# Phase transitions and critical phenomena

Theoretical Seminar in Statistical Physics

Ejona Syla

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**Abstract**

This report provides an overview of phase transitions and critical phenomena, focusing on their theoretical foundations. It covers the historical context, basic principles, and key concepts such as phase diagrams, order parameters, and critical exponents. The discussion includes the Ising model as an illustrative example and explores the mean-field theory's role in understanding phase transitions. The analysis includes an exploration of scaling and universality.

## 1 Introduction

Statistical physics provides a connection between the dynamics of a particle itself in a microscopic scale to the macroscopic collective behaviour in a many-body problem. To do so, the analysis of the systems properties are of interest. These properties can be conceptualized either as densities since they correspond to the first derivatives of the free energy w.r.t. the relevant field, or as susceptibilities, corresponding to the second derivatives of the relevant free energy, which is often realized in the absence of a field. In general, these functions are smooth but in the case of an abrupt change, there is a phase transition within the system [1].

This report will explain this vague formulation of phase transition in a more accurate description by first analyzing the phase diagram of water and then of a simple ferromagnet to understand the principles of phase transitions. Subsequently, the definition of phase transition are provided where Ehrenfest's and Landau's classification are introduced emphasized by the phase diagrams of a magnetic systems. Ehrenfest in particular uses the description of the discontinuity of the free energy derivatives as a definition for phase transitions while Landau describes a phase transition by a symmetry break allowing the notion of a order parameter. Focused on the Ising model for magnetic systems, phase diagrams illustrate phase transitions distinguishing between first and second-order transitions. This report delves into correlation functions and correlation length explaining how fluctuations influence the behaviour of the system. Further, critical exponents are introduced to describe thermodynamic potentials near criticality. Mean-field theory is presented as a tool for approximating many-body problems. Despite its limitations, it is useful to understand phase transitions. Equations and graphical analyses aid in understanding the scaling properties and universality of phase transitions.

## 1.1 Historical events

In order to better understand the development of the analysis of phase transitions and critical phenomena, a short summary of the history is given. The exploration of phase transitions and critical phenomena goes back far and is related to the development of thermodynamics, condensed matter physics and material sciences. It was only with the introduction of the concept of temperature through the development of the first thermometers by Galileo in 1592 and Fahrenheit between 1708 and 1724 that scientists such as Carnot, Mayer and Kelvin were able to lay the foundations for thermodynamics in the 19th century. At the end of the 19th century, the introduction of thermodynamic functions and state variables as internal energy and entropy by Gibbs and Duhem made it possible to systematically analyze phase transitions. In 1869, T. Andrews discovered a second order phase transition in carbon dioxide with a light-scattering experiment. In the following years, P. Curie studied ferromagnets in high-temperature regions. In 1911, K. Onnes discovered the superconductivity of mercury for low-temperature regions. P. Ehrenfest and L. Landau investigated in the 1930s the analogy of fluid and magnetic phase transitions and introduced the notion of an order parameter. In 1944, L. Onsager solved the two-dimensional Ising model for zero-field. L. P. Kadanoff, C. Domb and M. E. Fisher introduced sclaing and universality laws which was a new approach to study critical phenomena in the 1960s. By this, K. G. Wilson could developed the renormalization approach. Since then the interest of phase transition and critical phenomena increased especially in the industry why this topic is still relevant until today [2].

## 1.2 Ising model

This report will mostly deal with magnetic systems. The ferromagnet Ising model is a common model to describe interacting systems due to its simplicity and use for analog system. The model divides the magnetic system into a grid with $N$ sites each occupied by a spin $\vec{s}$ pointing up indicated by the value $+1$, or down corresponding to the value −1. The according Hamiltonian to this model reads

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i, \tag{1}$$

where $J > 0$ denotes the ferromagnetic interaction, $H$ the magnetic field and the subscript $\langle ij \rangle$ a sum over pairs of nearest neighbours at site $i$ and $j$. The model prefers parallel alignment of the spins which is indicated by the interaction term in Eq.(1) [3].

The Ising model helps to understand the phase diagrams for ferromagnetic systems which will be analyzed in the following. In addition, it will be used later to calculate the critical exponents of the mean-field theory.

# 2 Phase Transitions

## 2.1 Phase diagrams

The most intuitive way to understand phase transitions is through phase diagrams. Figure 1 shows the phase diagram of water. In this case, pressure is plotted against temperature but the axes can describe any macroscopic parameter which describes the thermodynamic potential. The lines in the diagram are *phase boundaries* separating different phases and by crossing them an abrupt change occurs in the system. At a triple-point, as depicted in Figure 1 with $(T_3, p_3)$, all three phases coexist indicating there are in equilibrium. A disappearing phase boundary at a so-called *critical point* indicates rather that the two phases become indistinguishable leading to a continuous transition. For water this takes place for temperatures above $T_C = 647\,\text{K}$. Above this critical temperature both phases, liquid and gaseous, cannot be distinguished anymore becoming a supercritical fluid [4].

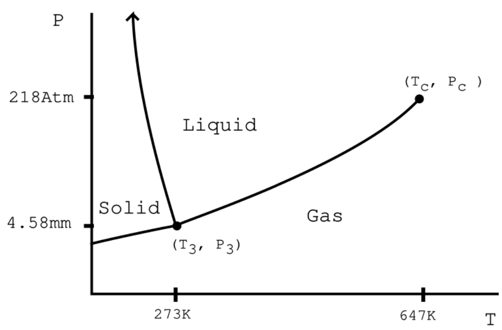


**Fig. 1:** Phase diagram of water [1].

Magnetic system show similar behaviour in phase transitions. In Figure 2a)-d), phase diagrams for a simple ferromagnet are depicted. The relevant thermodynamic control parameters for a magnetic system are temperature $T$, magnetic field $H$ and magnetization $M$. First, the behavior for temperatures below the critical point is evaluated. Figure 2a) shows that a sign change of the magnetic field leads to a spontaneous magnetization. This can also be seen in Figure 2b) where the magnetization for $T < T_C$ has a jump at $H = 0$. The spins are either showing up or down for the Ising model and thus the magnetization is either positive or negative which is also illustrated in Figure 2c). This type of transition is later called *first order* phase transition.
For $T > T_C$ one can continuously move from a negative magnetization to a positive. Due to the high temperature the spins do not prefer a direction and thus the net magnetization in Figure 2c) is zero.
Examining the transition from high to low temperatures, crossing $T_C$ at zero-field, one can see that the symmetry starts to break. The magnetization changes continuously to either positive (for $H \to 0^+$) or negative (for $H \to 0^-$). The susceptibility on the other hand, which is the derivative of the magnetization w.r.t. the magnetic field, diverges at $H = 0$ for $T = T_C$, as depicted in Figure 2d). This is why this type of transition is later called *second-order* or *continuous* [3].

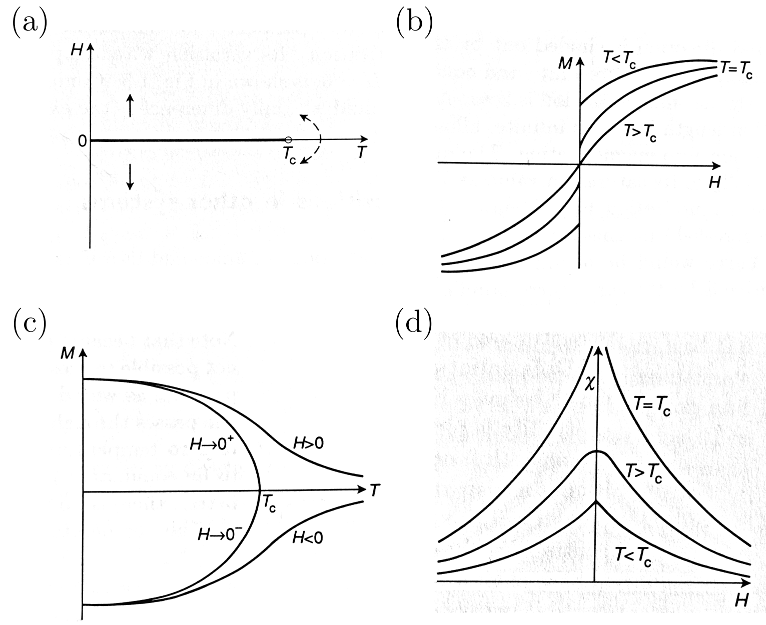


**Fig. 2:** Phase diagrams for a ferromagnet in the **(a)** $H-T$ plane, **(b)** the $M-H$ plane, **(c)** the $M-T$ plane and **(d)** $\chi - T$ plase [3].

The following section goes into more detail on how phase transitions are defined. In the past, two classification have been established, one from Ehrenfest and one from Landau. Former one makes a distinction through the corresponding derivatives of the thermal potential and if latent heat is absorbed/released or not, the latter one through a break in the symmetry allowing the introduction of an order parameter [2].

## 2.2 Ehrenfest classification

In 1933 P. Ehrenfest distinguished between two types of phase transitions. If a discontinuity is observed in one of the first derivatives of the thermodynamic potential, the phase transition is classified as **first-order transition**. This is often associated with the absorption or realese of latent heat.
However, if the first derivatives are continuous but higher-order derivatives are discontinuous or even infinite, the transition is called **continuous**. Once again, the ferromagnet serves as a fitting example here. Examining Figure 3a), the

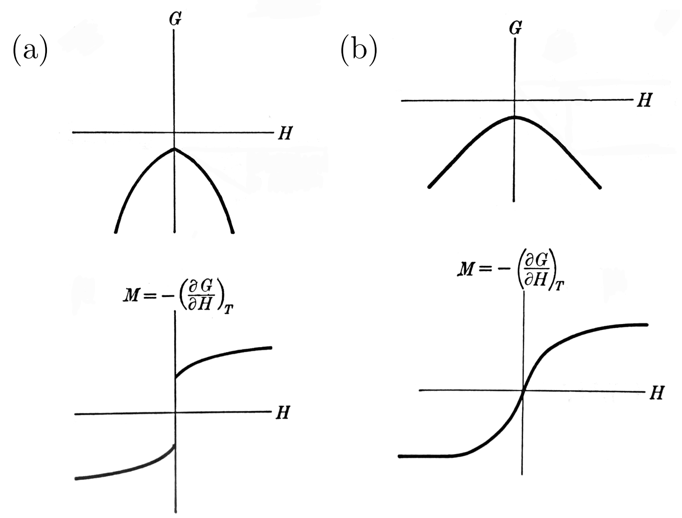


**Fig. 3:** The Gibb's free energy and its first derivative plotted against $H$ for **(a)** $T < T_C$ and **(b)** $T > T_C$ [4].

thermodynamic potential, in this case the Gibb's free energy, shows a discontinuity at $H = 0$ for $T < T_C$. This results in a jump of the magnetization $M$, which is the first derivative of $G$ w.r.t. the magnetic field $H$. Remember that this jump has already been discussed above. Therefore, for temperatures below the critical one $T < T_C$ the phase transition caused by a change in the magnetic field in a ferromagnet is of first-order.
For $T > T_C$ Figure 3b) shows a continuous function for both $G$ and its derivative

$M$. But for $T = T_C$, it was already discussed above that the susceptibility which is the first derivative of $M$ w.r.t. $H$ and thus the second derivative of $G$ w.r.t. $H$ approaches infinity leading to a continuous phase transition [4].

## 2.3 Landau classification

Landau suggested 1937 a more general way of defining phase transitions. Rather than coupling them to a thermodynamic potential, he defined phase transitions through a breaking in the symmetry. In the case of ferromagnets, symmetry is broken when the critical temperature is crossed, also known as the *Curie point*. One can describe the ferromagnet for $T < T_C$ as *ordered* and for $T > T_C$ as *disordered*. Landau proposed the notion of an order parameter to distinguish between a ordered phase indicated by a non-equal to zero order parameter and disordered phases which are characterized by a equal to zero order parameter. In magnetic system, the magnetization is a good indicator if the system is ordered or not and thus the order parameter. Phase transitions which have no order parameter are always *first-order* phase transitions as well as phase transitions with a discontinuous order parameter. Note that this exactly the case for $M$ when the magnetic field is changed, see Figure 2b). Phase transitions with a continuous order parameter, as it is indicated in Figure 2c) where the temperature is changed, are *continuous* phase transitions [2].

## 2.4 Correlation functions

To get a deeper understanding of phase transitions, it is useful to examine the system on a microscopic level. What is exactly meant is instead of considering the magnetization, which is a macroscopic variable, the system is analyzed from a microscopic view by focusing on the spins. For this, correlation functions are a helpful tool.

The **spin-spin correlation function** measures how much the fluctuation of spin at site $i$ influences the fluctuation of the spin at site $j$

$$C(\vec{r_i}, \vec{r_j}) \equiv \langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle. \tag{2}$$

The $\langle ... \rangle$ corresponds to the average at zero field. For a translationally invariant system applies $\langle s_i \rangle = \langle s_j \rangle$ and thus

$$C(\vec{r_i} - \vec{r_j}) \equiv C_{ij} = \langle s_i s_j \rangle - \langle s \rangle^2. \tag{3}$$

One can intuitively make some assessments about the correlation function, like if the spin at site $i$ is up, the neighbouring spin is forced to be up too due to interactions. But this effect decreases exponentially with distance between the spins

$$C(\vec{r}) \sim r^{\theta}\, e^{-r/\xi} \qquad \text{for } T \neq T_C, \tag{4}$$

where $\theta$ is a number. The $\xi$ represents the correlation length, which is temperature depending. Large temperatures leads to a small correlation length due to Boltzmann probabilities. At the critical temperature, the system develops long-range order and spontaneous magnetization which leads to a divergence of the correlation length. The relation between the correlation function and the distance is [3]

$$C(\vec{r}) \propto \frac{1}{r^{d-2+\eta}} \qquad \text{for } T = T_C \tag{5}$$

and for the correlation length

$$\xi(T) \sim |T - T_C|^{-\nu} \tag{6}$$

for temperatures near the critical point $T = T_C$. In Eq.(6), the first critical exponents is introduced describing the behaviour of the correlation length at criticality [1]. The report discusses in the following the characterization of criticality and the description of thermodynamic potentials near the critical temperature. Since these show divergences at $T = T_C$ as discussed above, further critical exponents will be introduced.

# 3 Critical phenomena

## 3.1 Critical exponents

In the classification of phase transitions, it was already discussed that continuous phase transitions are associated with critical phenomena around a critical point. One example was the divergence in the susceptibility for $H = 0$, see Figure 2d). To better understand the behaviour of the thermodynamic potentials near criticality, so-called **critical exponents** are introduced. For a thermodynamic function $F(t)$ with the reduced temperature $t \equiv \frac{T-T_C}{T_C}$, the critical exponent is defined by

$$\lambda = \lim_{t\to 0} \frac{\ln|F(t)|}{\ln|t|} \tag{7}$$

or in a more convenient way in the form of a power law

$$F(t) \sim |t|^{\lambda}. \tag{8}$$

| Quantitiy | Exponent | Power law | Conditions |
|---|---|---|---|
| Specific heat | $\alpha$ | $C_H \sim \|\tau\|^{-\alpha}$ | $\tau < 0$ and $H = 0$ |
| | $\alpha'$ | $C_H \sim \|\tau\|^{-\alpha'}$ | $\tau < 0$ and $H = 0$ |
| Zero-field magnetization | $\beta$ | $m \sim (-\tau)^{\beta}$ | $\tau < 0$ and $H = 0$ |
| Zero-field susceptibility | $\gamma$ | $\chi \sim \|\tau\|^{-\gamma}$ | $\tau > 0$ and $H = 0$ |
| | $\gamma'$ | $\chi \sim \|\tau\|^{-\gamma'}$ | $\tau < 0$ and $H = 0$ |
| Critical Isotherm | $\delta$ | $m \sim \|H\|^{\frac{1}{\delta}}$ | $\tau = 0$ and $H \neq 0$ |
| Correlation function | $\xi$ | $\mathcal{C}(\vec{r}) \sim r^{-x} e^{-r/\xi}$ | $\tau \neq 0$ |
| | $\eta$ | $\mathcal{C}(\vec{r}) \sim r^{-d+2-\eta}$ | $\tau = 0$ and $H = 0$ |
| Correlation length | $\nu$ | $\xi \sim \|\tau\|^{-\nu}$ | $\tau < 0$ and $H = 0$ |
| | $\nu'$ | $\sim \|\tau\|^{-\nu'}$ | $\tau < 0$ and $H = 0$ |

**Table 1:** Most important critical exponents for a magnetic system. ($d$ is the number of spatial dimensions in the system [4].

The most important critical exponents used for magnetic systems are shown in Table 1. One advantage of the critical exponents is that its sign shows how the thermodynamic potential behaves near criticality. A negative $\lambda$ in Eq. (7) leads to a potential which diverges to infinity at the critical point, shown in Figure 4a), while a positive critical exponent indicates that the function approaches zero, as shown in Figure 4b). Note that Figure 4a) has a similar form to the zero-field susceptibility in Figure 2d). In the next chapter the critical exponents for the mean-field theory are derived yielding a value of $-1$ for the susceptibility, which is one example which reaffirms the theory. For $\lambda = 0$ there is no unique behaviour of the thermodynamic potential. Examples are shown in Figure 4c) and d), where a logarithmic divergence and a cusp-like singularity are shown [4].

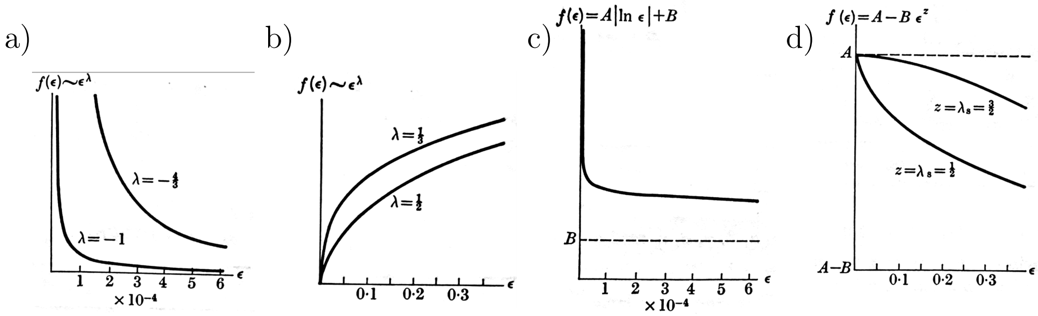


**Fig. 4:** Examples of a thermodynamic potential for different signs of the critical exponent. The thermodynamic potential **a)** diverges to infinity for $\lambda < 0$,**b)** approaches zero for $\lambda > 0$, **c)** diverges logarithmically for $\lambda = 0$ or **d)** has a cusp-like singularity [4].

The other advantage of critical exponents is that there are *universal*, i.e. they do not depend on many details the system, but systems which share the same critical exponents show the same critical behaviour [3].

## 3.2 Universality

Hocken and Moldover showed 1976 experimentally that the liquid-gas transition of Xe, $SF_6$ and $CO_2$ are close to the one expected for the three dimensional Ising model [5]. They are grouped in the same *universality class*. This has the advantage that instead of studying a complicated Hamiltonian of a fluid, one can study the simple Ising model. The renormalization theory was an important concept regarding universality. It changes the scale of a system and thus focuses on a more macroscopic view than on the short-range details [3].

## 3.3 Scaling

The physical phenomena which leads to a non-analytic behaviour at the critical temperature is that fluctuations of all length scales occur.

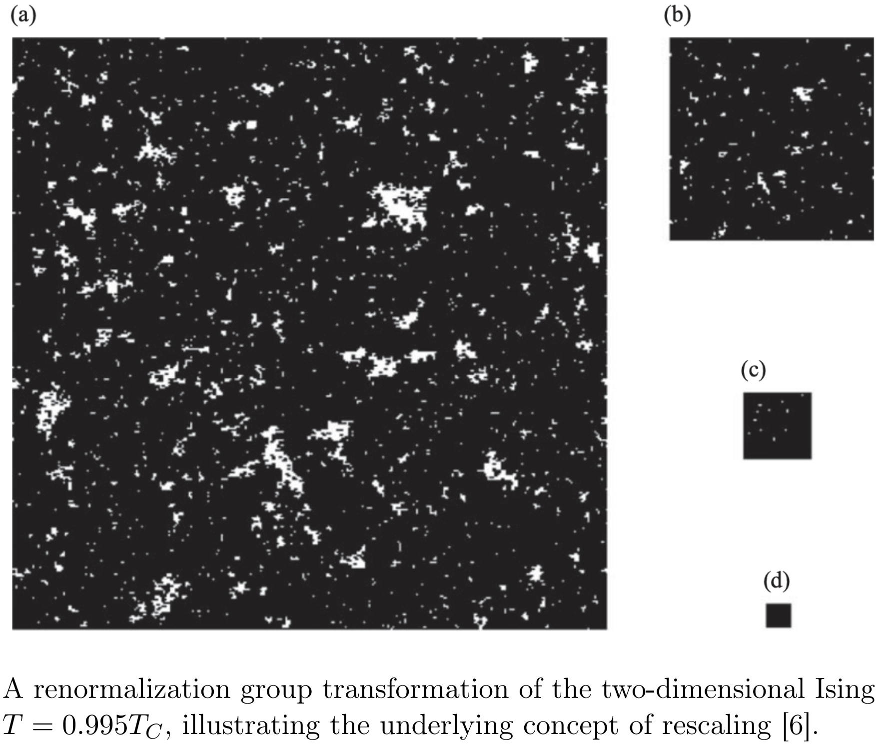


**Fig. 5:** A renormalization group transformation of the two-dimensional Ising model at $T = 0.995T_C$, illustrating the underlying concept of rescaling [6].

As already mentioned, by *rescaling* the system and observing its behaviour, one can extract information of the initial system. The underlying formula is following

$$f_s(\lambda^{y_T}\tau, \lambda^{y_H}H) = \lambda^d f_s(\tau, H), \tag{9}$$

where $\lambda^{y_T}$ and $\lambda^{y_H}$ are called thermal and magnetic exponent, $d$ the dimensionality of the system and $f_s$ is the singular part of the total free energy per spin [1]. The rescaling is demonstrated in the following in Figures 5-7 by a Monte Carlo simulation of the two-dimensional ferromagnetic Ising model. By applying a block-transformation, $3\times 3 = 9$ neighboring spins are replaced by a single spin indicating the direction/color that most of the 9 spins have shown.

Figure 5 shows the spin configuration for a temperature just below the critical one $T = 0.995T_C$. Rescaling the lattice eliminates any short-range order so that in the end in Figure 5d) only one block remains, where all spins are the same and the system is completely ordered. (Remember that it was already discussed that $T < T_C$ leads to an ordered system).

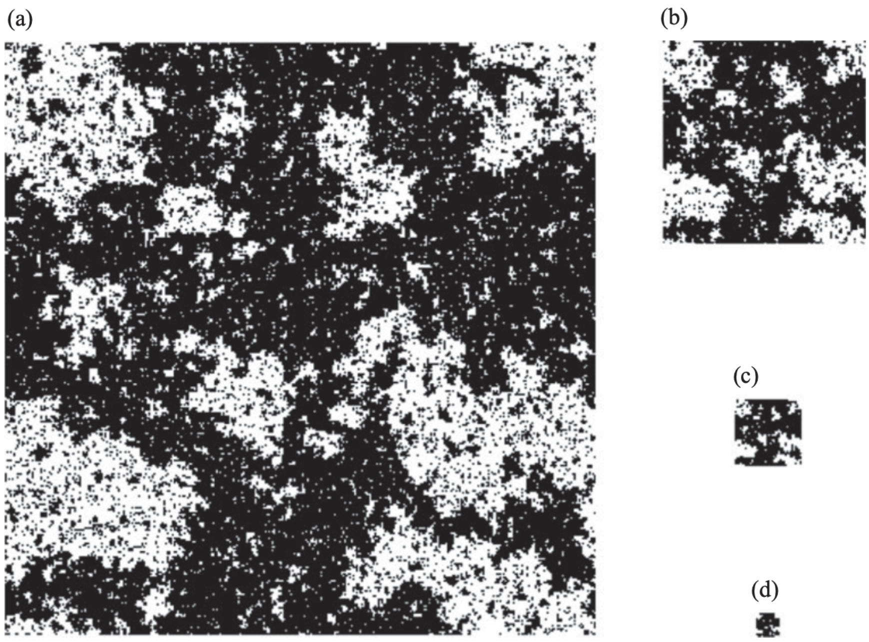


**Fig. 6:** Same as in Figure 5 but for $T = T_C$, where the system is invariant under renormalization [6].

If the temperature of the system is exactly the critical one $T = T_C$ and the renormalization process is iterated, the system remains the same with no preference between white or black spins. The reason is that at criticality fluctuations of all length scales exist. This behaviour is shown in Figure 6.

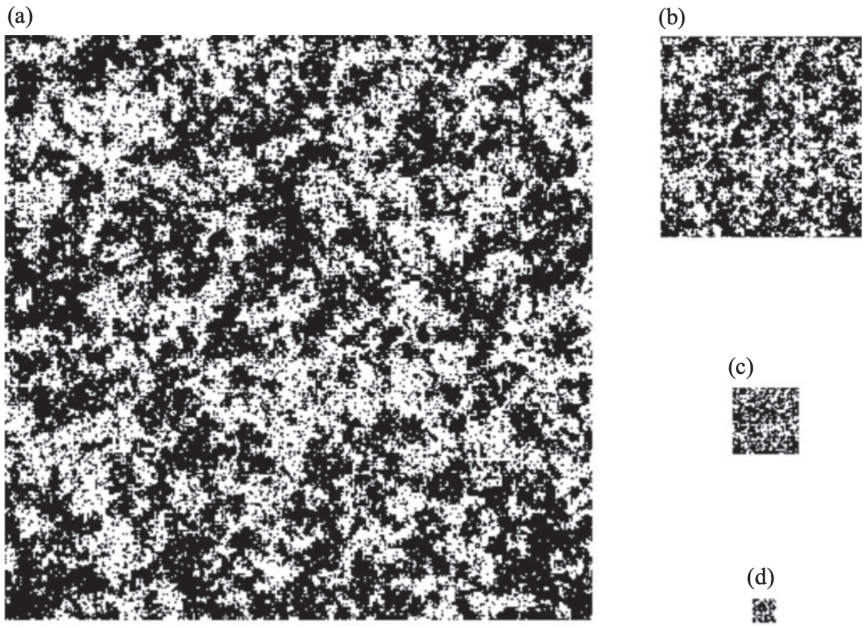


**Fig. 7:** Same as in Figure 5 but for $T = 1.05T_C$, where the system becomes completely disordered [6].

For temperatures above the criticality $T = 1.05T_C$, shown in Figure 7, the repeated application eliminates the short-range order yielding a completely disordered and uncorrelated system in Figure 7d). This is intuitive since physically for high temperatures the spins start to fluctuate rapidly [6].

The definition of the renormalization group is provided in the appendix and further details can be found in the following references [3], [1] and [7].

## 3.4 Relation between the exponents

The scaling hypothesis can be used to calculate the relations between the critical exponents. The derivation is spared here but can be looked up in [1] and [4] where it is demonstrated that with just two critical exponents, the remaining ones can be calculated. Following relations can be extracted

$$2 = \alpha + 2\beta + \gamma \tag{10}$$

$$\gamma = \beta(\delta - 1) \tag{11}$$

$$\gamma = \nu(2 - \eta) \tag{12}$$

$$d\nu = 2 - \alpha \tag{13}$$

with $\alpha = \alpha' = 2 - \frac{d}{y_T}$, $\gamma = \gamma'$ and $\nu = \nu'$ [2].

# 4 Mean-field theory

As already explained in the introduction, statistical physics deals with many-body problems and thus with many degrees of freedoms. For example, the $N-$spin Ising model, where each spin can have two states, has a total number of states of $2^N$. For increasing $N$, it gets more difficult to calculate an exact solution, why approximations are used. The most common one is the mean-field theory (MFT) which analyzes the behaviour of many-body problems by neglecting the interaction of the correlation between fluctuations of different particles. Although its prediction for the critical exponents are not as accurate like the renormalization group, it serves as a good intuitive model to understand phase transitions. In this section, the Ising model is again used as an example. Its Hamiltonian is defined in Eq. (1). By decomposing the spin in a mean part $\langle S \rangle$ and a fluctuation part

$$S_i = \langle S_i \rangle + \delta S_i, \tag{14}$$

the Hamiltonian for a translationally invariant system, i.e. $\langle S_i \rangle = \langle S_j \rangle \equiv m$, can be rewritten as

$$\begin{aligned}
\mathcal{H} &= -J \sum_{\langle ij \rangle} (m + \delta s_i)(m + \delta s_j) - H \sum_i s_i && (15) \\
&= -J \sum_{\langle ij \rangle} \Big[ m^2 + \underbrace{m(\delta S_i + \delta S_j)}_{=m(S_i+S_j)-2m^2} + \underbrace{\cancel{\delta S_i \delta S_j}}_{\text{MF approx.}} \Big] - H \sum_i S_i && (16) \\
&= -J \sum_{\langle ij \rangle} m^2 - Jm \sum_{\langle ij \rangle} (\delta S_i + \delta S_j) - H \sum_i S_i. && (17)
\end{aligned}$$

The **mean-field (MF) approximation** neglects the second order fluctuation term in Eq. (16). Assuming a lattice with $N$ sites and $z$ nearest neighbors, called coordinate number, which leads to a number of $\sum_{\langle ij \rangle} = \frac{Nz}{2}$ nearest neighbors pairs, one obtains

$$\begin{aligned}
\mathcal{H}_{MF} &= -\frac{JNz}{2} m^2 - Jmz \sum_i \underbrace{\delta S_i}_{=S_i - m} - H \sum S_i && (18) \\
&= \frac{JNz}{2} m^2 - \underbrace{(Jzm + H)}_{H_{\text{eff}}} \sum_i S_i. && (19)
\end{aligned}$$

At this point, the MF approximation decoupled the Ising Hamiltonian into an effective mean field $H_{\text{eff}}$ with only one undetermined parameter $m$ and no interaction between the spins anymore. The magnetization $m$ can be determined via the partition function and the density operator. The ansatz is shown but the derivation is spared here.

$$m = \frac{\sum\limits_{S_i=\pm 1} \left(S_i e^{\beta H_{\text{eff}} S_i}\right)}{\sum\limits_{S_i=\pm 1} \left(e^{\beta H_{\text{eff}} S_i}\right)} = \tanh \beta(Jzm + H), \tag{20}$$

called *self-consistency relation* [6].

## 4.1 Graphical solution of the self-consistency equation

To obtain the solution of the self-consistency relation, the variable $x \equiv \beta(Jzm+H)$ is introduced to rewrite Eq. (20)

$$\frac{kT}{Jz}x - \frac{H}{Jz} = \tanh x. \tag{21}$$

By plotting both sides of Eq. (21), see Figure 8a) for $H = 0$, the intersection leads to $m$.

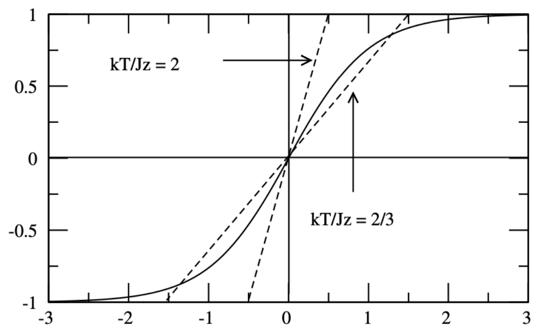


**Fig. 8:** Graphical analysis of the mean-field equation [4].

There are two interesting cases now.

$$\text{for } \frac{kT}{Jz} \equiv \frac{T}{T_C} \begin{cases} > 1 \Rightarrow \text{one solution: } m = 0, \\ < 1 \Rightarrow \text{three solutions: } m \neq 0 \text{ and } m = 0 \text{ (unstable).} \end{cases} \tag{22}$$

This result is also confirmed in the $M-T$ phase diagram in Figure 2c). For $T > T_C$ the net magnetization is zero, for $T < T_C$ there are two solution which are not equal to zero [6].

## 4.2 Critical exponents of the MFT

Next, the critical exponents of the MFT are determined. The first one is the critical exponent corresponding to the zero-field magnetization $m(\tau, H = 0) \sim (-\tau)^{\beta}$, shown in Table 1.
The interesting region is near criticality at $T \lesssim T_C$ where $m$ is small. Thus the tanh in Eq. (20) can be expanded

$$m \approx \beta Jzm - \frac{1}{3}(\beta Jz)^3 m^3 + \mathcal{O}(m^5) \tag{23}$$

Since $m \neq 0$, one can divide by $m$, which results into

$$m = \pm\left(\frac{3(Jz - T)}{\beta^2 (Jz)^3}\right)^{1/2} \approx \pm\left(\frac{3(T_C - T)}{T_C}\right)^{1/2} \qquad \text{for } \frac{T}{T_C} \approx 1 \tag{24}$$

$$\sim (-\tau)^{1/2} \Rightarrow \beta = \frac{1}{2}. \tag{25}$$

Therefore, the critical exponent for the magnetization is $\beta = \frac{1}{2}$.

The next critical exponent which will be determined is the one corresponding to the magnetic susceptibility $\chi(\tau) \sim |\tau|^{-\gamma}$. For that, temperatures above the critical one are examined with $H \neq 0$. The right-hand side of Eq.(20) can again be expanded to

$$m \approx \beta Jzm + \beta H \Leftrightarrow m = \frac{H}{k(T - T_C)} \tag{26}$$

$$\Rightarrow \chi = \left(\frac{\partial m}{\partial H}\right) \sim \frac{1}{T - T_C} \sim |\tau|^{-1} \Rightarrow \gamma = 1 \tag{27}$$

The critical exponent in the mean-field approximation is determined to be $\gamma = 1$.

One can obtain the other exponents in an analog way or by making use of the relations Eqs.(10)-(13). This leads to the specific heat critical exponent $C_H(\tau, H = 0) \sim |\tau|^{-\alpha}$ with $\alpha = 0$ and the critical isotherm $m(\tau = 0, H) \sim |H|^{1/\delta}$ where $\delta = 3$ [6].

## 4.3 Limitation of the MFT

The MFT is a good approximation if the system mostly deals with short-range order where the correlation between the spins are small and can thus be neglected.

Ginzburg formulated 1961 the criterion that the fluctuation of the order parameter $m$ are much less than the mean order parameter near criticality

$$\langle(\delta m)^2\rangle \ll \langle m\rangle^2. \tag{28}$$

Conducting the derivations results in the following relation for the critical exponents

$$\frac{2\beta}{\nu} + (2-\nu) < d. \tag{29}$$

For the mean-field approximation, this leads to $4 < d$ thus for dimensions less than 4, the mean-field approximation is not accurate. It is still a good theory to understand the underlying concepts qualitative [1].

# 5 Conclusion

Phase transitions and critical phenomena are subjects of increasing interest in both research and industry. Phase transitions are characterized by a significant change in the properties of a system. Above the critical temperature the system shows even anomalies. The identification of critical behavior involves analyzing the discontinuities and divergences derivatives of thermodynamic potentials at the critical temperature. Hence, critical exponents are introduced to describe the thermodynamic potential near criticality. The advantage of these critical exponents is that these are universal, meaning systems with the same critical exponents belong to the same universality class and exhibit similar phase transitions. Moreover, these exponents are interrelated, thus only two of them are required to deduce the rest. These relations are derived from scaling behaviours where rescaling the system allows extracting the important information by elimating details from short range order. There are different models, which are not covered in this report due to its introductory nature. Examples include Landau's theory, which serves as a mean-field approach, and a more detailed perspective of the renormalization group.

# A Renormalization Group

The following points provide the conceptual foundation of a renormalization group transformation.
a) The idea behind the renormalization group approach is to transform or *renormalize* the initial reduced Hamiltonian $\bar{\mathcal{H}} \equiv \mathcal{H}/kT$ to a new Hamiltonian $\bar{\mathcal{H}}'$

$$\bar{\mathcal{H}}' = \mathbf{R}\bar{\mathcal{H}}. \tag{30}$$

b) In doing so, the number of degrees of freedom are reduced by the renormalization group operator $\mathbf{R}$ from $N$ to

$$N' = N/b^d, \tag{31}$$

where $d$ is the spatial dimensionality and $b$ the (re-)scale factor.
c) Preservation of the partition function is the fundamental condition to be fulfilled by any renormalization operator $\mathbf{R}$

$$\mathcal{Z}_{N'}(\bar{\mathcal{H}}') = \mathcal{Z}_N(\bar{\mathcal{H}}). \tag{32}$$

d) Since the spatial density of degrees of freedom need to be preserved, all lengths have to be rescaled, i.e. lengths are now determined by the new lattice spacing

$$\vec{x}' = \vec{x}/b. \tag{33}$$

Analog momenta can be renormalized reciprocally by

$$\vec{q}' = b\vec{q}. \tag{34}$$

e) Lastly, the spin vectors need to be rescaled to preserve the properties of the spin correlation function

$$\vec{s}'_{\vec{x}'} = \vec{s}_{\vec{x}}/c. \tag{35}$$

According to Eq.(32), the reduced free energy $\bar{f} \equiv f/kT$ transforms as

$$\bar{f}(\bar{\mathcal{H}}') = b^d \bar{f}(\bar{\mathcal{H}}). \tag{36}$$

In a similar way, one can conclude from Eq. (35) that the spin-spin correlation function transforms as

$$C(\vec{x}; \bar{\mathcal{H}}) = c^2 C(\vec{x}/b; \bar{\mathcal{H}}'). \tag{37}$$

These are both important equation, which will later lead to scaling properties.

The reason of this whole procedure is to locate *fixed points*

$$\bar{\mathcal{H}}' = \bar{\mathcal{H}} \equiv \bar{\mathcal{H}}^*, \tag{38}$$

i.e. points where scale invariance applies, which describe the system at criticality. For example, according to Eq.(33) the correlation length should be reduced by a factor $b$

$$\xi' = \xi/b, \tag{39}$$

but otherwise at criticality, the correlation length should be the same for the initial and the renormalized system

$$\xi' = \xi \equiv \xi^*. \tag{40}$$

This can only be the case if $\xi$ is infinite (or trivially zero), as it is expected at the critical point.

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