

Topological phase transitions

Lea Bartels

Seminar in statistical physics - Professor Wolschin

Summer semester 2024

Abstract

In two dimensions, phase transitions caused by symmetry breaking are forbidden by the Mermin-Wagner-Hohenberg theorem. However, there exists another kind of phase transition that is caused by topological defects in the system. Following the calculations done by Kosterlitz and Thouless, the energy of different vortex configurations in the two-dimensional XY model is derived. Topological phase transitions are found to be the result of the unbinding of closely bound vortex-antivortex pairs. Finally, two experiments that observe the topological phase transition are discussed.

Contents

1	Introduction	3
2	The XY model	4
3	Correlation function at different temperatures	6
3.1	High temperature limit	6
3.2	Low temperature limit	7
4	Topological defects	8
4.1	Energy of a single vortex	10
4.2	Energy of a vortex-antivortex pair	11
4.3	The BKT transition	12
5	Experiments	14
5.1	BKT crossover in a trapped atomic gas	14
5.2	First and second sound in a BKT superfluid	16

1 Introduction

A phase transition marks the point of transition from the unordered state of a system to an ordered state. In the ordered state, there will be a non-zero order parameter, often associated with the spontaneous breaking of a symmetry.

We can distinguish between first order phase transitions and second order phase transitions. In first order phase transitions, or discontinuous phase transitions, the order parameter shows a jump at the critical temperature. These can be caused by symmetry breaking but do not always have to be, such as in the case of the gas-to-liquid transition.

Second order phase transitions are continuous in the order parameter. At the critical temperature, there will be a smooth onset of a non-vanishing order parameter. However, the correlation length diverges at the critical temperature. These phase transitions are generally related to symmetry breaking, a famous example being magnetisation.

In a 1966 paper, Mermin and Wagner proved that there cannot be any long range order at temperatures above absolute zero in the one- and two-dimensional Heisenberg model [9]. The Heisenberg model is a toy model with three-dimensional spin vectors. In the same year, Hohenberg extended this absence of long range order to general two-dimensional Bose and Fermi systems [7]. This can be generalized to the Mermin-Wagner-Hohenberg theorem which states that continuous symmetries cannot be spontaneously broken in one- or two-dimensional systems at non-zero temperature. This means there cannot be a non-vanishing order parameter either.

This generally led to the conclusion that there was no continuous phase transition in two-dimensional systems with non-discrete symmetries.

But there were some indications that this might not be the case. Using high temperature extrapolation methods, Stanley and Kaplan found that the correlation shows algebraic behaviour in the two-dimensional Heisenberg model at some critical temperature, just as it does in the three-dimensional case [11].

They drew attention to the fact that the absence of long range order is not the same as the absence of a phase transition. There could be another phase transition that relies on a mechanism other than spontaneous symmetry breaking.

In his 1971 paper, Soviet scientist Berezinskii was the first to recognize the importance of topological excitations for this new mechanism [2]. Independently of him, Kosterlitz and Thouless developed a full theory of a phase transition caused by topological defects in the two-dimensional XY model and multiple related systems [8]. The theory explains the new type of phase transition by the unbinding of pairs of topological excitations into separate entities. The transition is often called the BKT-transition after these three.

This phenomenon has since found widespread application to a large number of physical systems. The concept of universality classes makes it possible to

apply findings from the two-dimensional XY model to systems with the same critical exponents such as two-dimensional crystals or superfluids.

In 2016, David Thouless and Michael Kosterlitz, along with Duncan Haldane, were awarded the Nobel Prize in Physics for their discovery. Vadim Berezinskii had already died at that point and thus could not be awarded.

Following Kosterlitz and Thouless, we want to arrive at an understanding of the topological phase transition in the 2-dimensional XY model.

2 The XY model

The XY model describes an n -dimensional lattice of size L . The spacing between the lattice sites is given by the lattice constant a . There is a two-dimensional spin vector at each lattice site. The spin is confined to the XY plane, hence the name "XY model".

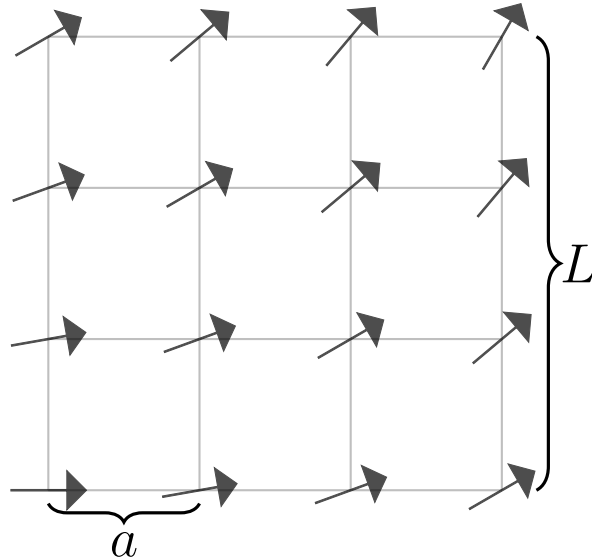


Figure 1: Visualisation of a two-dimensional XY model, a unit length spin vector sits at each lattice site.

It is important to distinguish between the dimensionality of the lattice and that of the spins. The spins always have dimension two and can move freely in the plane. The lattice can have any dimension; we will be working with a two-dimensional lattice to investigate the properties of phase transitions in two dimensions.

The spin vectors all have unit length so they can be described by one angle

each,

$$\vec{s}_i = \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix} . \quad (1)$$

The system possesses a continuous symmetry so with a two-dimensional lattice we have exactly one of the situations governed by the Mermin-Wagner-Hohenberg theorem. We know there cannot be any long range order so we are looking for another type of phase transition.

Similarly to the Ising model, the Hamiltonian only takes next neighbour interaction into account. With a coupling constant J , that gives us

$$H = -J \sum_{\langle i j \rangle} \vec{s}_i \cdot \vec{s}_j , \quad (2)$$

where $\langle i j \rangle$ denotes a sum over all next-neighbour-pairs on the lattice.

For positive J , the energy gets lowered for aligned spins, which is the ferromagnetic case. In the following, we will implicitly assume ferromagnetism in the figures. The equations hold for a more general but constant coupling.

Inserting the spin definition (1), we find

$$H = -J \sum_{\langle i j \rangle} \sin \theta_i \sin \theta_j + \cos \theta_i \cos \theta_j = -J \sum_{\langle i j \rangle} \cos(\theta_i - \theta_j) , \quad (3)$$

where in the last step we used a trigonometric identity.

Later we will be computing expectation values using the canonical partition function

$$Z = \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} e^{-\beta H} , \quad (4)$$

where $\beta = 1/T$ and the Boltzmann factor has been set to $k_B = 1$.

The theory of topological phase transitions in the XY model has had a large impact on many fields in physics. That is because the XY model shares a universality class with a range of other systems such as the superfluid transition in helium-4. The behaviour of physical models is to a large extent determined by their symmetries, especially around the relevant phase transition. This allows us to group models into universality classes. Systems within the same universality class will have the same critical exponents at the phase transition and relations that are derived for one of them hold for the others as well. This is a powerful concept that makes it possible to map the results for the very abstract XY model onto experimentally measurable systems.

3 Correlation function at different temperatures

We want to investigate the correlation behaviour in both the high and the low temperature regime. So we calculate the correlation between two spins, one at an arbitrarily chosen point s_0 , and the other at a point s_r which has a distance of r steps to the first one.

The spin-spin correlation function is thus given by

$$\langle \vec{s}_0 \cdot \vec{s}_r \rangle = \frac{1}{Z} \prod_{i=1}^N \int_0^{2\pi} \frac{d\theta_i}{2\pi} \cos(\theta_0 - \theta_r) \exp \left(\frac{J}{T} \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right) . \quad (5)$$

3.1 High temperature limit

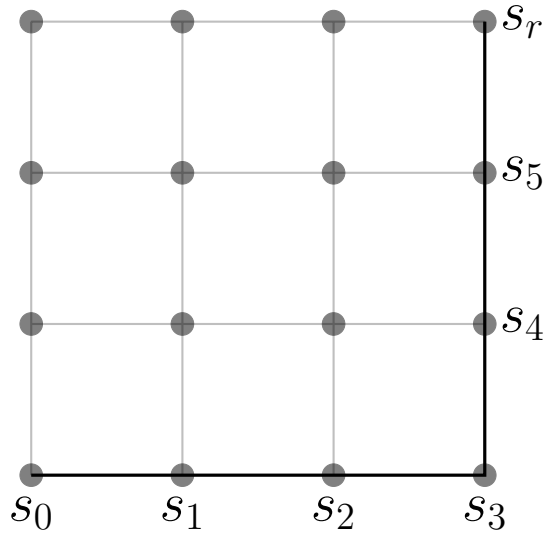


Figure 2: Shown in black is one of the possible shortest paths that connect spin site s_0 with spin site s_r .

At very large temperatures, $K \equiv \frac{J}{T} \ll 1$ so we can Taylor expand the exponential function in (5) to first order. This gives us

$$\langle \vec{s}_0 \cdot \vec{s}_r \rangle = \frac{1}{Z} \prod_{i=1}^N \int_0^{2\pi} \frac{d\theta_i}{2\pi} \cos(\theta_0 - \theta_r) \prod_{\langle ij \rangle} (1 + K \cos(\theta_i - \theta_j)) . \quad (6)$$

This equation contains a product over all next neighbour pairs on the lattice. Using

$$\int_0^{2\pi} \frac{d\theta_1}{2\pi} \cos(\theta_1 - \theta_2) = 0 , \quad (7)$$

we can immediately see that all terms of order K will vanish. With the identity

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_3) = \frac{1}{2} \cos(\theta_1 - \theta_3) , \quad (8)$$

we can essentially integrate out the intermediate steps of connected paths between lattice sites. This will give us a factor of $1/2$ for each of the integrals and the cosine of the spins at the two endpoints. Combining this with (7), we can see that the vast majority of the terms will vanish.

Only paths from lattice site s_0 to lattice site s_r provide non-vanishing contributions since the resulting integral over $\cos^2(\theta_0 - \theta_r)$ does not vanish over a 2π period. The remaining integral then looks like

$$\int_0^{2\pi} \frac{d\theta_0}{(2\pi)^2} \frac{d\theta_r}{(2\pi)^2} \cos^2(\theta_0 - \theta_r) = \frac{1}{2} . \quad (9)$$

Each step along a path contributes a factor of $K/2$. Since K is very small in the high temperature limit, the leading order contribution is given by the direct path between s_0 and s_r . By definition, this will have the length r . There can of course be multiple paths of length r but we will only be considering one of these as we are interested in the qualitative behaviour of the correlation function.

With these considerations, we find the spin-spin correlation to be

$$\langle \vec{s}_0 \vec{s}_r \rangle = \left(\frac{K}{2} \right)^r = e^{-r/\xi} , \quad \text{with } \xi = \frac{1}{\ln(2/K)} . \quad (10)$$

So at high temperatures, the correlation decays exponentially with a characteristic correlation length ξ . This matches our expectations, at high temperatures there should be short range order and very fast decay.

3.2 Low temperature limit

We compare this result to the correlation behaviour at low temperatures.

Here, we make use of the fact that the cosine is the real part of the exponential function and get

$$\langle \vec{s}_0 \vec{s}_r \rangle = \langle \cos(\theta_0 - \theta_r) \rangle = \langle \text{Re } e^{i(\theta_0 - \theta_r)} \rangle . \quad (11)$$

We will now assume that the difference between the angle θ_0 and θ_r is very small, i.e. the spins are highly correlated. This does not mean that we have

long range order in our system so it does not contradict the Mermin-Wagner-Hohenberg theorem. At arbitrarily low temperatures, spins can be correlated over any finite distance. So for any finite length r , we can find a temperature above zero, where this approximation holds.

Knowing that $(\theta_0 - \theta_r)$ is close to zero, we can extend the integration range from $[-\pi, \pi]$ to $(-\infty, \infty)$, thus making $(\theta_0 - \theta_r)$ a variable with Gaussian distribution. This is called the spin-wave-approximation. Now we can use a rule for Gaussian distributions that relates the expectation value of the exponential function to the exponential function of a different expectation value. Applying this rule gives us

$$\langle \text{Re } e^{i(\theta_0 - \theta_r)} \rangle = \text{Re } e^{-\frac{1}{2} \langle (\theta_0 - \theta_r)^2 \rangle} . \quad (12)$$

We can expand the term in the exponent into $-\langle \theta_0^2 \rangle + \langle \theta_0 \theta_r \rangle$, where we will be focusing on the last part as it contains the spatial information. This expectation value is easiest to calculate in Fourier space. Solving it leads us to

$$\langle \vec{s}_0 \cdot \vec{s}_r \rangle = \left(\frac{r}{a} \right)^{-T/4\pi J} . \quad (13)$$

In the low temperature limit, we find algebraic decay instead of the exponential decay at high temperatures. So there is a disordered phase and another phase that is less disordered than the high temperature limit but does not show long range order either. This intermediate state is therefore called quasi long range order (QLRO).

The relevant conclusion we can draw from this is the existence of a phase transition between the two regimes. Since there is no possibility to analytically continue the power law at low temperatures into the exponential function, a smooth crossover is excluded. So there must be a phase transition at the critical point where the correlation function becomes non-analytic.

4 Topological defects

Knowing that a phase transition due to symmetry breaking is excluded, we will consider topological defects and see that these can cause another type of phase transition. We begin by looking at excitations in the system which will introduce topological defects.

A thermal excitation or phonon in the XY model corresponds to a plane wave that makes the spin angles oscillate along its axis as it moves through the system. In the case of long-wavelength excitations, neighbouring spins will be nearly parallel so it is energetically quite cheap to put these into the system. It is precisely these waves that destroy any long range order in our two-dimensional models.

There is also a second type of excitation where the spins point around in a circle rather than a straight line. These are known as vortices and they form a topological defect in the system.

They have the characteristic property that the spin angle changes by 2π on any closed loop around the center of the vortex. This notion can be generalized to a definition of the topological charge

$$\oint d\theta = 2\pi q , \quad (14)$$

where the charge q can take integer values. Positive topological charges are called vortices, negative charges are antivortices.

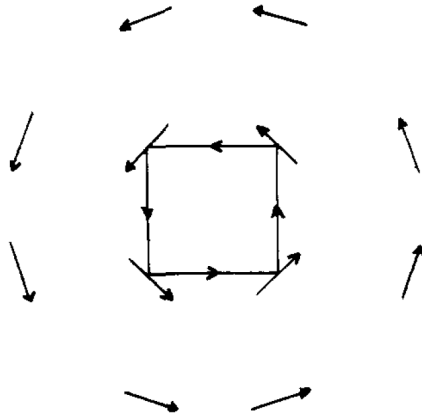


Figure 3: A vortex with topological charge $q = 1$, the spin angle θ changes by 2π when following a closed contour around the center, figure adapted from [8].

We can see that these excitations are of topological nature by looking at the contour of the integration. The charge does not change when the contour is deformed as long as the center is enclosed in the loop. If we shrink the contour down to a single point, we can see that the spin angle right at the vortex center cannot be defined. The system has a singularity there.

In other physical systems that belong to the same universality class as the XY model, topological defects may present themselves in a shape different from vortices. In a two-dimensional crystal for example, they show up as free dislocations.

A vortex is a configuration distinct from the ground state. This means that a vortex cannot be brought into the ground state by symmetry transformation. There are many different configurations with topological charge $q = 1$ that can all be transformed into one another by a rotation of all spins. But the charge itself is conserved and thus a $q = 0$ state like the ground state can never be reached by such a transformation.

This property cannot be generalized to systems with other dimensionalities. In a two-dimensional system with three-dimensional spin vectors like the Heisenberg model, the spins can escape into the third dimension outside the plane. So there exists a continuous transformation that transforms a vortex into the ground state, as can be seen in figure 4.

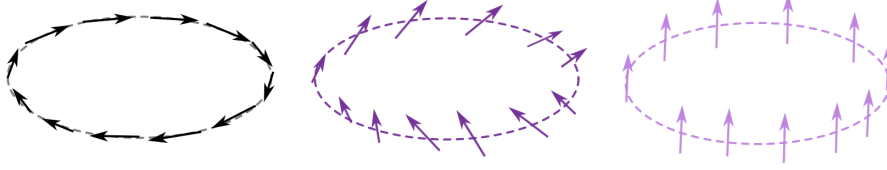


Figure 4: A vortex can be removed if the spin can also reach the third dimension, figure adapted from [1].

This once again illustrates how the symmetries of the system shape its properties.

4.1 Energy of a single vortex

We are interested to see how the presence of the vortex changes the energy of the system. This will also tell us how likely a vortex is to occur.

For convenience, we will be working in the continuum limit of the Hamiltonian which we need to derive from the Hamiltonian given in (3)

$$H = -J \sum_{\langle i j \rangle} \cos(\theta_i - \theta_j) . \quad (15)$$

To take the continuum limit, we can assume the changes between neighbouring spin sites to be infinitesimally small since we want to move from separate sites to a smooth field. Therefore we can Taylor expand the cosine to find

$$H = -J \sum_{\langle i j \rangle} 1 - \frac{1}{2}(\theta_i - \theta_j)^2 . \quad (16)$$

The constant term will add a ground state energy. Since we are interested in energy changes, we can neglect this and only focus on the quadratic term. The continuum limit of the Hamiltonian then is

$$H = \frac{J}{2} \sum_{\langle i j \rangle} a^2 \frac{(\theta_i - \theta_j)^2}{a^2} \xrightarrow{\text{cont. limit}} H = \frac{J}{2} \int d^2x (\nabla\theta)^2 . \quad (17)$$

Depending on the way the limit is taken, there might be an additional factor of $1/2$ so there might be slightly varying results in different textbooks.

To calculate the energy for a certain configuration, we need to find $\nabla\theta$ for that specific spin setup. In the case of a single vortex, we go back to the definition of the topological charge (14) to determine this.

$$2\pi q = \oint d\theta = \oint dr \nabla\theta = 2\pi r \nabla\theta , \quad (18)$$

where in the last step we used that the topological charge does not depend on the integration contour so we assumed a circle of radius r . This leaves us with

$$\nabla\theta = \frac{q}{r} . \quad (19)$$

We set $q = 1$ to find the energy cost of the lowest excitation of a single vortex above ground state. Then the energy for a single vortex is

$$E_V = \frac{J}{2} \int d^2x \frac{1}{r^2} = \frac{J}{2} \int_a^L dr \, 2\pi r \frac{1}{r^2} = \pi J \ln\left(\frac{L}{a}\right) , \quad (20)$$

where we chose the lattice constant a to be the lower integration boundary as we can not resolve anything below that scale.

The result scales with the system size and will diverge for a smooth system of infinite size. This makes it extremely unlikely that a single vortex is spontaneously created by thermal fluctuations and thus this configuration will be highly suppressed.

Another interesting thing to remark upon is the logarithmic scaling of the energy. This is reminiscent of a Coulomb potential in two dimensions. A monopole in a two-dimensional electric field will show exactly the same behaviour. With this similarity in mind, we take a look at the energy of a vortex-antivortex pair.

4.2 Energy of a vortex-antivortex pair

Drawing inspiration from this comparison, we will now consider the vortex-equivalent of a dipole which is a bound pair of a vortex and an antivortex. The idea is that like with an electric dipole, the opposing charges will shield each other and the lattice gets only locally disturbed.

For the electric dipole, we know that

$$\nabla\theta = \frac{d}{r^2} , \quad (21)$$

where d is the separation between the two vortex centers. Inserting this expression into the Hamiltonian gives us

$$E_D = \frac{J}{2} \int_a^L dr \, 2\pi r \left(\frac{d}{r^2}\right)^2 = \frac{J\pi}{2} d^2 \left(\frac{1}{a^2} - \frac{1}{L^2}\right) . \quad (22)$$

Since L grows large in the thermodynamic limit, we drop the $1/L^2$ term and are left with the dipole energy

$$E_D = \frac{J\pi}{2} \frac{d^2}{a^2} . \quad (23)$$

When a becomes very small in the thermodynamic limit, d has to follow, otherwise the dipole energy would diverge, too. So for closely bound dipoles with very small distance d , the dipole energy stays finite. The energy cost scales with the separation, not with the system size. It is therefore common for vortex-antivortex pairs to occur as thermal excitations.

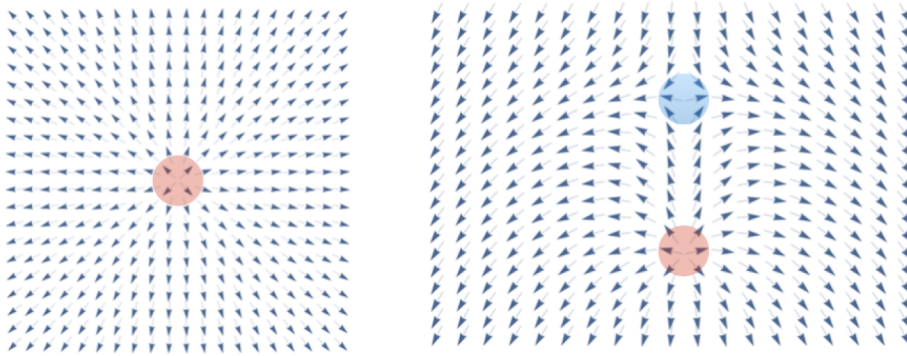


Figure 5: A single vortex affects the orientation of every spin in the system and thus the energy scales with the system size. A vortex-antivortex pair shows shielding and affects the system locally, the energy scales with the separation, figure adapted from [5].

This difference between single vortex and dipole also makes sense intuitively, for a graphical representation see figure 5. To introduce a single vortex into the system, every single spin needs to align itself. So the energy scales with the system size and it is very unfavourable for such a configuration to spontaneously manifest. By the same logic, a vortex will not spontaneously vanish once it has formed. In a vortex-antivortex pair however, the two opposing charges can shield each other and spins at large distances will not be affected. So only a very limited amount of spin vectors need to be aligned as a response and the energy stays finite.

4.3 The BKT transition

We are now in a position to understand Kosterlitz's and Thouless' argument on how topological defects can lead to a phase transition.

As the vortex-antivortex pairs only have a finite energy cost, they will appear in systems with non-zero temperatures as thermal excitations. However, closely bound pairs that move as one entity are not entropically favourable over

two separate vortices. So with increasing temperature, the entropy becomes increasingly relevant as well and there is a point where the contribution of the entropy becomes dominant and the vortex-antivortex-pair dissolves. This unbinding of vortices causes the phase transition. Below the critical temperature the system is topologically ordered, above there is no order, instead there is the disordered state we derived in (10).

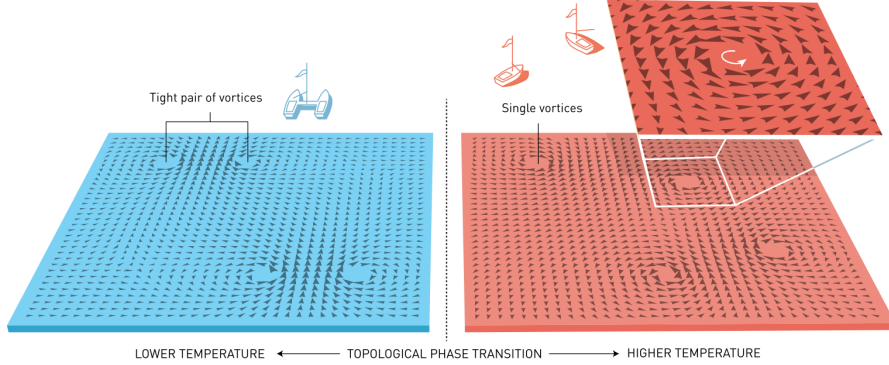


Figure 6: Tightly bound vortex-antivortex pairs in the topologically ordered state unbind into separate vortices during the phase transition, figure adapted from [6].

With the notion that vortices unbind once the entropy dominates, we can make a rough approximation of the critical temperature. For this, we look at the free energy which compares energy to entropy. The free energy is given by

$$F = E - TS . \quad (24)$$

We can find an approximation for the entropy with a combinatorial argument. In a system with size L^2 , there will be L^2/a^2 possibilities to place a vortex of the size a^2 . So we conclude that the entropy of a single vortex is

$$S_V = \ln \left(\frac{L^2}{a^2} \right) . \quad (25)$$

This no longer holds if we take multiple vortices into account but we limit ourselves to one vortex here.

For the energy, we will use the energy cost of a single vortex as derived in (20). This is a reasonable choice because we want to find the point where the energy cost of a single vortex is overruled by its entropy. So we find

$$F = E_v - TS_v = (\pi J - 2T) \ln \left(\frac{L}{a} \right) , \quad (26)$$

and solve for the critical temperature T_c

$$T_c = \frac{\pi J}{2} . \quad (27)$$

Note that this only works because both energy and entropy depend logarithmically on the system size. This is to be expected for the entropy. In case of the energy however, this can be led back to the fact that the Coulomb potential has a logarithmic solution in two dimensions. So it already becomes evident that the BKT transition is particular to two dimensional models.

5 Experiments

The two-dimensional XY model and two-dimensional quantum gases belong to the same universality class so it is possible to use experimental results from such quantum gases to check the results derived in the purely theoretical XY model.

5.1 BKT crossover in a trapped atomic gas

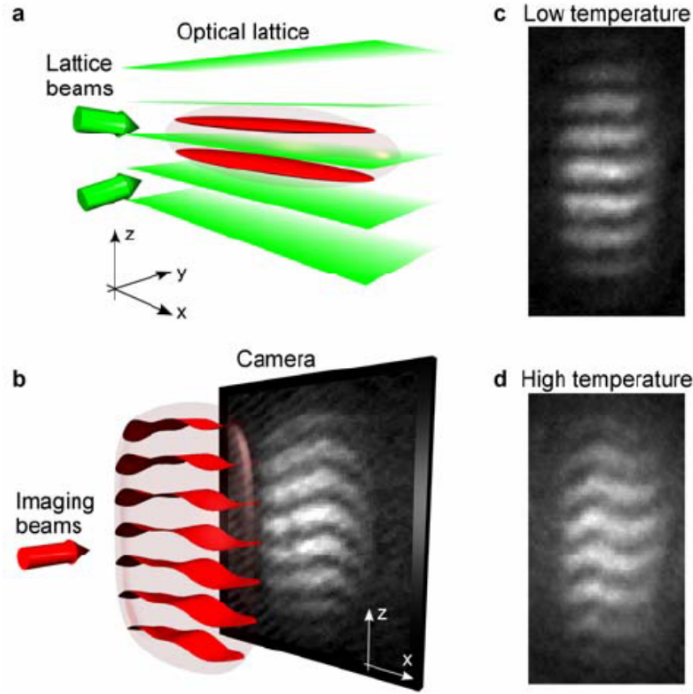


Figure 7: In the experimental setup, a cold gas gets confined to a plane, images of the interference are taken after the slices have mixed, figure adapted from [4].

The 2006 paper by Hadzibabic et al. is one of the most famous experimental results regarding topological phase transitions. They observed the emergence of topological defects at the critical temperature in a two-dimensional quantum gas [4]. The spin-equivalent in a quantum gas is the phase as it also has a

continuous symmetry. So a jump in the phase of the quantum gas is equivalent to a vortex in the XY model.

In their experimental setup, the cold gas is compressed into multiple two-dimensional slices by an optical lattice. After a given equilibration time, the lattice potential gets turned off. The gas now expands in z -direction and mixes with the other two-dimensional slices. This leads to interference of the phases and a CCD camera is used to take images of the resulting interference patterns. Figure 7 shows examples of such patterns for low and high temperatures. The wavy pattern in the high temperature image **d** is caused by phonons, the low temperature image **c** shows visibly less of these thermal excitations.

We can also see that the contrast is lower in the high temperature image than in the low temperature image. It is inversely related to the temperature and since it is complicated to directly measure the temperature of the two-dimensional slices, the contrast is measured instead.

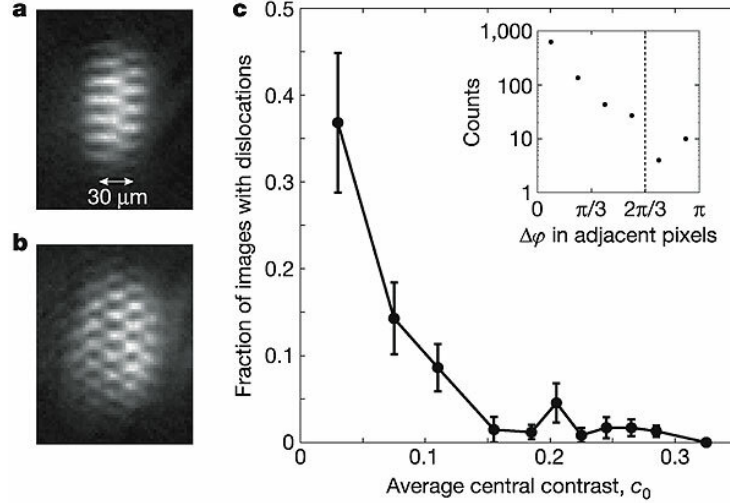


Figure 8: On the left are examples of interference patterns with one or multiple jumps in the phase due to topological defects. On the right, the proportion of the images that show a topological defect is plotted against the contrast. With increasing temperature (decreasing contrast) there is a sudden onset of vortices, figure adapted from [4].

Topological defects show up on the interference patterns as sharp dislocations, figure 8 shows example images. It is not possible to see the vortex-antivortex pairs below the critical temperature because they are shielded too well and thus do not show up in the interference patterns. Once the dipoles unbind, the free vortices become visible as phase distortions. The ratio of images that show one or more of these distortions is plotted against the contrast. In the corresponding plot, we can see that the fraction of images with vortices increases sharply below

a certain value of the contrast so above a certain temperature. This is the critical temperature and the onset of free vortices is the unbinding phase transition.

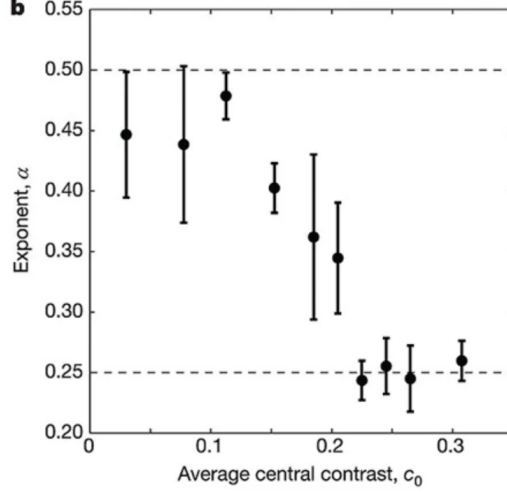


Figure 9: Transition between fast and slow correlation decay, the decay exponent α is large at low contrast/high temperature so correlations decay faster and the system is disordered, figure adapted from [4].

This result combines with the measurement of the decay exponent α at different temperatures. Figure 9 shows the phase transition in terms of the decay behaviour. At high temperatures, α is large so correlations decay fast and there is no order. When the temperature decreases, the decay exponent α also decreases and the gas transitions into the quasi long range ordered state where correlations decay more slowly.

The transition itself does not look sharp like a phase transition but smooth like a crossover instead. This can be traced back to the finite size of the system and other experimental limitations.

These two results combined are evidence that the topological phase transition was observed here. In particular, the emergence of vortices above the critical temperature becomes visible which strongly supports the notion of unbinding to describe the BKT phase transition.

5.2 First and second sound in a BKT superfluid

Christodoulou et al. were the first to observe the two sound modes for a two-dimensional superfluid and could also identify a theoretically predicted jump in the superfluid phase space density at the topological phase transition [3][10].

In three dimensions, the liquid-to-superfluid transition is a second order phase transition that is associated with symmetry breaking and the emergence of long range order, while the Mermin-Wagner-Hohenberg theorem excludes such long range order in two dimensions. Still, there can be two-dimensional superfluids at finite temperature due to the BKT phase transition. It has also been predicted that there is a jump in the superfluid phase space density at the critical temperature [10].

A characteristic property of three-dimensional superfluids is the existence of two different sound excitations called first and second sound. These are a consequence of describing the fluid below the critical temperature as a mixture of two components, a superfluid and a viscous one. The two-sound-phenomenon was predicted to also occur in two-dimensional superfluids but had not been observed experimentally before. The authors of this paper were the first to experimentally observe first and second sound in a two-dimensional superfluid [3].

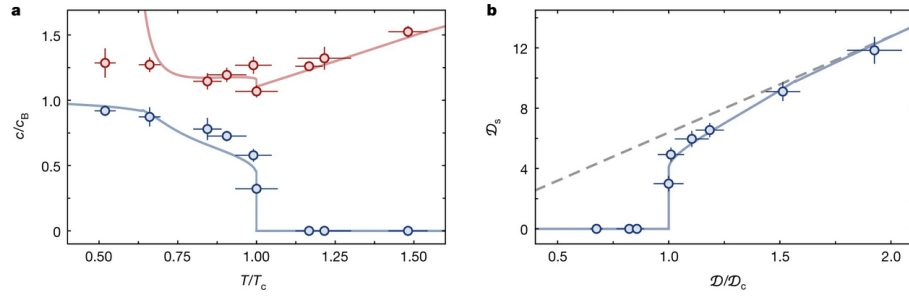


Figure 10: **a** First (red) and second (blue) sound speed in a two dimensional superfluid, **b** a jump in the superfluid phase space density can be observed, figure adapted from [3].

Figure 10 shows the measured sound speeds of both first sound (red) and second sound (blue) plotted against the temperature. At the critical temperature, there are discontinuities in both sound speeds.

The sound speeds are then used to deduce the phase space density. This shows a jump from 0 to 4 at the critical point which is according to theoretical predictions.

References

- [1] Aron Beekman, Louk Rademaker, and Jasper Wezel. “An introduction to spontaneous symmetry breaking”. In: *SciPost Physics Lecture Notes* (Dec. 2019). DOI: 10.21468/SciPostPhysLectNotes.11.
- [2] V. L. Berezinsky. “Destruction of long range order in one-dimensional and two-dimensional systems having a continuous symmetry group. I. Classical systems”. In: *Sov. Phys. JETP* 32 (1971), pp. 493–500.
- [3] Panagiotis Christodoulou et al. “Observation of first and second sound in a BKT superfluid”. In: *Nature* 594 (June 2021), pp. 191–194. DOI: 10.1038/s41586-021-03537-9.
- [4] Zoran Hadzibabic et al. “Berezinskii-Kosterlitz-Thouless Crossover in a Trapped Atomic Gas”. In: *Nature* 441 (July 2006), pp. 1118–21. DOI: 10.1038/nature04851.
- [5] Thors Hans Hansson, David Haviland, and Gunnar Ingelman. “Advanced information. NobelPrize.org. Nobel Prize Outreach AB 2024. Tue. 16 Apr 2024.” In: (). URL: <<https://www.nobelprize.org/prizes/physics/2016/advanced-information/>>(last_accessed_18.04.24).
- [6] Thors Hans Hansson, David Haviland, and Gunnar Ingelman. “Popular information. NobelPrize.org. Nobel Prize Outreach AB 2024.” In: (). URL: <<https://www.nobelprize.org/prizes/physics/2016/popular-information/>>(last_accessed_18.04.24).
- [7] P. C. Hohenberg. “Existence of Long-Range Order in One and Two Dimensions”. In: *Phys. Rev.* 158 (2 June 1967), pp. 383–386. DOI: 10.1103/PhysRev.158.383. URL: <https://link.aps.org/doi/10.1103/PhysRev.158.383>.
- [8] J. M. Kosterlitz and D. J. Thouless. “Ordering, metastability and phase transitions in two-dimensional systems”. In: *J. Phys. C* 6 (1973), pp. 1181–1203. DOI: 10.1088/0022-3719/6/7/010.
- [9] N. D. Mermin and H. Wagner. “Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models”. In: *Phys. Rev. Lett.* 17 (22 1966), pp. 1133–1136. DOI: 10.1103/PhysRevLett.17.1133. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.17.1133>.
- [10] David R. Nelson and J. M. Kosterlitz. “Universal Jump in the Superfluid Density of Two-Dimensional Superfluids”. In: *Phys. Rev. Lett.* 39 (19 Nov. 1977), pp. 1201–1205. DOI: 10.1103/PhysRevLett.39.1201. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.39.1201>.
- [11] H. E. Stanley and T. A. Kaplan. “Possibility of a Phase Transition for the Two-Dimensional Heisenberg Model”. In: *Phys. Rev. Lett.* 17 (17 1966), pp. 913–915. DOI: 10.1103/PhysRevLett.17.913. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.17.913>.