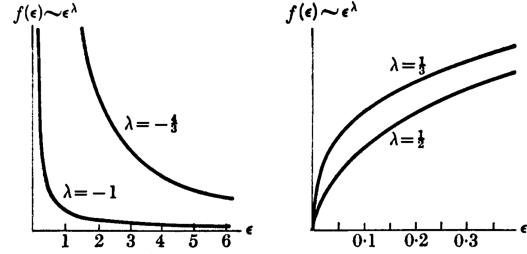


Figure 2: cases of $\lambda < 0$ (left) and $\lambda > 0$ (right), by [11]

$$\lambda_s \equiv j + \lim_{\epsilon \rightarrow 0} \frac{\ln |f^{(j)}(\epsilon)|}{\ln \epsilon} \quad (16)$$

with j the smallest integer such that the derivative $\partial^j f / \partial \epsilon^j = f^{(j)}$ diverges. Depending on the system we now list some common critical exponents but by no means all

exponent	relationship	quantity
α	$C_V \propto (\epsilon)^{-\alpha}$	specific heat at constant volume
α'	$C_V \propto (-\epsilon)^{-\alpha'}$	
β	$\rho_G - \rho_L \propto (-\epsilon)^{-\beta}$	liquid-gas density difference
γ	$K_T \propto (\epsilon)^{-\gamma}$	
γ'	$K_T \propto (-\epsilon)^{-\gamma'}$	

Table 2: critical exponents for a liquid system with $\epsilon = \frac{T-T_C}{T}$, by [11]

exponent	relationship	quantity
α	$C_H \propto (\epsilon)^{-\alpha}$	specific heat at constant magnetic field
α'	$C_H \propto (-\epsilon)^{-\alpha'}$	
β	$M \propto (-\epsilon)^{-\beta}$	zero field magnetization
γ	$\chi_T \propto (\epsilon)^{-\gamma}$	
γ'	$\chi_T \propto (-\epsilon)^{-\gamma'}$	

Table 3: critical exponents for a magnetic system with $\epsilon = \frac{T-T_C}{T}$, by [11]

As an example of mean field critical exponents for the Ising model one finds

$$m \sim (T_C - T)^\beta, \quad \beta = \frac{1}{2} \quad (17)$$

$$m \sim h^{\frac{1}{\delta}}, \quad \delta = 3 \quad (18)$$

$$\chi \sim \frac{1}{|T - T_C|^\gamma}, \quad \gamma = 1 \quad (19)$$

which are exactly the values which later on will be calculated. It is important to note, that these mean field critical exponents do not in general correspond to the exact solution as it can be seen for the magnetic system by comparing it to Onsagers solutions in Figure 3. An interesting remark would be, that for $d \geq 4$ these mean field solutions fall in line with many other solutions of higher dimensional systems. As a final remake for this section; One usually splits up the thermodynamical quantity of interest in one part containing the singularity as $\epsilon \rightarrow 0$ and one part which behaves analytically. Thus, only the singularity part is assumed to be i.e. proportional to $\pm \epsilon^\alpha$, α being the critical exponent.

4.3.2 Universality

Looking at the values of these critical exponents, one finds remarkable result; microscopically very different systems share the same values of critical exponents. Thus it is like all microscopical physical details are washed away near a second order phase transition (if one is able to find critical exponents for first order phase transitions, they most often do not agree with different systems). The behaviour of the system near the critical point consequently seems to be describable with a single, underlying theory. This phenomenon is called universality; the fact that many different systems are described by the same set of critical exponents. Looking at this from a practical point of view (this is as practical as it gets in theoretical physics), calculating the critical exponents for one fairly simple system and identifying all the other systems which can be described by the same critical exponent, one saves a lot of time and computational power. Systems that can be described by the same critical exponents are said to lie in the same universality class. Now of course the question arises, how to find out which systems lie in the same universality class. In general the universality classes depends largely on the symmetries, dimensions and interaction range of the systems. Yet, the challenge often lies in exactly finding out which models lie in the same universality class and why.

	MF	$d = 2$	$d = 3$
α	0 (disc.)	0 (log)	0.1101
β	$\frac{1}{2}$	$\frac{1}{8}$	0.3264
γ	1	$\frac{7}{4}$	1.2371
δ	3	15	4.7898

Figure 3: mean field critical exponents for the Lenz-Ising model, by [12]

4.3.3 Static Scaling Laws

The formulation of the scaling law hypothesis in critical phenomena was motivated by the desire to better understand the relations between critical exponents. Inequalities were known for a longer time and people like Fischer et al. calculated them by simple thermodynamics assumptions. The scaling law hypothesis was first formulated by Kadanoff, Domb and Widom as they proposed the following; near the critical point, the thermodynamic quantities obey a scaling law behaviour. This idea underlies the physical idea of the divergence of the correlation length $\zeta(T)$ as $T \rightarrow T_C$ and thus the loss of any microscopical behaviour. In general one believes that one can write, i.e. the thermodynamics potential G as a $G = G_S + G_N$ in vicinity of the critical point, G_S being the part with a singularity at T_C and G_N a part without a singularity. One now assumes that the potential G_S obey the following scaling behaviour

$$\lambda G_S(\epsilon, H) = G_S(\lambda^n \epsilon, \lambda^m H) \quad (20)$$

with $\epsilon = \frac{T - T_C}{T}$, H being the presence of an external field. Note that this scaling law behaviour here is formulated for the Gibbs free energy of a magnetic system, however it can be easily applied to any potential, which are linked via the Legendre transformation. G_S is defined to be a general homogeneous function of the first degree with respect to the variables². Using this form one can

²A function $f(x)$ is said to be homogeneous of degree k if it satisfies the relation $f(x) = b k f(bx)$

relate the critical exponents of the system as functions of m and n . One finds, amongst others, the following relations:

$$\alpha + 2\beta + \gamma = 2 \quad (\text{Rushbrooke}) \quad (21)$$

$$\gamma = \beta(\delta - 1) \quad (\text{Widom}) \quad (22)$$

$$\alpha + \beta(\gamma + 1) = 2 \quad (\text{Griffiths}). \quad (23)$$

An extensive list can be found in [11]. Lets consider a fairly simple example on how one could derive these exponential relations. We start by differentiating equation (20) on both sides, such that

$$\lambda \frac{\partial G_S(\epsilon, H)}{\partial H} = \lambda^m \frac{\partial G(\lambda^n \epsilon, \lambda^m H)}{\partial \lambda^m H} \quad (24)$$

$$\Rightarrow \lambda^m M(\lambda^n \epsilon, \lambda^m H) = \lambda M(\epsilon, H). \quad (25)$$

Now there are two critical exponents related to the behaviour of the magnetization near the critical point, β and $1/\delta$. Considering β we set $H = 0$ and $\epsilon \rightarrow 0$ and we get

$$M(\epsilon, 0) = \lambda^{m-1} M(\lambda^m \epsilon, 0). \quad (26)$$

Now the argument is, that this equation must be valid for all λ and thus especially for $\lambda = (-1/\epsilon)^{1/m}$ hence

$$M(\epsilon, 0) = (-\epsilon)^{1-m/n} M(-1, 0) \quad (27)$$

and taking $\epsilon \rightarrow 0$ we have that $M(\epsilon, 0) \sim (-\epsilon)^\beta$ such that $\beta = \frac{1-m}{n}$. The other exponents can be obtained in a similar fashion and one obtains a set linear equations depending on n and m . Solving this set of equations via n and m , one obtains the desired equalities.

4.4 Landau Theory

L.D. Landau gave a new perspective on phase transitions by highlighting the role of symmetries. Additionally to the importance of thermodynamical potentials and the behaviour of their response functions, he noticed that phase transitions often come along with a change in the symmetry of the system. A fairly well known example would be the spontaneous magnetization for a ferromagnet when moving from temperatures above T_C to temperatures below T_C , where the magnetization corresponds to now correlated and ordered spins. One main assumption of the Landau theory is the assumption that one can expand the so called Landau free energy L , a function of coupling parameters K_i and the order parameter η , around small values of η , in powers of η . L has the property, that the state of the system is specified by the global minimum of L . Usually one enforces the following constraints on L ³

1. L has consistencies with the symmetries of the system
2. near T_C L can be expanded in powers of η , meaning L is an analytical function in both η and K_i
3. near T_C η is small and thus one can obtain all the essential physics by a truncated power expansion

For our specific calculation we take the nearest neighbour interacting Ising model, given by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_{i=1}^N s_i. \quad (28)$$

We then advocate a mean field approach by sending s_i to $\langle s_i \rangle = m$, thus obtaining

$$\Rightarrow H_{MF} = -Jm^2 \frac{N}{2} z - hmN, \quad \sum_{\langle ij \rangle} = \frac{1}{2} \sum_{i=1}^N \sum_{j \in NN(i)} = Nz/2. \quad (29)$$

³further discussion can be found in [5]

Since the key component is the Landau function however, we are now interested in

$$f(m) = \frac{F(m)}{N} = \frac{E(m) - TS(m)}{N} \quad (30)$$

and after calculating

$$\frac{S(m)}{N} = \log(2) - \frac{1+m}{2} \log(1+m) - \frac{1-m}{2} \log(1-m) \quad (31)$$

and simply plugging it in, one obtains

$$f(m) = -hm - J \frac{m^2}{2} z - T \left(\log(2) - \frac{1+m}{2} \log(1+m) - \frac{1-m}{2} \log(1-m) \right). \quad (32)$$

Now Taylor expanding in small powers of m , our order parameter, gives

$$f(m) \approx -T \log(2) - hm + \frac{1}{2}(T - zJ)m^2 + \frac{T}{12}m^4 + \dots \quad (33)$$

With this result one is perfectly equip to calculate the critical exponents. I.e. minimising equation (33) w.r.t. m and setting it to 0, one obtains

$$m \sim |T - T_C|^{1/2} \quad (34)$$

with thus $\beta = 1/2$. The other values follow similarly as seen in the presentation and can be found in Figure 3.

5 Conclusion and Outlook

Concluding this report, a brief summary as well as an outlook of recent applications of phase transitions and critical phenomena will be given.

After a short introduction and brief history we started our discussion by two different definitions of phase transitions; the historical Ehrenfest definition and a more modern and we saw that the more modern one comprises the too restricted Ehrenfest definition. Consequently we began to investigate different critical phenomena and we began our journey with the discussion of fluctuations and correlations in the vicinity of a phase transition. We encountered the concept of the correlation length ζ which generally diverges for second order phase transitions but remains finite for first order phase transitions. Next we discussed the importance of symmetries and order parameters and saw that they are key components of the so called Landau theory, a qualitative theory. We learned about critical exponents and the fact that we can describe various very different systems near a phase transition by the same set of critical exponents, a phenomena called universality. Furthermore we discussed how one could obtain relationships between these critical exponents by advocated the static scaling law hypothesis, which finds foundation in the renormalization group and in the end we used the Landau theory to calculate some of the critical mean field Ising exponents ourselves.

As already mentioned in the introduction, the theory and applications of phase transitions one can find all over science; let it be material sciences, mathematics, biology or physics, where it originated in. This short paragraph is dedicated to give the reader a brief outlook of applications mostly beyond physics. An application of great influence which we already mentioned is the paper by Duminil-Copin et al. [2], which contributed strongly to the award of the fields medal in 2022. In this paper they were able to prove that the large scale properties of a number of two-dimensional lattice models are rotationally invariant. Another application would be in neural networks. If one comes to terms with the similarities of the Ising model and neural networks by mapping a spin s_i to a neuron n_i , the occurrence of phase transitions comes with no surprise. In [13], Ziyin et al. find deep-learning-unique first-order and second-order phase transitions. The second order phase transition is due to the competition between prediction error and model complexity for nets with one hidden layer and the first order phase transitions for nets with more than one hidden layer. But the application of phase transitions by far exceeds standard examples. For instance one can also apply it to the financial sector as pointed out by [8]. Yet another quiet extraordinary application, which however should be viewed with a bit of scepticism, would be the paper by [10] which illustrates the possibility of love being a second order phase transition.

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